

# Chapter 8.1.1-8.1.2. Generating Functions

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# Ordinary Generating Functions (OGF)

- Let  $a_n$  ( $n = 0, 1, \dots$ ) be a sequence.
- The *ordinary generating function* (OGF) of this sequence is

$$G(x) = \sum_{n=0}^{\infty} a_n x^n .$$

Note: If the range of  $n$  is different, sum over the range of  $n$  instead.

- This is the *Maclaurin series* of  $G(x)$ , which is the Taylor series of  $G(x)$  centered at  $x = 0$ .
- Chapter 8.2 has another kind of generating function called an *exponential generating function*. But “generating function” without specifying which type usually refers to the “ordinary generating function” defined above.

# Example: $3^n$

- Let  $a_n = 3^n$  for  $n \geq 0$ : 1, 3, 9, 27, 81, 243, ...

$$G(x) = \sum_{n=0}^{\infty} 3^n x^n = \frac{1}{1-3x}$$

- This is a geometric series:

**First term** ( $n=0$ )  $3^0 x^0 = 1$

**Ratio**  $\frac{\text{Term } n+1}{\text{Term } n} = \frac{3^{n+1} x^{n+1}}{3^n x^n} = 3x$

**Sum**  $\frac{\text{First term}}{1-\text{ratio}} = \frac{1}{1-3x}$

- This series converges for  $|3x| < 1$ ; that is, for  $|x| < 1/3$ .

# Example: $n$ doesn't have to start at 0

- Let  $a_n = 3$  for  $n \geq 1$ .

$$G(x) = \sum_{n=1}^{\infty} 3x^n = \frac{3x}{1-x}$$

- This is a geometric series:

**First term** ( $n=1$ )  $3x$

**Ratio**  $\frac{\text{Term } n+1}{\text{Term } n} = \frac{3x^{n+1}}{3x^n} = x$

**Sum**  $\frac{\text{First term}}{1-\text{ratio}} = \frac{3x}{1-x}$

- This series converges for  $|x| < 1$ .

- $b_n = 10$  for  $n \geq -2$  has generating function

$$B(x) = \sum_{n=-2}^{\infty} 10x^n = \frac{10/x^2}{1-x} = \frac{10}{x^2(1-x)},$$

which converges for  $0 < |x| < 1$ .

# Example: $1/n!$

- Let  $a_n = \frac{1}{n!}$  for  $n \geq 0$ .

$$G(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \quad (\text{using Taylor series})$$

- This series converges for all  $x$ .
- We're focusing on nonnegative integer sequences. However, generating functions are defined for any sequence. The series above arises in Exponential Generating Functions (Ch. 8.2).
- In Probability (Math 180 series), generating functions are used for sequences  $a_n$  of real numbers in the range  $0 \leq a_n \leq 1$ .

# Example: $\binom{10}{n}$

- Let  $a_n = \binom{10}{n}$  for  $n \geq 0$ .
- Since  $\binom{10}{n} = 0$  for  $n > 10$ , we can restrict the sum to  $n = 0, \dots, 10$ .
- By the Binomial Theorem,

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \binom{10}{n} x^n = \sum_{n=0}^{10} \binom{10}{n} x^n = (x + 1)^{10}$$

- This is a polynomial. It converges for all  $x$ .

- Let  $b_n = \binom{10}{n}$  for  $n \geq 2$ .

$$\begin{aligned} B(x) &= \sum_{n=2}^{\infty} \binom{10}{n} x^n = \left( \sum_{n=0}^{\infty} \binom{10}{n} x^n \right) - \left( \sum_{n=0}^1 \binom{10}{n} x^n \right) \\ &= (x + 1)^{10} - (1 + 10x) \end{aligned}$$

## Example: $n!$

- Let  $a_n = n!$  for  $n \geq 0$ . This is the # permutations of  $n$  elements.

$$G(x) = \sum_{n=0}^{\infty} n! x^n$$

- This diverges at all  $x \neq 0$ .

### Divergence is not necessarily a problem

- We are using Taylor series to encode infinite sequences  $a_n$ .
- We do various operations ( $G(x) + H(x)$ ,  $G(x) \cdot H(x)$ ,  $G'(x)$ , ...) on Taylor series to obtain new Taylor series. Then we determine the sequences they represent.
- These operations usually don't involve plugging in values of  $x$ , so the radius of convergence doesn't matter.
- Convergence only matters if we have to plug in a value of  $x$ . There's an example at the end of these slides.

# Finding a coefficient

- Find the coefficient of  $x^3$  in the Maclaurin series of this function:

$$f(x) = \frac{1+x}{1-3x} + (x+10)^8 = \sum_{n=0}^{\infty} a_n x^n$$

- Compute just enough terms:

$$\frac{1+x}{1-3x} = (1+x) \cdot (1-3x)^{-1} = (1+x)(1+3x+9x^2+27x^3+\dots)$$

$$(x+10)^8 = \dots + \binom{8}{3} x^3 \cdot 10^5 + \dots$$

- In  $f(x)$ , the term  $x^3$  arises from

$$\begin{aligned} 1 \cdot 27x^3 + x \cdot 9x^2 + \binom{8}{3} x^3 \cdot 10^5 &= 27x^3 + 9x^3 + (56x^3)(100000) \\ &= 5600036x^3. \end{aligned}$$

- The coefficient is **5600036**.

# Using Calculus to find generating function of $a_n = n$

- Let  $a_n = n$  for  $n \geq 0$ .

$$G(x) = \sum_{n=0}^{\infty} n x^n = ?$$

- Use Calculus:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Differentiate ( $d/dx$ ):  $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}$

Times  $x$ :  $\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} n x^n$

So  $G(x) = \frac{x}{(1-x)^2}$ .

# Using Calculus to find generating function of $a_n = n$

- Let  $a_n = n$  for  $n \geq 0$ .

$$G(x) = \sum_{n=0}^{\infty} n x^n = ?$$

- Note that  $\frac{d}{dx}x^n = n x^{n-1}$  for all values of  $n$ , including  $n = 0$ .

Sometimes the constant term  $x^0 = 1$  is separated out for derivatives. Here, that's valid but adds more work:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + \sum_{n=1}^{\infty} x^n$$

Differentiate ( $d/dx$ ):

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

Times  $x$ :

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n$$

$G(x)$  should include term  $n = 0$ , but this sum starts at  $n = 1$ .

So, add the  $n = 0^{\text{th}}$  term  $n x^n$  as  $0 x^0$  on the right and 0 on the left:

$$0 + \frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} n x^n$$

- So again,  $G(x) = x/(1-x)^2$ .

# Using Calculus to find generating function of $a_n = n$

- Let  $a_n = n$  for  $n \geq 0$ .

$$G(x) = \sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

- By the Ratio Test, this converges for  $|x| < 1$ .
- Example:

$$\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} + \dots = \sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1/3}{(1-1/3)^2} = \frac{3}{4}$$

# Solving a recursion

- Consider the recursion

$$a_0 = 1$$
$$a_{n+1} = 2a_n + 1 \quad \text{for } n \geq 0$$

- The first few terms are

$$a_0 = 1$$
$$a_1 = 2a_0 + 1 = 2(1) + 1 = 3$$
$$a_2 = 2a_1 + 1 = 2(3) + 1 = 7$$
$$a_3 = 2a_2 + 1 = 2(7) + 1 = 15$$

- Any conjectures on the formula?
- We will study this using induction and using generating functions.

# Solving a recursion

- Consider the recursion

$$a_0 = 1 \qquad a_{n+1} = 2a_n + 1 \quad \text{for } n \geq 0$$

- The first few terms are

$$a_0 = 1 \qquad a_1 = 3 \qquad a_2 = 7 \qquad a_3 = 15$$

- Conjecture:  $a_n = 2^{n+1} - 1$  for  $n \geq 0$ .
- We will prove this using induction.

# Solving a recursion

## Theorem

*The recursion*

$$a_0 = 1, \quad a_{n+1} = 2a_n + 1 \quad \text{for } n \geq 0$$

*has solution*

$$a_n = 2^{n+1} - 1 \quad \text{for } n \geq 0$$

## Base case:

- The base case is  $n = 0$ .
- $a_0 = 1$  is given.
- The formula gives  $2^{0+1} - 1 = 2^1 - 1 = 1$ . ✓
- They agree, so the base case holds.

# Solving a recursion

## Theorem

The recursion  $a_0 = 1$ ,  $a_{n+1} = 2a_n + 1$  for  $n \geq 0$   
has solution  $(*) a_n = 2^{n+1} - 1$  for  $n \geq 0$ .

## Induction step:

- Assume  $(*)$  holds at  $n = k$ :  $a_k = 2^{k+1} - 1$ .

We will prove it also holds for  $n = k + 1$ .

- Apply the recursion:

$$\begin{aligned} a_{k+1} &= 2a_k + 1 \\ &= 2(2^{k+1} - 1) + 1 \quad \text{by the induction hypothesis} \\ &= 2^{k+2} - 2 + 1 = 2^{k+2} - 1 = 2^{(k+1)+1} - 1 \end{aligned}$$

- Thus,  $(*)$  holds for  $n = k + 1$  as well.
- Thus,  $(*)$  holds for all integers  $n \geq 0$ .

# Solving a recursion using generating functions

$$a_0 = 1, a_{n+1} = 2a_n + 1 \text{ for } n \geq 0$$

- That method required us to conjecture the solution. Now we'll use generating functions, which will solve it without guessing.
- Define  $A(x) = \sum_{n=0}^{\infty} a_n x^n$  and substitute the recursion into it.
- Separate  $a_0$  vs.  $a_n$  for  $n \geq 1$ , since they're defined differently:

$$A(x) = a_0 x^0 + \sum_{n=1}^{\infty} a_n x^n$$

- Rewrite  $a_{n+1} = 2a_n + 1$  for  $n \geq 0$   
as  $a_n = 2a_{n-1} + 1$  for  $n \geq 1$ , and plug in:

$$\begin{aligned} &= 1 \cdot 1 + \sum_{n=1}^{\infty} (2a_{n-1} + 1) x^n \\ &= 1 + 2 \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} x^n \end{aligned}$$

# Solving a recursion using generating functions

$$a_0 = 1, a_{n+1} = 2a_n + 1 \text{ for } n \geq 0$$

$$A(x) = 1 + 2 \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} x^n$$

Evaluate this in terms of  $A(x)$  and known Taylor series.

- $1 + \sum_{n=1}^{\infty} x^n = 1 + \frac{x}{1-x} = \frac{1}{1-x}$

- $\sum_{n=1}^{\infty} a_{n-1} x^n = a_0 x^1 + a_1 x^2 + a_2 x^3 + \dots = xA(x)$

- Or, substitute  $m = n - 1$ , so  $n = m + 1$ :
  - Lower limit  $n = 1$  becomes  $m = 1 - 1 = 0$ .
  - Upper limit  $n = \infty$  becomes  $m = \infty - 1 = \infty$ .
  - Term  $a_{n-1} x^n$  becomes  $a_m x^{m+1}$ .

$$\sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{m=0}^{\infty} a_m x^{m+1} = x \sum_{m=0}^{\infty} a_m x^m = xA(x)$$

# Solving a recursion using generating functions

$$a_0 = 1, a_{n+1} = 2a_n + 1 \text{ for } n \geq 0$$

$$A(x) = 1 + 2 \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} x^n = \frac{1}{1-x} + 2xA(x)$$

- Solve for  $A(x)$ :

$$A(x) \cdot (1 - 2x) = \frac{1}{1-x} \qquad A(x) = \frac{1}{(1-x)(1-2x)}$$

# Solving a recursion using generating functions

$$a_0 = 1, a_{n+1} = 2a_n + 1 \text{ for } n \geq 0$$

$$A(x) = \frac{1}{(1-x)(1-2x)}$$

- Use Calculus to find the Taylor series of  $A(x)$ .
- This is a rational function. We will use partial fractions to show that

$$A(x) = \frac{2}{1-2x} - \frac{1}{1-x}$$

- Expand these with the geometric series:

$$A(x) = 2 \sum_{n=0}^{\infty} (2x)^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (2(2^n) - 1)x^n = \sum_{n=0}^{\infty} (2^{n+1} - 1)x^n$$

- So  $a_n = 2^{n+1} - 1$  for  $n \geq 0$ .

- We did not need to conjecture the formula; this method found the formula for us.
- We performed operations on Taylor series representing sequences, in order to get other Taylor series representing other sequences. We did not need to plug in values for  $x$ , so convergence at certain  $x$ 's is not an issue.

# Partial fractions

- We seek to rewrite

$$\frac{1}{(1-x)(1-2x)} = \frac{B}{1-x} + \frac{C}{1-2x}$$

with  $B, C$  constants. This should hold for all values of  $x$ .

- Clear denominators by multiplying by  $(1-x)(1-2x)$ , and collect:

$$1 = B(1-2x) + C(1-x) = (B+C) + (-2B-C)x$$

- Since this is true for *all*  $x$ , we need  $B+C=1$  and  $-2B-C=0$ .

- Solve:  $C = -2B$  so  $1 = B + C = B + (-2B) = -B$ .

Then  $B = -1$  and  $C = -2B = -2(-1) = 2$ , giving

$$\frac{1}{(1-x)(1-2x)} = -\frac{1}{1-x} + \frac{2}{1-2x}$$

- Double-check that (next slide) and expand using geometric series to get  $a_n$  (already shown).

# Partial fractions

Double-check the partial fraction expansion:

$$\begin{aligned} -\frac{1}{1-x} + \frac{2}{1-2x} &= \frac{-(1-2x) + 2(1-x)}{(1-x)(1-2x)} \\ &= \frac{-1 + 2x + 2 - 2x}{(1-x)(1-2x)} \\ &= \frac{1}{(1-x)(1-2x)} \quad \checkmark \end{aligned}$$

# Multiplying generating functions

- What's the coefficient of  $x^2$  in this product?

$$(1 + 2x + 3x^2)(4 + 5x + 6x^2)$$

- Work out all the ways  $x^2$  arises in the product:

$$\begin{aligned} & 1 \cdot 6x^2 + 2x \cdot 5x + 3x^2 \cdot 4 \\ &= 6x^2 + 10x^2 + 12x^2 \\ &= 28x^2 \end{aligned}$$

so the coefficient is 28.

# Multiplication rule

- Let  $A(x) = \sum_{n=0}^{\infty} a_n x^n$        $B(x) = \sum_{n=0}^{\infty} b_n x^n$

and define  $C(x) = A(x)B(x) = \sum_{n=0}^{\infty} c_n x^n$

- Then for all  $n \geq 0$ ,

$$c_n = \sum_{k=0}^n a_k b_{n-k}$$

- This is the sum of contributions to the coefficient of  $x^n$  from

$$(a_k x^k \text{ in } A(x)) \cdot (b_{n-k} x^{n-k} \text{ in } B(x)) = a_k b_{n-k} x^n$$

- For any specific  $n$ , computing  $c_n$  only requires a finite # of arithmetic operations, even though the series may be infinite. So  $c_n$  is well-defined regardless of whether the series converge.

# Coefficient of $x^{13}$ in $(1+x)^{10} \cdot (1+x)^5 = (1+x)^{15}$

$$(1+x)^{10} \cdot (1+x)^5 = (1+x)^{15}$$

$$\sum_{i=0}^{10} \binom{10}{i} x^i \cdot \sum_{j=0}^5 \binom{5}{j} x^j = \sum_{k=0}^{15} \binom{15}{k} x^k$$

- **Left side:**  $x^{13}$  arises from  $x^{10} \cdot x^3$ ,  $x^9 \cdot x^4$ ,  $x^8 \cdot x^5$ , with coefficients:

$$\begin{array}{r} x^{10} \cdot x^3 \qquad x^9 \cdot x^4 \qquad x^8 \cdot x^5 \\ \hline \binom{10}{10} \binom{5}{3} \quad + \quad \binom{10}{9} \binom{5}{4} \quad + \quad \binom{10}{8} \binom{5}{5} \\ = \quad 1 \cdot 10 \quad + \quad 10 \cdot 5 \quad + \quad 45 \cdot 1 \\ = \quad 10 \quad + \quad 50 \quad + \quad 45 \quad = \quad \boxed{105} \end{array}$$

- **Right side:**  $\binom{15}{13} = \frac{15 \cdot 14}{2} = \boxed{105}$  ✓

# Coefficient of $x^k$ in $(1+x)^m \cdot (1+x)^n = (1+x)^{m+n}$

First method: Multiplying as polynomials

$$(1+x)^m \cdot (1+x)^n = (1+x)^{m+n}$$

$$\sum_{i=0}^m \binom{m}{i} x^i \cdot \sum_{j=0}^n \binom{n}{j} x^j = \sum_{k=0}^{m+n} \binom{m+n}{k} x^k$$

## Coefficient of $x^k$ on the left:

- Need  $\underbrace{i+j=k}_{j=k-i}$  with  $0 \leq i \leq m$  and  $\underbrace{0 \leq j \leq n}_{0 \leq k-i \leq n}$   
 $-k \leq -i \leq n-k$ , so  $k-n \leq i \leq k$

- **Limits:**  $i \geq 0$  and  $i \geq k-n$  give  $i \geq \max\{0, k-n\}$ .  
 $i \leq m$  and  $i \leq k$  give  $i \leq \min\{m, k\}$ .

- Coefficient of  $x^k$  on the left vs. on the right:

$$\sum_{i=\max\{0, k-n\}}^{\min\{k, m\}} \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}$$

# Coefficient of $x^k$ in $(1+x)^m \cdot (1+x)^n = (1+x)^{m+n}$

First method: Multiplying as polynomials

$$\sum_{i=\max\{0, k-n\}}^{\min\{k, m\}} \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}$$

For the coefficient of  $x^{13}$  in  $(1+x)^{10}(1+x)^5 = (1+x)^{15}$ :

- $m = 10, n = 5, k = 13$
- Lower limit  $\max\{0, 13 - 5\} = \max\{0, 8\} = 8$
- Upper limit  $\min\{13, 10\} = 10$

$$\sum_{i=8}^{10} \binom{10}{i} \binom{5}{13-i} = \binom{15}{13}$$

- This matches the equation from a few slides ago,

$$\binom{10}{10} \binom{5}{3} + \binom{10}{9} \binom{5}{4} + \binom{10}{8} \binom{5}{5} = \binom{15}{13}$$

# Coefficient of $x^k$ in $(1+x)^m \cdot (1+x)^n = (1+x)^{m+n}$

*Second method: Multiplying as infinite series*

$$(1+x)^m \cdot (1+x)^n = (1+x)^{m+n}$$
$$\sum_{i=0}^{\infty} \binom{m}{i} x^i \cdot \sum_{j=0}^{\infty} \binom{n}{j} x^j = \sum_{k=0}^{\infty} \binom{m+n}{k} x^k$$

- We changed the limits to 0 to  $\infty$ .
- For integers  $m, n, k \geq 0$ , the multiplication rule gives

$$\sum_{i=0}^k \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}$$

- Reconcile this with multiplying as polynomials:
  - Any terms with  $i > m$  have  $\binom{m}{i} = 0$ .
  - Any terms with  $k - i > n$  have  $\binom{n}{k-i} = 0$ .
  - All additional terms here are 0, so the sum comes out the same.
- This method is easier.

# Counting in two ways / bijective proof

*Third method* to prove the same identity (though not using generating functions)

$$\sum_{i=0}^k \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}$$

How many  $k$  element subsets are there of  $[m+n]$ ?

- It's  $\binom{m+n}{k}$ .

- Or, pick  $i = 0, \dots, k$  and then

pick  $i$  elements from  $\{1, \dots, m\}$  in  $\binom{m}{i}$  ways

and  $k - i$  elements from  $\{m+1, \dots, m+n\}$  in  $\binom{n}{k-i}$  ways.

- Total:  $\sum_{i=0}^k \binom{m}{i} \binom{n}{k-i}$

- E.g., for  $m = 10$ ,  $n = 5$ , and  $i = 3$ :

picking  $\{1, 3, 10\} \subseteq \{1, \dots, 10\}$  of size 3

and  $\{11, 13\} \subseteq \{11, \dots, 15\}$  of size 2

gives  $\{1, 3, 10, 11, 13\} \subseteq \{1, \dots, 15\}$  of size  $3 + 2 = 5$ .

# Formalizing the bijection

- We'll formalize the bijection just used, and turn it into a generating function proof.

- Let

$$\mathcal{A} = \mathcal{P}(\{1, \dots, m\})$$

$$\mathcal{B} = \mathcal{P}(\{m + 1, \dots, m + n\})$$

$$\mathcal{C} = \mathcal{P}(\{1, \dots, m + n\})$$

- $\mathcal{A} \times \mathcal{B}$  is isomorphic to  $\mathcal{C}$ : There is a bijection respecting set size.

- Given  $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$ , then  $\gamma = \alpha \cup \beta \in \mathcal{C}$ .

- Given  $\gamma \in \mathcal{C}$ , form  $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$  by

$$\alpha = \gamma \cap \{1, \dots, m\}$$

$$\beta = \gamma \cap \{m + 1, \dots, m + n\}.$$

- Sizes satisfy  $|\alpha| + |\beta| = |\gamma|$ .

- E.g., for  $m = 10$  and  $n = 5$ ,

$$(\alpha, \beta) = (\{1, 3, 10\}, \{11, 13\}) \quad \leftrightarrow \quad \gamma = \{1, 3, 10, 11, 13\}$$

sizes  $3 + 2 = 5$ .

# Combining the bijection with generating functions

$$\mathcal{A} = \mathcal{P}(\{1, \dots, m\}), \quad \mathcal{B} = \mathcal{P}(\{m+1, \dots, m+n\}), \quad \mathcal{C} = \mathcal{P}(\{1, \dots, m+n\}).$$

- Let

$$a_i = \# \text{ elements in } \mathcal{A} \text{ of size } i = \binom{m}{i}$$

$$b_j = \dots \quad \mathcal{B} \quad \dots \quad j = \binom{n}{j}$$

$$c_k = \dots \quad \mathcal{C} \quad \dots \quad k = \binom{m+n}{k}$$

**Gen. fn.**

$$A(x) = \sum_{i \geq 0} a_i x^i$$

$$B(x) = \sum_{j \geq 0} b_j x^j$$

$$C(x) = \sum_{k \geq 0} c_k x^k$$

- The number of elements in  $\mathcal{C}$  of size  $k$  is

$$c_k = \sum_{i=0}^k (\# \text{ in } \mathcal{A} \text{ of size } i)(\# \text{ in } \mathcal{B} \text{ of size } k-i) = \sum_{i=0}^k a_i b_{k-i}$$

- These match the coefficient of  $x^k$  on both sides of  $C(x) = A(x)B(x)$ .

# Generating function for # of subsets of a set, by size

- Let  $\mathcal{A} = \mathcal{P}([3]) = \{ \alpha : \alpha \subseteq [3] \}$ .
- There are  $a_i = \binom{3}{i}$  elements in  $\mathcal{A}$  of size  $i$ .
- Three equivalent formulas for the generating function of this:

$$A(x) = \sum_{i=0}^3 a_i x^i = 1 + 3x + 3x^2 + x^3 = (1 + x)^3$$

$A(x) = \sum_{\alpha \in \mathcal{A}} x^{ \alpha } = x^0$	$\alpha$
$+ x^1 + x^1 + x^1$	$\emptyset,$
$+ x^2 + x^2 + x^2$	$\{1\}, \{2\}, \{3\},$
$+ x^3$	$\{1, 2\}, \{1, 3\}, \{2, 3\},$
	$\{1, 2, 3\}$
$= 1 + 3x + 3x^2 + x^3 = (1 + x)^3$	

$$A(x) = \sum_{\alpha \subseteq [3]} x^{|\alpha|} \quad \text{since } \alpha \in \mathcal{A} \text{ iff } \alpha \subseteq [3]$$

# Generating function for # of subsets of a set, by size

- Let  $S$  be a set of size  $m$ , and  $\mathcal{A} = \mathcal{P}(S)$  (the power set of  $S$ ).
- Let  $a_i$  be the number of elements of  $\mathcal{A}$  with size  $i$ :

$$\begin{aligned} a_i &= |\{\alpha \in \mathcal{A} : |\alpha| = i\}| \\ &= |\{\alpha \subseteq S : |\alpha| = i\}| \quad \text{using } \mathcal{A} = \mathcal{P}(S) = \{\alpha : \alpha \subseteq S\}. \end{aligned}$$

- The generating function for the # of sets in  $\mathcal{A}$  having each size is

$$G(x) = \sum_{i \geq 0} a_i x^i = \sum_{i=0}^m \binom{m}{i} x^i = \boxed{(1+x)^m}$$

It can also be written

$$\begin{aligned} &= \sum_{i \geq 0} (\# \alpha \in \mathcal{A} \text{ with } |\alpha| = i) x^i = \boxed{\sum_{\alpha \in \mathcal{A}} x^{|\alpha|}} \\ &= \sum_{i \geq 0} (\# \alpha \subseteq S \text{ with } |\alpha| = i) x^i = \boxed{\sum_{\alpha \subseteq S} x^{|\alpha|}} \end{aligned}$$

# Structures

We generalize this method:

- Let  $\mathcal{A}$  be a set, whose elements we call *structures*.
- Define a *weight* function  $w : \mathcal{A} \rightarrow \mathbb{N}$ , where  $\mathbb{N} = \{0, 1, 2, \dots\}$  is the set of nonnegative integers.
- The generating function for the # of structures in  $\mathcal{A}$  by weight is

$$A(x) = \sum_{\alpha \in \mathcal{A}} x^{w(\alpha)} = \sum_{i=0}^{\infty} a_i x^i$$

where  $a_i$  is the # of structures in  $\mathcal{A}$  with weight  $i$ :

$$a_i = \left| \{ \alpha \in \mathcal{A} : w(\alpha) = i \} \right|$$

## Example

$\mathcal{A}$  is a set of sets, and the weight of  $\alpha \in \mathcal{A}$  is  $w(\alpha) = |\alpha|$ .

$$\begin{aligned} \mathcal{A} &= \{ \{1\}, \{2\}, \{1, 3\}, \{1, 3, 5\} \} \\ A(x) &= x^1 + x^1 + x^2 + x^3 = 2x + x^2 + x^3 \end{aligned}$$

# Multiplication rule for generating functions of structures

## Earlier example with subsets of $[m + n]$

$$\mathcal{A} = \mathcal{P}(\{1, \dots, m\})$$

$$A(x) = (1 + x)^m$$

$$\mathcal{B} = \mathcal{P}(\{m + 1, \dots, m + n\})$$

$$B(x) = (1 + x)^n$$

$$\mathcal{C} = \mathcal{P}(\{1, \dots, m + n\})$$

$$C(x) = (1 + x)^{m+n} = A(x)B(x)$$

Isomorphism  $\mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$

$(\alpha, \beta) \mapsto \gamma = \alpha \cup \beta$  has weights  $|\alpha| + |\beta| = |\gamma|$

## Generalization

- Let  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  be sets of structures, with weight functions  $w_{\mathcal{A}}, w_{\mathcal{B}}, w_{\mathcal{C}}$  and generating functions  $A(x), B(x), C(x)$ .
- Suppose there's an isomorphism  $\mathcal{A} \times \mathcal{B} \cong \mathcal{C}$ , and when  $(\alpha, \beta) \leftrightarrow \gamma$  under this isomorphism,  
$$w_{\mathcal{A}}(\alpha) + w_{\mathcal{B}}(\beta) = w_{\mathcal{C}}(\gamma).$$
Then  $A(x)B(x) = C(x)$ .

# Proof of multiplication rule

To simplify notation, assume all weight functions are the same,  $w$

$$C(x) = \sum_{\gamma \in \mathcal{C}} x^{w(\gamma)}$$

Definition of generating function

$$= \sum_{(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}} x^{w(\alpha) + w(\beta)}$$

Using isomorphism

$$= \sum_{\alpha \in \mathcal{A}} \sum_{\beta \in \mathcal{B}} x^{w(\alpha)} \cdot x^{w(\beta)}$$

Unravelling Cartesian product

$$= \left( \sum_{\alpha \in \mathcal{A}} x^{w(\alpha)} \right) \left( \sum_{\beta \in \mathcal{B}} x^{w(\beta)} \right)$$

Separate variables

$$= A(x) B(x)$$

Definition of generating function

# Addition rule for generating functions of structures

- Let  $\mathcal{A}, \mathcal{B}$  be disjoint sets of structures:  $\mathcal{A} \cap \mathcal{B} = \emptyset$
- Let  $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ .
- Then their generating functions  $A(x), B(x), C(x)$  satisfy
$$C(x) = A(x) + B(x).$$

## Example

Consider these collections of sets, with weight  $w(S) = |S|$ .

$$\mathcal{A} = \left\{ \{1\}, \{2\}, \{1, 3\}, \{1, 3, 5\} \right\}$$

$$A(x) = 2x + x^2 + x^3$$

$$\mathcal{B} = \left\{ \{4\}, \{5\}, \{6\}, \{4, 5\} \right\}$$

$$B(x) = 3x + x^2$$

$$\mathcal{A} \cap \mathcal{B} = \emptyset$$

$$\mathcal{C} = \mathcal{A} \cup \mathcal{B}$$

$$C(x) = 5x + 2x^2 + x^3$$

# How many ways can we make change for $n$ cents?

- Let  $c_n$  be the number of ways to make change for  $n\text{¢}$  from 5 pennies and 2 nickels.

$n$		$c_n$
$0, \dots, 4$	$n$ pennies	1
5	5 pennies, or 1 nickel	2
$6, 7, 8, 9$	1 nickel and $n-5$ pennies	1
10	1 nickel and 5 pennies, or 2 nickels	2
$11, \dots, 15$	2 nickels and $n-10$ pennies	1
otherwise		0

- Coins of the same type are considered indistinguishable for this count. For  $1\text{¢}$ , we just use one penny, and count  $c_1 = 1$ ; we don't count the 5 ways to choose which penny.

# How many ways can we make change for $n$ cents?

- We have 5 pennies and 2 nickels.
- Make a choice from each column, and combine them.

Column A	Column B
0 pennies	0 nickels
1 penny	1 nickel
2 pennies	2 nickels
3 pennies	
4 pennies	
5 pennies	

- The *weight* of a combination of coins is the total value in cents.
- The weight of  $r$  pennies and  $s$  nickels is  $(r + 5s)\text{¢}$ .

# How many ways can we make change for $n$ cents?

	Column A Pennies	Column B Nickels
# coins	$r = 0, 1, 2, 3, 4, 5$	$s = 0, 1, 2$
Weight ( $\text{¢}$ )	$1r = 0, 1, 2, 3, 4, 5$	$5s = 0, 5, 10$

- Let  $a_n = \#$  ways to make  $n \text{ ¢}$  using pennies (Column A).  
 $a_n = 1$  for  $0 \leq n \leq 5$  and  $a_n = 0$  otherwise.

$$A(x) = \sum_{n=0}^5 a_n x^n = 1 + x + x^2 + x^3 + x^4 + x^5$$

- Let  $b_n = \#$  ways to make  $n \text{ ¢}$  using nickels (Column B).  
 $b_n = 1$  for  $n = 0, 5, 10$  and  $b_n = 0$  otherwise.

$$B(x) = \sum_{n=0}^{10} b_n x^n = 1 + x^5 + x^{10}$$

$A(x)B(x) = 1 + x + \dots + x^4 + x^5$	0 nickels
+	$x^5 + x^6 + \dots + x^9 + x^{10}$
+	$x^{10} + x^{11} + \dots + x^{15}$
$C(x) = 1 + x + \dots + x^4 + 2x^5 + x^6 + \dots + x^9 + 2x^{10} + x^{11} + \dots + x^{15}$	Total

# Divisibility notation, for integers $a$ and $b$

- $a \mid b$  means that  $a$  divides into  $b$ ; that is,  $\frac{b}{a}$  is an integer.
- $a \nmid b$  means that  $a$  does not divide into  $b$ .
- $5 \mid 20$  but  $5 \nmid 19$ .

# How many ways to make change with unlimited nickels?

- How many ways can we make change for  $j$  cents with an unlimited number of nickels allowed?
- One way if  $j$  is a multiple of 5, and no ways otherwise:

$$b_j = \begin{cases} 1 & \text{if } 5 \mid j, \\ 0 & \text{otherwise.} \end{cases} \quad \text{Gen. fn.: } B(x) = \sum_{j=0}^{\infty} b_j x^j \\ = 1 + x^5 + x^{10} + \dots = \frac{1}{1-x^5}$$

- The set of numbers of nickels permitted is  $\mathcal{B} = \mathbb{N} = \{0, 1, 2, \dots\}$ .
- The weight of  $s$  nickels is the value in cents,  $5s$ .
- The generating function is also

$$B(x) = \sum_{s \in \mathcal{B}} x^{5s} = \sum_{s=0}^{\infty} x^{5s} = 1 + x^5 + x^{10} + \dots = \frac{1}{1-x^5}$$

# How many ways can we make change from any combination of pennies, nickels, dimes, and quarters?

- Let  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$ , and  $\mathcal{D}$  denote the sets of allowed numbers of pennies, nickels, dimes, and quarters. Set them all to  $\mathbb{N}$ .

- The *weight* is the value in cents.

The generating functions for the weights for each kind of coin are

$$A(x) = \frac{1}{1-x} \quad B(x) = \frac{1}{1-x^5} \quad C(x) = \frac{1}{1-x^{10}} \quad D(x) = \frac{1}{1-x^{25}}$$

- Let  $\mathcal{T} = \mathcal{A} \times \mathcal{B} \times \mathcal{C} \times \mathcal{D}$ .

The generating function for weights of combining coins is

$$T(x) = A(x)B(x)C(x)D(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})} = \sum_{n \geq 0} t_n x^n$$

where  $t_n$  is the number of ways to make change for  $n\text{¢}$  by choosing one option in each of  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$ ,  $\mathcal{D}$ .

# How many ways can we make change?

- How many ways can we make change for 13¢ from pennies, nickels, dimes, and quarters?
- It's the # solutions of  $i + 5j + 10k + 25\ell = 13$  in integers  $i, j, k, \ell \geq 0$ , with  $i, j, k, \ell$  the numbers of pennies, nickels, dimes, and quarters.
- Expand  $T(x)$  and compute the coefficient of  $x^{13}$ :

$$T(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}$$
$$= (1+x+\dots+x^{13}+\dots)(1+x^5+x^{10}+\dots)(1+x^{10}+\dots)(1+\dots)$$

- Only terms in each factor of degree  $\leq 13$  contribute to  $x^{13}$ , so terms with higher exponents are ignored above. Then

$$\underbrace{x^{13} \cdot 1 \cdot 1 \cdot 1}_{13 \text{ pennies}} + \underbrace{x^8 \cdot x^5 \cdot 1 \cdot 1}_{8 \text{ pennies \& 1 nickel}} + \underbrace{x^3 \cdot x^{10} \cdot 1 \cdot 1}_{3 \text{ pennies \& 2 nickels}} + \underbrace{x^3 \cdot 1 \cdot x^{10} \cdot 1}_{3 \text{ pennies \& 1 dime}} = 4x^{13}$$

so  $t_{13} = 4$ .

# Alternative formulation

- The *weight* of an integer partition is the sum of its parts. It's also the number of dots in its Ferrers diagram.

$$w((5, 5, 1, 1)) = w \left( \begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \\ \bullet & & & & & \\ \bullet & & & & & \end{array} \right) = 12$$

- Reformulate problem: use integer partitions instead of coin counts:

$$\mathcal{A} = \left\{ \underbrace{(1, \dots, 1)}_{i \text{ ones, weight } i} : i \geq 0 \right\} \quad \mathcal{C} = \left\{ \underbrace{(10, \dots, 10)}_{k \text{ tens, weight } 10k} : k \geq 0 \right\}$$

$$\mathcal{B} = \left\{ \underbrace{(5, \dots, 5)}_{j \text{ fives, weight } 5j} : j \geq 0 \right\} \quad \mathcal{D} = \left\{ \underbrace{(25, \dots, 25)}_{\ell \text{ 25's, weight } 25\ell} : \ell \geq 0 \right\}$$

$$\begin{aligned} \mathcal{T} &= \left\{ \underbrace{(25, \dots, 25)}_{\ell} \underbrace{(10, \dots, 10)}_k \underbrace{(5, \dots, 5)}_j \underbrace{(1, \dots, 1)}_i : i, j, k, \ell \geq 0 \right\} \\ &\cong \mathcal{A} \times \mathcal{B} \times \mathcal{C} \times \mathcal{D} \quad \text{weight } i + 5j + 10k + 25\ell \end{aligned}$$

- E.g.,  $((1, 1, 1), (5), (), (25)) \in \mathcal{A} \times \mathcal{B} \times \mathcal{C} \times \mathcal{D} \leftrightarrow (25, 5, 1, 1, 1) \in \mathcal{T}$ .
- The bijection respects the weights.

# Notation for products

- $\prod$  notation for products is like  $\sum$  notation for sums:

$$\sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n \quad \prod_{i=1}^n a_i = a_1 a_2 \cdots a_n \quad n! = \prod_{i=1}^n i$$

- An empty sum is 0:  $\sum_{i=1}^0 a_i = 0$

An empty product is 1:  $\prod_{i=1}^0 a_i = 1$

# Number of partitions of an integer

- In place of coins of value  $i$  cents, we use parts of all sizes  $i \geq 1$ . Each part size  $i$  can be repeated any number of times.
- Partitions using just  $i$  are

$$(), (i), (i, i), (i, i, i), \dots$$

(where  $()$  is a partition of 0.)

- The number of partitions of  $n \geq 0$  using parts of size  $i$  is 1 if  $i \mid n$  and 0 otherwise. The generating function for this is

$$1 + x^i + x^{2i} + \dots = \frac{1}{1 - x^i}$$

- The generating function for  $p(n)$  = number of partitions of  $n$  is

$$\sum_{n=0}^{\infty} p(n) x^n = \prod_{i=1}^{\infty} \frac{1}{1 - x^i} = \frac{1}{(1 - x)(1 - x^2)(1 - x^3) \dots}$$

# How many integer partitions does 4 have?

$$\begin{aligned}
 \sum_{n=0}^{\infty} p(n) x^n &= \prod_{i=1}^{\infty} \frac{1}{1-x^i} \\
 &= (1 + x + x^2 + x^3 + x^4 + \dots) && \text{from } \frac{1}{1-x} \\
 &\quad \cdot (1 + x^2 + x^4 + \dots) && \text{from } \frac{1}{1-x^2} \\
 &\quad \cdot (1 + x^3 + \dots) && \text{from } \frac{1}{1-x^3} \\
 &\quad \cdot (1 + x^4 + \dots) && \text{from } \frac{1}{1-x^4} \\
 &\quad \cdot (1 + \dots) && \text{from } \frac{1}{1-x^i} \text{ for } i \geq 5
 \end{aligned}$$

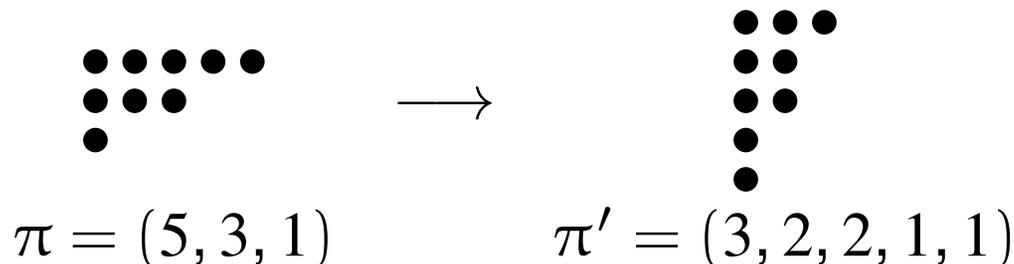
$x^4$  arises from

$$\underbrace{x^4 \cdot 1 \cdot 1 \cdot 1}_{(1,1,1,1)} + \underbrace{x^2 \cdot x^2 \cdot 1 \cdot 1}_{(1,1,2)} + \underbrace{x \cdot 1 \cdot x^3 \cdot 1}_{(1,3)} + \underbrace{1 \cdot x^4 \cdot 1 \cdot 1}_{(2,2)} + \underbrace{1 \cdot 1 \cdot 1 \cdot x^4}_{(4)} = 5x^4 \Rightarrow p(4) = 5$$

- It's a finite computation even though there's an infinite number of factors.
- To match the order of the factors, parts are shown in increasing order instead of decreasing order, e.g., (1, 3) instead of (3, 1).

# Number of partitions with exactly $k$ parts

- Recall that integer partition  $\pi$  has  $k$  parts iff  $\pi'$  has largest part  $k$ .



- Let  $p_k(n) = \#$  partitions of  $n$  with exactly  $k$  parts  
 $= \#$  partitions of  $n$  whose largest part is  $k$  (we'll use this).
- Form partitions with any # (including zero) of 1's,  $\dots$ ,  $(k - 1)$ 's;  
 at least one  $k$ ;  
 no larger part sizes.

- For  $1 \leq i \leq k - 1$ :

The g.f. of zero or more  $i$ 's is  $1 + x^i + x^{2i} + \dots = \frac{1}{1-x^i}$ .

- The g.f. for one or more  $k$ 's is  $x^k + x^{2k} + x^{3k} + \dots = \frac{x^k}{1-x^k}$ .

- This gives

$$\sum_{n=0}^{\infty} p_k(n) x^n = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdots \frac{1}{1-x^{k-1}} \cdot \frac{x^k}{1-x^k} = \frac{x^k}{(1-x) \cdots (1-x^k)}$$

# Partitions with odd parts or distinct parts

- Let  $p_{\text{odd}}(n) = \#$  partitions of  $n$  into odd parts (meaning all part sizes are odd). E.g.,

$$n = 5 : (5), (3, 1, 1), (1, 1, 1, 1, 1) \quad p_{\text{odd}}(5) = 3$$

$$n = 6 : (5, 1), (3, 3), (3, 1, 1, 1), (1, 1, 1, 1, 1, 1) \quad p_{\text{odd}}(6) = 4$$

- Let  $p_d(n) = \#$  partitions of  $n$  with all parts distinct. E.g.,

$$n = 5 : (5), (4, 1), (3, 2) \quad p_d(5) = 3$$

$$n = 6 : (6), (5, 1), (4, 2), (3, 2, 1) \quad p_d(6) = 4$$

- **Theorem:** For all  $n$ ,  $p_{\text{odd}}(n) = p_d(n)$ .
- A bijection is known, but is rather complicated. We'll give a proof using generating functions.

# Partitions with odd parts or distinct parts

- The generating functions for  $p_{\text{odd}}(n)$  and  $p_d(n)$  are

$$F(x) = \sum_{n=0}^{\infty} p_{\text{odd}}(n) x^n = \frac{1}{(1-x)(1-x^3)(1-x^5)\dots}$$

$$G(x) = \sum_{n=0}^{\infty} p_d(n) x^n = (1+x)(1+x^2)(1+x^3)\dots$$

- We'll show  $F(x) = G(x)$ , which proves  $p_{\text{odd}}(n) = p_d(n)$  for all  $n$ .
- Note that  $1 - x^{2i} = (1 - x^i)(1 + x^i)$  gives  $1 + x^i = \frac{1-x^{2i}}{1-x^i}$ .
- In  $G(x)$ , for each  $i$ , replace  $1 + x^i$  by  $\frac{1-x^{2i}}{1-x^i}$ , and cancel like terms:

$$\begin{aligned} G(x) &= \prod_{i=1}^{\infty} (1 + x^i) = \prod_{i=1}^{\infty} \frac{1 - x^{2i}}{1 - x^i} = \frac{\cancel{(1-x^2)} \cancel{(1-x^4)} (1-x^6)(1-x^8)\dots}{(1-x) \cancel{(1-x^2)} (1-x^3) \cancel{(1-x^4)} \dots} \\ &= \frac{1}{(1-x)(1-x^3)(1-x^5)\dots} = F(x) \end{aligned}$$

- Terms  $\frac{1-x^n}{1-x^n}$  with  $n$  even cancel on top and bottom!
- Since  $F(x) = G(x)$ , their Taylor series are equal, so  $p_{\text{odd}}(n) = p_d(n)$ .

# Generating functions for strict compositions

- Solutions  $(i_1, i_2, i_3)$  of  $i_1 + i_2 + i_3 = 12$  with  $i_1, i_2, i_3 \in \{2, 5, 10\}$ :

$$(2, 5, 5), (5, 2, 5), (5, 5, 2)$$

- The # such solutions is the coefficient of  $x^{12}$  in

$$(x^2 + x^5 + x^{10})(x^2 + x^5 + x^{10})(x^2 + x^5 + x^{10}) = (x^2 + x^5 + x^{10})^3$$

- In the expansion of this product,  $x^{12}$  arises from

$$x^2 \cdot x^5 \cdot x^5 + x^5 \cdot x^2 \cdot x^5 + x^5 \cdot x^5 \cdot x^2 = 3x^{12}$$

so there are 3 solutions.

# Generating functions for strict compositions

- Let  $\mathcal{A} \subseteq \mathbb{N}$  and  $k, n \in \mathbb{N}$ .
- The # solutions  $(i_1, \dots, i_k)$  of  $i_1 + \dots + i_k = n$  with  $i_1, \dots, i_k \in \mathcal{A}$  is the coefficient of  $x^n$  in  $(\sum_{j \in \mathcal{A}} x^j)^k$ .
- For strict compositions of  $n$  into  $k$  parts:

$$\mathcal{A} = \mathbb{Z}^+ = \{1, 2, 3, \dots\}$$

$$A(x) = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$$

$$A(x)^k = \frac{x^k}{(1-x)^k} = x^k \sum_{m=k}^{\infty} \binom{m-1}{k-1} x^{m-k} = \sum_{m=k}^{\infty} \binom{m-1}{k-1} x^m$$

- The coefficient of  $x^n$  is at  $m = n$ , so  $\binom{n-1}{k-1}$ , as expected.

# Total # strict compositions of $n$

Now count the total number of strict compositions of  $n$ .

E.g., there are 8 strict compositions of  $n = 4$ :

# Parts	Compositions	Count
1	(4)	1
2	(3, 1), (1, 3), (2, 2)	3
3	(2, 1, 1), (1, 2, 1), (1, 1, 2)	3
4	(1, 1, 1, 1)	1
		<b>Total: 8</b>

# Total # strict compositions of $n$

- $A(x)^k = \frac{x^k}{(1-x)^k}$  is the g.f. for the # strict compositions of  $n$  into  $k$  parts.
- Sum it over  $k$  to get the total # strict compositions of  $n$ :

$$\begin{aligned}\sum_{k=0}^{\infty} A(x)^k &= \frac{1}{1 - A(x)} = \frac{1}{1 - \frac{x}{1-x}} = \frac{1-x}{1-2x} = \frac{1}{1-2x} - \frac{x}{1-2x} \\ &= \sum_{n=0}^{\infty} 2^n x^n - x \sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} 2^n x^n - \sum_{n=0}^{\infty} 2^n x^{n+1}\end{aligned}$$

- To collect coefficients, express these in terms of  $x^n$  rather than mixing  $x^n$  and  $x^{n+1}$ .
  - In the rightmost summation, substitute  $m = n + 1$ , so  $n = m - 1$ :

$$\sum_{n=0}^{\infty} 2^n x^{n+1} = \sum_{m=1}^{\infty} 2^{m-1} x^m$$

- Then rename  $m$  back to  $n$ .

# Total # strict compositions of $n$

- Continuing:

$$\begin{aligned}\sum_{k=0}^{\infty} A(x)^k &= \dots = \sum_{n=0}^{\infty} 2^n x^n - \sum_{n=1}^{\infty} 2^{n-1} x^n \\ &= 2^0 x^0 + \sum_{n=1}^{\infty} (2^n - 2^{n-1}) x^n \\ &= 1 x^0 + \sum_{n=1}^{\infty} 2^{n-1} x^n\end{aligned}$$

- Thus, for  $n = 0$ , there is 1 strict composition of 0, and for  $n > 0$ , there are  $2^{n-1}$  strict compositions of  $n$ .

# Summary of examples

<b>Object</b>	<b>Weight</b>
Sets	Set size
Integer partitions	Sum of part sizes or number of dots in Ferrers diagram
Integer compositions	Sum of part sizes
Coins	Value in cents

# Averages

- What is the average size of a subset of  $[n]$ ?
- For  $n = 2$ , the subsets are

$$\emptyset \quad \{1\} \quad \{2\} \quad \{1, 2\}$$

The average size is  $(0 + 1 + 1 + 2)/4 = 1$ .

- We'll use two methods. The first is specific to this example of subsets of  $[n]$ . The second uses generating functions and can be applied to other structures.

# Average size of a subset of $[n]$

First method: specific to subsets of  $[n]$

- For  $n = 0$ , there is just  $\emptyset$ , with size 0, so the average size is 0.
- For  $n > 0$ , pair all subsets of  $[n]$  using set complements:  $\{S, S^c\}$ .
- E.g., for  $n = 2$ :

$S$	$S^c$	sum of sizes
$\emptyset$	$\{1, 2\}$	$0 + 2 = 2$
$\{1\}$	$\{2\}$	$1 + 1 = 2$

- We get  $2^{n-1}$  pairs  $\{S, S^c\}$ , each with  $|S| + |S^c| = n$ .
- There are  $2^{n-1}$  pairs, so the sum of sizes of subsets is  $n \cdot 2^{n-1}$ .
- Divide by the number of subsets,  $2^n$ , to get average  $\frac{n2^{n-1}}{2^n} = \frac{n}{2}$ .

# Average size of a subset of $[n]$

*Second method:* more general, based on generating functions

- Let  $\mathcal{A} \neq \emptyset$  be a finite set of structures,

$a_k = \#$  of structures in  $\mathcal{A}$  with weight  $k$ , and

$A(x) = \sum_{k=0}^{\infty} a_k x^k$ . This is a polynomial since  $\mathcal{A}$  is finite.

- Average weight =  $\frac{\text{sum of all weights}}{\text{number of structures}} = \frac{\sum_{s \in \mathcal{A}} w(s)}{|\mathcal{A}|}$ .

- The numerator is sum of  $a_0$  0's,  $a_1$  1's,  $a_2$  2's,  $\dots$ , so it equals

$$\sum_{s \in \mathcal{A}} w(s) = 0 a_0 + 1 a_1 + 2 a_2 + \dots = \sum_{k=0}^{\infty} k a_k$$

Notice that  $A'(x) = \sum_{k=0}^{\infty} k a_k x^{k-1}$ , so numerator =  $A'(1) = \sum_{k=0}^{\infty} k a_k$ .

- The denominator is  $|\mathcal{A}| = \sum_{k=0}^{\infty} a_k = A(1)$  since  $A(1) = \sum_{k=0}^{\infty} a_k 1^k$ .

- Thus, the average is  $A'(1)/A(1)$ .

No convergence issues since  $A(x)$  is a polynomial.

# Average size of a subset of $[n]$

*Second method:* more general, based on generating functions

- The generating function for # subsets of  $[n]$  by weight is

$$G(x) = (1 + x)^n = \sum_{k=0}^n \binom{n}{k} x^k.$$

- For  $n > 0$ :

- $G'(x) = n(1 + x)^{n-1}$  and  $G'(1) = n \cdot 2^{n-1}$
- $G(1) = (1 + 1)^n = 2^n$
- Average =  $G'(1)/G(1) = n \cdot 2^{n-1}/2^n = n/2$ .

- For  $n = 0$ , the above is still valid, but we'll do it separately:

- $G(x) = 1$  so  $G'(x) = 0$
- Average =  $G'(1)/G(1) = 0/1 = 0$ .

# Generating functions in probability

- Keep flipping a fair coin until you get heads.

Let  $U = \#$  flips until the first heads.

The probability in this scenario is called the *Geometric Distribution*.

Flips	$U$	Probability
H	1	$1/2$
TH	2	$1/4$
TTH	3	$1/8$
...	...	...
$T^{n-1}H$	$n$	$1/2^n$ for $n \geq 1$

- **Generating function:**

$$G(x) = \sum_{n=0}^{\infty} P(U = n) x^n = \sum_{n=1}^{\infty} \frac{1}{2^n} x^n = \frac{x/2}{1 - (x/2)} = \frac{x}{2 - x}$$

It converges in  $|x/2| < 1$ , which is  $|x| < 2$ .

- **Total probability:**  $\sum_{n=0}^{\infty} P(U = n) = G(1) = \frac{1/2}{1 - (1/2)} = 1.$

# Generating functions in probability

- **Theoretical average of  $U$ :**

$U=n$  a fraction  $1/2^n$  of the time, giving

$$(1/2)1 + (1/4)2 + (1/8)3 + \dots = \sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-(1/2))^2} = 2.$$

- In general, this theoretical average is  $\sum_{n=0}^{\infty} P(U = n) \cdot n$ .

<b>Generating function</b>	$G(x) = \sum_{n=0}^{\infty} P(U = n) \cdot x^n$	$= x/(2 - x)$
<b>Derivative</b>	$G'(x) = \sum_{n=0}^{\infty} P(U = n) \cdot n x^{n-1}$	$= 2/(2 - x)^2$
<b>Theoretical average</b>	$G'(1) = \sum_{n=0}^{\infty} P(U = n) \cdot n$	$= 2/1^2 = 2$

- Since  $G(1)$  and  $G'(1)$  involve plugging  $x = 1$  into an infinite series, convergence does have to be considered.
- In probability, this type of average is called the *expected value*,  $E(U)$ . Repeat an experiment a huge number of times and average the results together. This experimental average varies since it's a random process, but is usually close to the expected value.