Chapter 10.3
Counting walks in directed graphs

Prof. Tesler

Math 184A
Winter 2017

Review of dot product

- Take two vectors of the same length.
- Their *dot product* is defined as:

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} = a_1\, b_1 + a_2\, b_2 + \cdots + a_n\, b_n$$

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 & -5 & 6 \end{bmatrix} &= (1)(4) + (2)(-5) + (3(6)) \\ &= 4 - 10 + 18 = 12 \end{aligned}$$

- The result is a *scalar* (number or number with units).
- The following is invalid since the lengths are different:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 & -5 \end{bmatrix}$$

Review of matrix multiplication

- Let A be a $p \times q$ matrix (p rows, q columns)
  and B be a $q \times r$ matrix (q rows, r columns).
  The number of columns of A must equal the number of rows of B.
- The *matrix product* $C = AB$ is a $p \times r$ matrix.
  Entry c_{ij} (row i, column j) is this dot product:
  $$c_{ij} = (i^{\text{th}} \text{ row of } A) \cdot (j^{\text{th}} \text{ column of } B)$$

$$\begin{array}{ccccc}
A \quad (2 \times 3) & B \quad (3 \times 4) & = & C \quad (2 \times 4) \\
\begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix} & \begin{bmatrix} 0 & -3 & 4 & -1 \\ 8 & 2 & -5 & 7 \\ 0 & -1 & 6 & 0 \end{bmatrix} & = & \begin{bmatrix} 16 & -2 & 12 & 13 \\ 160 & -20 & 120 & 130 \end{bmatrix} \\
\text{Row 1} & \text{Column 3} & = & \text{Entry in row 1, column 3}
\end{array}$$

Symbolic entries in matrix multiplication

In terms of symbolic entries:

- Let $A = (a_{ij})$ be a $p \times q$ matrix
  and $B = (b_{ij})$ be a $q \times r$ matrix.
- The *matrix product* $C = AB$ is a $p \times r$ matrix.
  Entry c_{ij} (row i, column j) is this dot product:

$$
\begin{aligned}
c_{ij} &= (i^{\text{th}} \text{ row of } A) \cdot (j^{\text{th}} \text{ column of } B) \\
&= \begin{bmatrix} a_{i,1} & a_{i,2} & \cdots & a_{i,q} \end{bmatrix} \cdot \begin{bmatrix} b_{1,j} \\ b_{2,j} \\ \vdots \\ b_{q,j} \end{bmatrix} \\
&= a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \cdots + a_{i,q}b_{q,j} = \sum_{k=1}^{q} a_{i,k}b_{k,j}
\end{aligned}
$$

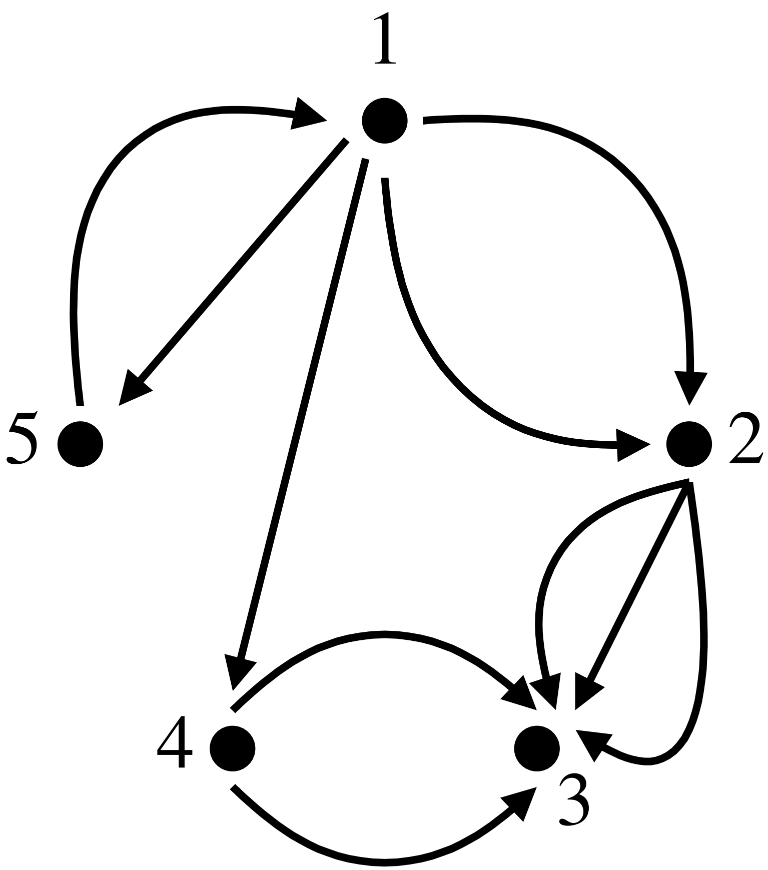
Product of multiple matrices

- Let $A = (a_{ij})$ be $p \times q$
  $B = (b_{ij})$ be $q \times r$
  $C = (c_{ij})$ be $r \times s$
  $D = (d_{ij})$ be $s \times t$
- The # of columns in each matrix = # of rows in the next matrix.
- The product $ABCD$ has dimensions $p \times t$
  (number of rows in the first matrix $\times$ number of columns in the last)
- Compute $ABCD = ((AB)C)D$ to show that row i, column j is

$$(ABCD)_{ij} = \sum_{k=1}^{q} \sum_{\ell=1}^{r} \sum_{m=1}^{s} a_{ik}\, b_{k\ell}\, c_{\ell m}\, d_{mj}$$

  (first index is i; last index is j; do a multiple sum over the others)
- Repeated factors: $AAAA = A^4$.
  This isn't the same as raising individual entries to the power of 4.

10.3. Counting walks in a directed graph

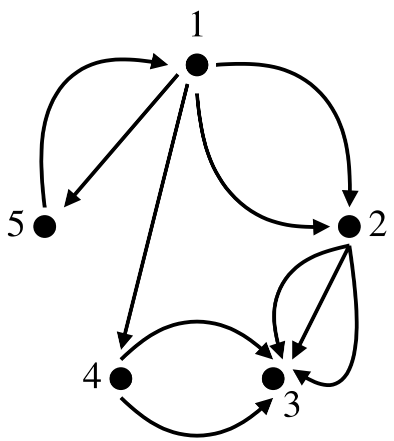


Adjacency matrix

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \text{To } 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} \text{From} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{array}{c} \\ A = \begin{bmatrix} 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}
\end{array}
$$

Entry a_{ij} is the number of directed edges $i \longrightarrow j$.

Counting walks in a directed graph



$$A = \begin{array}{rc} \text{From} \quad \text{To} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

- How many walks of length 2 are there from vertex 1 to vertex 3?
- Recall that walks allow reusing vertices and edges.
- The number of walks $1 \to k \to 3$ is $a_{1k}\, a_{k3}$:

$$\begin{array}{r|r} 1 \longrightarrow 2 \longrightarrow 3 & 2 \cdot 3 = 6 \\ 1 \longrightarrow 4 \longrightarrow 3 & 1 \cdot 2 = 2 \\ 1 \longrightarrow \text{(1, 3, or 5)} \longrightarrow 3 & 0 \\ \hline \text{Total} & 8 \end{array}$$

- Formula: $\sum_{k=1}^{5} a_{1k}\, a_{k3} = (AA)_{1,3} = (A^2)_{1,3}$

How many walks of length 3 from i to j?

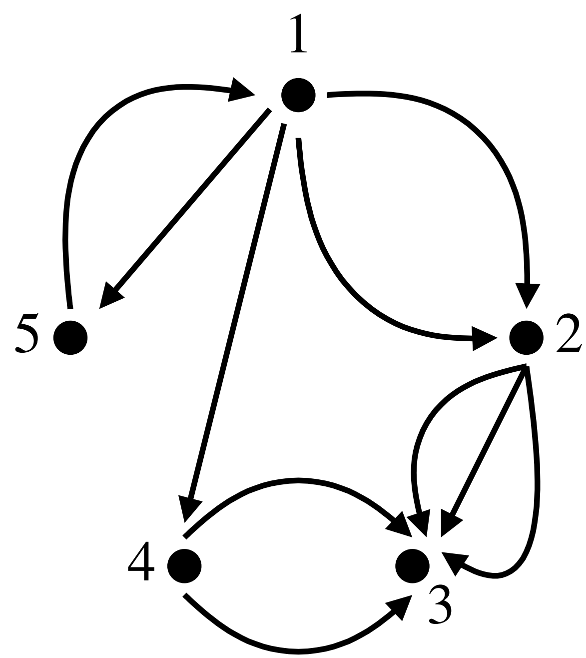
- Such walks have two intermediate vertices:
  $i \to k \to \ell \to j$ occurs $a_{ik}\, a_{k\ell}\, a_{\ell j}$ times

- Sum over all intermediate vertices k, ℓ:

$$\text{Total \# walks} = \sum_{k=1}^{5} \sum_{\ell=1}^{5} a_{ik}\, a_{k\ell}\, a_{\ell j} = (AAA)_{ij} = (A^3)_{ij}$$

- In general, the number of walks of length m from i to j is $(A^m)_{ij}$:
  Raise matrix A to the mth power by multiplying m factors of A.
  Take the entry in row i, column j.

Counting walks in a directed graph



$$A = \begin{bmatrix} 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 1 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 8 & 0 & 0 \end{bmatrix}$$

Application: Counting sequences with restrictions

Problem

- We will consider sequences of the form $d_1\, d_2\, \ldots\, d_n$ where each $d_i \in \{1,2,3\}$ and consecutive entries 11, 23, and 32 are forbidden.
- 12213 is such a sequence.
- 1$\underline{232}$1 and 12$\underline{11}$3 are not, since they have forbidden patterns.
- Let $f(n)$ be the number of such sequences. Compute $f(n)$.

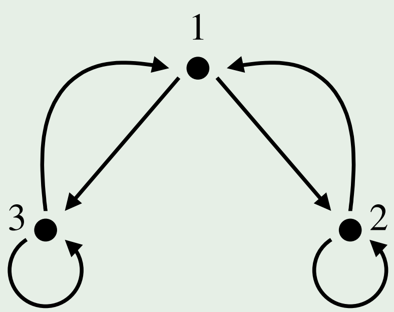
Application: Counting sequences with restrictions

Problem

Compute $f(n)$, the number of sequences of the form $d_1\, d_2\, \ldots\, d_n$ where each $d_i \in \{1, 2, 3\}$ and consecutive entries 11, 23, and 32 are forbidden.

Solution

- Form a directed graph on vertices $1, 2, 3$, with edges $i \to j$ when consecutive symbols ij are allowed.



$$A = \begin{array}{cc} \text{From} & \begin{array}{ccc} \text{To } 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

- $(A^4)_{ij}$ counts allowed sequences of the form $i\, d_2\, d_3\, d_4\, j$.
- Note that A^n counts walks with n edges, which have $n + 1$ vertices, leading to sequences of length $n + 1$.
- For $n \geqslant 1$, $f(n)$ is the sum of all 9 entries in A^{n-1}.

Matrix powers

- For a specific value of n, one may compute A^{n-1}.
- We would also like a general formula in n.
  We'll outline the general method (which requires advanced linear algebra) as well as a method specific to this problem.
- Using the *Jordan Canonical Form* (Math 100, 102, or 180C), which is based on eigenvalues and eigenvectors of A, one can show that for this matrix,

$$A^n = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \frac{(-1)^n}{6}\begin{bmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{bmatrix} + \frac{2^n}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Summing up the 9 entries in A^n gives $3 \cdot 2^n$.
- $f(n)$ is the sum of the 9 entries in A^{n-1},
  so $f(n) = 3 \cdot 2^{n-1}$ for $n \geqslant 1$.

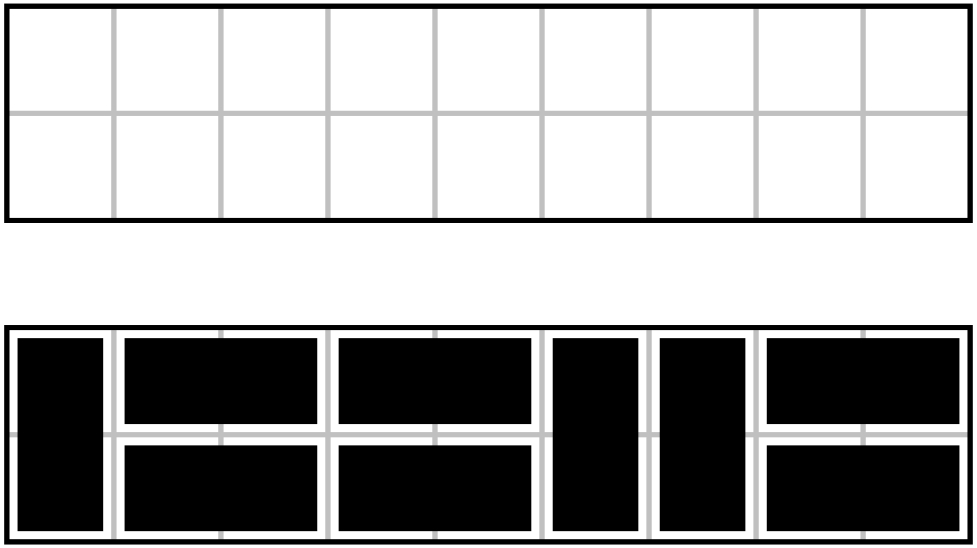
Direct proof that $f(n) = 3 \cdot 2^{n-1}$ for $n \geqslant 1$

- There are three choices of the first digit: 1, 2, or 3.
- There are two choices for each subsequent digit, depending on the previous digit:

  1 may be followed by 2 or 3;
  2 may be followed by 1 or 2;
  3 may be followed by 1 or 3.

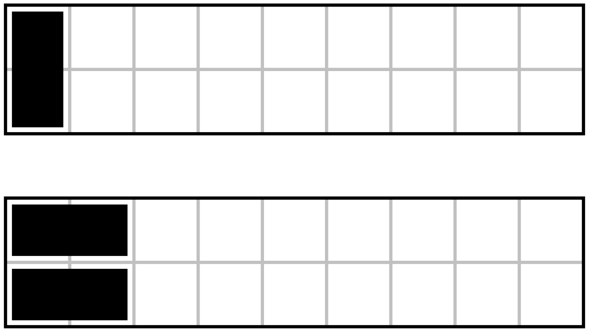
  We make $n-1$ choices of this type (for the 2nd through nth digits).
- Thus, $f(n) = 3 \cdot 2^{n-1}$.
- That method is specific to this problem, while matrix powers are more general. If we disallow 11, 12, 23, 32, then there would be one option after a 1 and two options after a 2 or 3. This way wouldn't work since the number of options would vary, but the transition matrix method would still work.

Application: Domino tilings of a $2 \times n$ board



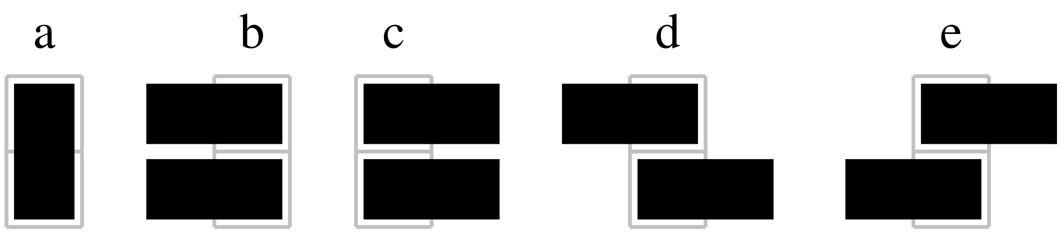
- How many ways are there to tile a $2 \times n$ board with dominos so that all squares are filled? Call it T_n.
- The dominos must not overlap.
  Each domino must be properly positioned to cover either two horizontally adjacent or two vertically adjacent squares.

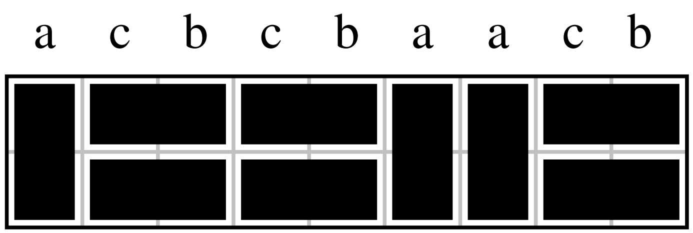
Domino tilings, first solution: induction



- The tiling starts with either a vertical domino or two stacked horizontal dominos.
- If $n \geqslant 1$ and the first column has a vertical domino, there are T_{n-1} ways to tile the remaining $n-1$ columns.
- If $n \geqslant 2$ and the first two columns have two stacked horizontal dominos, there are T_{n-2} ways to tile the remaining $n-2$ columns.
- This gives a recursion: $T_n = T_{n-1} + T_{n-2}$ for $n \geqslant 2$.
- **Base cases:** $n = 0$: $T_0 = 1$ $(2 \times 0$; no squares to fill)
  $n = 1$: $T_1 = 1$ $(2 \times 1$; one vertical domino)
- Same recursion and initial conditions as the Fibonacci numbers!
  $T_0 = 1,\ T_1 = 1,\ T_2 = 2,\ T_3 = 3,\ T_4 = 5,\ T_5 = 8,\ \ldots$

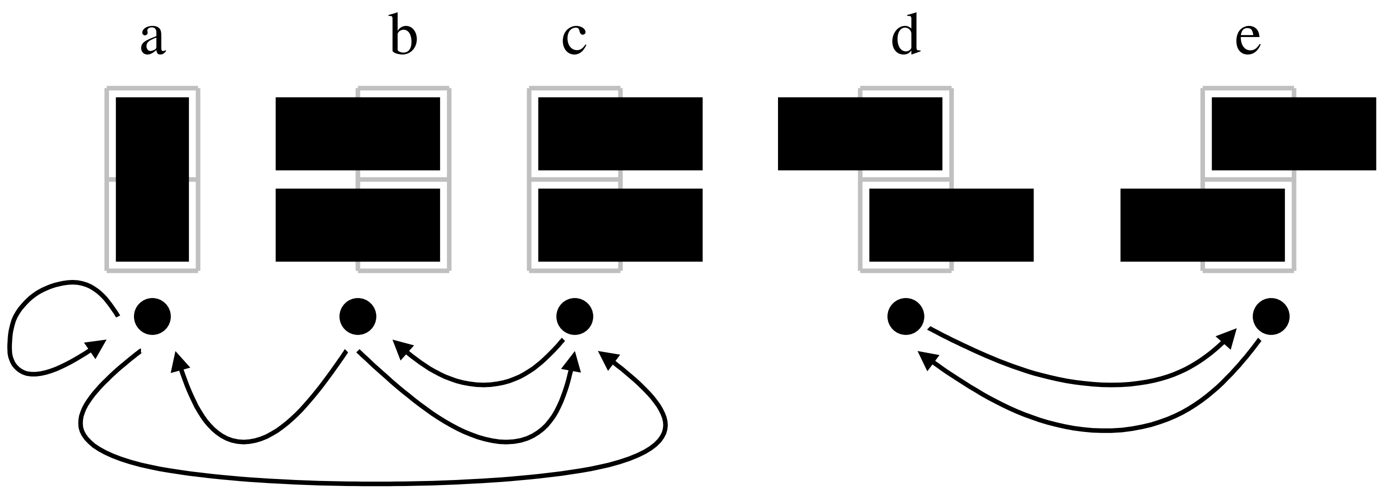
Domino tilings, second solution: Transition Matrices





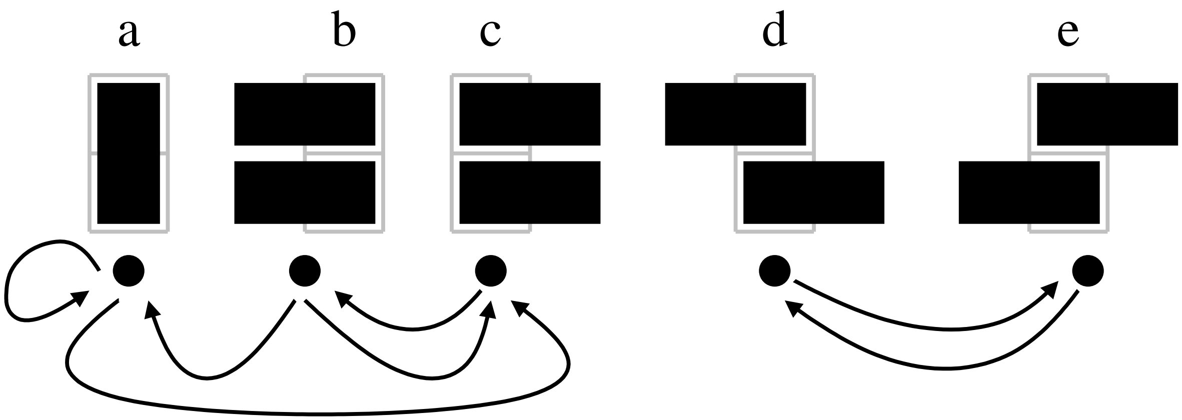
- This method is also called *Transfer Matrices*.
  It is closely related to *Markov Chains* in Probability (Math 180C) and to *Finite State Automata* in Computer Science (CSE 21).
- If we consider one column at a time, there are five possible states (a, b, c, d, e above) that it can be in.
- Adjacent columns can only have certain combinations of states.
  E.g., aa and ac are valid, but ab, ad, ae are not valid.
  Systematically determine all such rules.

Domino tilings, second solution: Transition Matrices



- We will make a graph representing valid combinations of states in adjacent columns.
- The vertices are the states a, b, c, d, e.
- If consecutive columns may have states k, ℓ then draw a directed edge from k to ℓ.

Domino tilings, second solution: Transition Matrices



From To

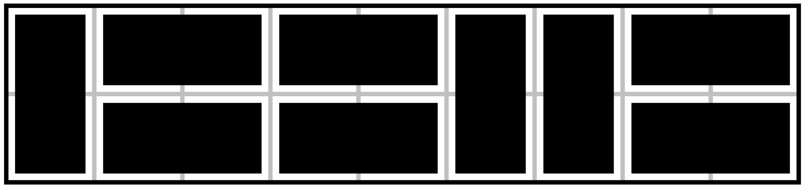
$$M = \begin{array}{c} \\ \text{a} \\ \text{b} \\ \text{c} \\ \text{d} \\ \text{e} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{a} & \text{b} & \text{c} & \text{d} & \text{e} \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

- The *transition matrix* M is the adjacency matrix of this graph.
- $(M^n)_{i,j}$ is the number of walks $i, k_2, k_3, \ldots, k_n, j$ in the graph.
  Again, it has n edges, corresponding to $n+1$ vertices.
  As a tiling, it represents $n+1$ columns: $i, k_2, k_3, \ldots, k_n, j$.
  For n columns, we need to use M^{n-1}.
- To avoid overflowing the board, a walk must start with a or c, and end with a or b, so

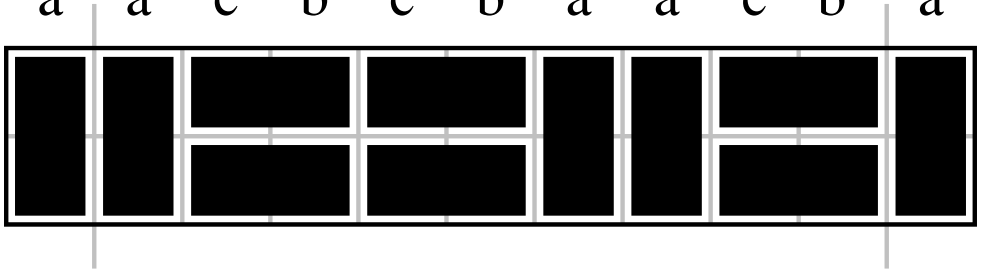
$$T_n = (M^{n-1})_{a,a} + (M^{n-1})_{a,b} + (M^{n-1})_{c,a} + (M^{n-1})_{c,b} \;.$$

Domino tilings, second solution: Transition Matrices

a c b c b a a c b



a a c b c b a a c b a

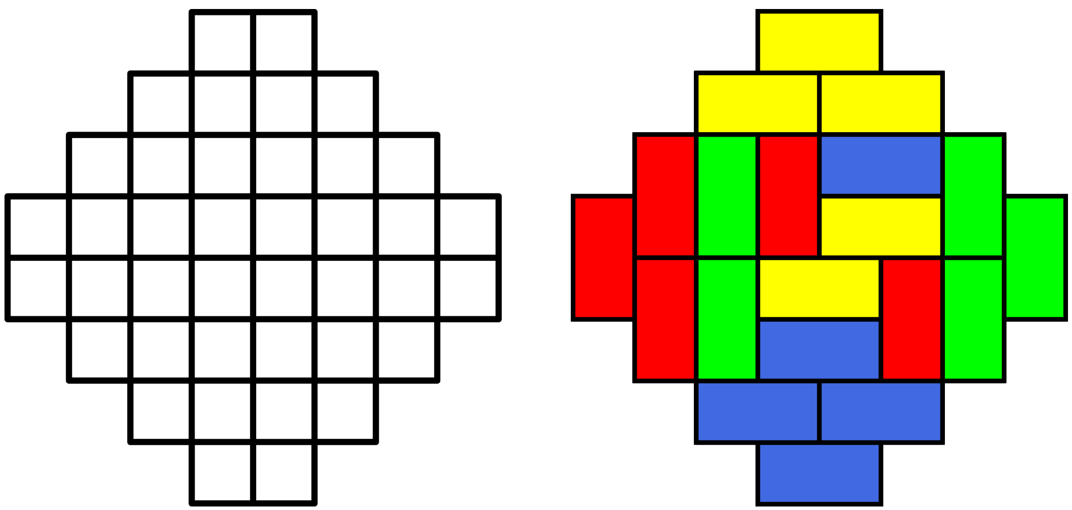


Alternate solution to splitting T_n into four subcases:

- Augment the $2 \times n$ board into a $2 \times (n+2)$ board by adding one extra column on each end, with a vertical domino in each.
- The number of ways to fill the middle n columns of the augmented board is the same as filling a $2 \times n$ board, T_n.
- In the augmented board, a tiling corresponds to a walk with $n+1$ edges from a to a, so $T_n = (M^{n+1})_{a,a}$.

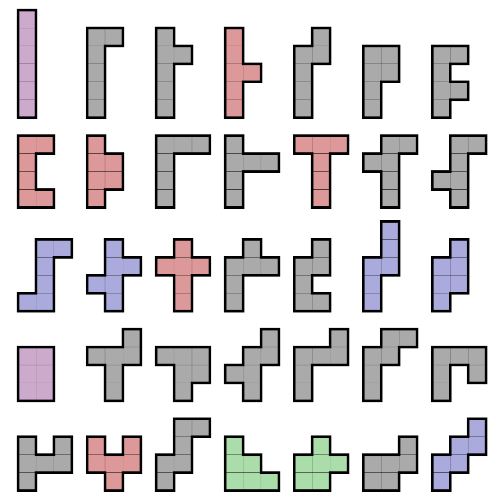
Generalizations of domino tilings

There are generalizations (using additional techniques) to count domino tilings of rectangular grids and other shapes, and to using *polyominoes* instead of dominos.



Aztec Diamond

http://en.wikipedia.org/wiki/Aztec_diamond



Polyominoes

http://en.wikipedia.org/wiki/Polyomino