

Math 184A, Winter 2019, Prof. Tesler
Homework #9, Due Wednesday March 13, 2019

If you would like to use your homework to study for the exam, please make a copy of it before turning it in.

Chapter 11# 17

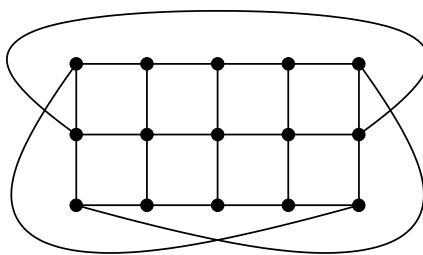
Chapter 12# 1, 3, 12 (Hint on 12.12: use a method similar to H-901(b)(iii))

Problems H-901 through H-904 (see both pages)

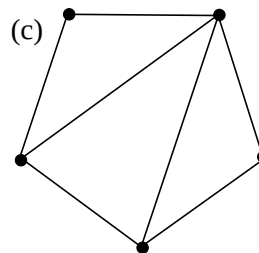
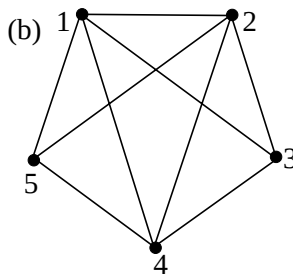
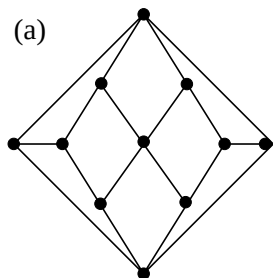
Problem H-901.

- (a) A connected graph is drawn in the plane without crossing edges. It has 20 vertices and 45 edges. How many faces does it have?
- (b) An ordinary soccer ball has 32 panels: 20 hexagons and 12 pentagons. (For a picture, see the Wikipedia article “Truncated icosahedron” in the Gallery section.) The vertices, edges, and faces of these panels give a planar graph. An edge and face are *incident* with each other if the edge is contained in the face (just touching at a corner doesn’t count). A vertex and edge or vertex and face are *incident* if the vertex is contained in the edge or face.
 - (i) What degrees do the vertices have? How many faces is each vertex incident with? How many faces is each edge incident with?
 - (ii) Consider the set $T = \{(v, f) : v \text{ is a vertex, } f \text{ is a face, } v \text{ is on } f\}$. Determine its size $|T|$ two ways: first, for each v , consider how many faces it’s in, and add that up over all v ; and second, for each f , consider how many vertices it has, and add that up over all f . Use this to determine the total number of vertices.
 - (iii) Consider the set $U = \{(e, f) : e \text{ is an edge, } f \text{ is a face, } e \text{ is on } f\}$. Determine its size in two ways. Use this to determine the total number of edges.
 - (iv) Show that Euler’s formula (for planar graphs) holds in this graph.

Problem H-902. Prove whether or not this graph is planar. If it is planar, prove it by redrawing it without edges crossing. If it is not planar, use Kuratowski’s Theorem: identify a subgraph homeomorphic to either K_5 or $K_{3,3}$, and draw the graph with that subgraph highlighted and its vertices identified.



Problem H-903. Determine the chromatic number of each of these graphs. Give an example of a proper coloring of each of these graphs achieving that number of colors.



- (d) Redraw graph (b) as a planar embedding. In the redrawn graph, determine V, E, F and show that they satisfy Euler’s formula (for planar graphs).

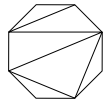
Problem H-904. See the lecture slides on Catalan numbers.

- (a) Compute the number of each of these types of objects (see the slides for definitions and examples):
- A string of 9 pairs of balanced parentheses.
 - Complete binary parenthesizations of a product of 9 variables.
 - Ordered full binary trees with 9 leaves.
 - Triangulations of a 9-sided polygon.
- (b) The slides give bijections between various structures counted by the Catalan number C_n , including:
- n pairs of balanced parentheses;
 - complete binary parenthesizations of a product of $n + 1$ variables;
 - ordered full binary trees with $n + 1$ leaves;
 - triangulations of an $(n + 2)$ -sided polygon.

For each object below, determine the corresponding objects in the other categories listed above via the appropriate bijections. Please do not draw multiple bijections on the same diagram, as that will be messy; please make separate copies of the diagrams for each computation.

(i) Balanced parentheses: $((()))()()$

(ii) Complete binary parenthesization: $(a(bc))(((de)f)g)$



(iii) Triangulation of a polygon:

- (c) An election has two candidates, A and B . The ballots are counted one-by-one. Suppose that the candidates are tied with n votes each.
- (i) If the order in which the votes are counted is random, what is the probability that A is never behind B during the counting?
- Hint:* Write a vote for A as “(” and a vote for B as “)”, in the order the votes are counted.
- (ii) If the order the votes are counted is random, what is the probability that A is always ahead of B while counting, except for the $0 - 0$ and $n - n$ ties at the start and end of the counting?