

Math 184A, Winter 2019, Prof. Tesler
Homework #6, Due Wednesday February 20, 2019

Chapter 8# 35* and the problems below, H-601 through H-603.

*Notes on Chapter 8# 35:

- The weight of a sequence of tiles is how many squares they cover.
- Make a generating function $A(x) = \sum_n a_n x^n$ for one tile, with $a_n = \#$ types of tiles of weight n .
- For a k -tuple of tiles, raise $A(x)$ to the k^{th} power; the coefficient of x^n in $A(x)^k$ is the number of sequences of k tiles with total weight n . This is based on the multiplication rule for generating functions: we're making a sequence of k choices, each with the same generating function $A(x)$. Multiplying the k generating functions $A(x) \cdot A(x) \cdots$ (repeated k times) gives $A(x)^k$.
- Then $\sum_{k \geq 0} A(x)^k$ is the generating function for any number of tiles. Find a closed form for this (you do not need to work out the Taylor series).

Problem H-601. (Based on Chapters 6 and 8.)

- (a) Consider all permutations of $[5]$. What is the average number of cycles per permutation?
- (b) Let $n \geq 0$. Compute the average number of cycles per permutation, over all permutations of $[n]$.

Hint: Near the end of the Chapter 8 slides, see "Average size of a subset of $[n]$." It gives a formula for the average in terms of generating functions, as $A'(1)/A(1)$. Apply this to Lemma 6.13. Also use $(d/dx) \ln(A(x)) = A'(x)/A(x)$.

Problem H-602. (Based on Chapter 8.) Consider what exact amounts of postage can be made out of 3 and 4 cent stamps.

- (a) There is an integer n_0 such that if $n \geq n_0$, then postage can be made for n cents using 3 and 4 cent stamps. Conjecture what n_0 is. Use the smallest correct value possible.
- (b) Prove by induction that postage for n cents can be made out of 3 and 4 cent stamps for all $n \geq n_0$.
- (c) Let a_n denote the number of ways to make postage for n cents out of 3 and 4 cent stamps, without regard to order; e.g., $a_7 = 1$ since $3 + 4 = 4 + 3$ both use one 3 cent stamp and one 4 cent stamp. Compute the generating function for a_n .
- (d) Compute a_{24} .

Hints on (a–b):

- As soon as you find three numbers in a row for which you can make postage, you can keep going up from those three by adding on additional 3 cent stamps.
- The induction has multiple base cases, not just one, and it doesn't start at 0 or 1.
- The induction step involves going down by 3 or 4 cents (the values of the stamps), not by 1 cent. Use "strong induction": the induction hypothesis should involve all relevant smaller values than n , not just $n - 1$.

Problem H-603. (Based on the Chapter 18.1 slides.)

- (a) How many 8-long circular sequences can be made using the ten digits $0, 1, \dots, 9$ if no digit can be used more than once? Rotations are considered equivalent, e.g., $3, 8, 9, 1, 5, 4, 2, 7 \equiv 9, 1, 5, 4, 2, 7, 3, 8$.
- (b) Generalize your answer to n -long circular sequences when there are k symbols available, where n and k are positive integers.
- (c) The *reverse* of a sequence is to reverse the order of its elements. E.g., the reverse of the sequence $3, 8, 9, 1, 5, 4, 2, 7$ is $7, 2, 4, 5, 1, 9, 8, 3$. How many n -long circular sequences made from k symbols are there when all combinations of rotations and reversals are considered equivalent? E.g., $3, 8, 9, 1, 5, 4, 2, 7 \equiv 9, 8, 3, 7, 2, 4, 5, 1$. Again, no symbol may be used more than once, and n and k are positive integers. *Be careful about $n = 1$ and $n = 2$.*