

Math 184A, Winter 2019, Prof. Tesler
Homework #5, Due Wednesday February 13, 2019

Chapter 8# 2*, 12, 23, 39*
and the problems below, H-501 through H-503 (see both pages).

*Notes:

- Chapter 8# 2: The info in Problem H-503 below may be helpful.
- Chapter 8# 39: There is a straightforward counting solution without using generating functions. It can also be done with techniques in Chapter 8.2, but we did not cover those.

The shelves are distinguishable (e.g., top & bottom shelves), and the order of the books matters. So for books A,B,C,D,E, these four arrangements are counted differently (and there are many other arrangements besides these):

$$\begin{array}{l} \text{Top shelf:} \\ \text{Bottom shelf:} \end{array} \left| \begin{array}{ccc} \text{A B C} \\ \text{D E} \end{array} \right| \left| \begin{array}{ccc} \text{C B A} \\ \text{E D} \end{array} \right| \left| \begin{array}{ccc} \text{E D} \\ \text{A C B} \end{array} \right| \left| \begin{array}{ccc} \text{D E} \\ \text{B A C} \end{array} \right|$$

Problem H-501. This is based on Chapter 8.1. However, you should already be able to do this problem using Taylor series from Math 20B and 20D.

- (a) Let $a_n = n^2$ for integers $n \geq 0$. The generating function of a_n is

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} n^2 x^n.$$

Derive an explicit formula for it as follows:

- Start with the geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$.
- Differentiate both sides with respect to x .
- Multiply both sides by x .
- Differentiate again.
- Multiply by x again.
- Simplify the result, if needed.
- One side is now $\sum_{n=0}^{\infty} n^2 x^n$. The other side is a rational function (ratio of polynomials in x); set $f(x)$ equal to that.

- (b) Use the above result to compute $\sum_{n=0}^{\infty} \frac{n^2}{4^n}$.

Problem H-502. Compute the coefficient of x^2 in each of these (expanded as a polynomial or as a Taylor series centered at 0):

- (a) $(1 + 2x + x^2)(1 + x + 3x^2)(4 + x + x^2)$
(b) $x(1 + x)^{20}(2 - 3x)^{10}$
(c) $(1 - x)(1 + 4x)/(1 - 100x)$

Homework problems continue on next page.

Problem H-503.

(a) Consider $\sum_{i=0}^{\infty} \sum_{j=0}^i f(i, j)$.

The summation ranges $0 \leq i < \infty$ and $0 \leq j \leq i$ can be combined into $0 \leq j \leq i < \infty$.

In turn, that can be separated into j ranging between two values not depending on i , and then i ranging between two values that may depend on j .

Use that to “reverse the order of the summation”: $\sum_{i=0}^{\infty} \sum_{j=0}^i f(i, j) = \sum_{j=?}^? \sum_{i=?}^? f(i, j)$.

(This is similar to reversing the order of integration in Math 20C.)

(b) Compute $\sum_{i=0}^{\infty} \sum_{j=0}^i x^{i+j}$.

Hint: Reverse the order of summation. Evaluate the inside sum using a geometric series, bringing it to a single summation. Simplify and evaluate using another geometric series.