

Math 184A, Winter 2019, Prof. Tesler
Homework #2, Due Wednesday January 23, 2019

Chapter 4# 1, 3*, 4*, 10, 16, 28*, 42, 43, 46*

Chapter 5# 23*, 24*

and the problem below, H-201.

*Notes:

- Chapter 4# 3 and 4: Give two proofs for each problem. One proof should be based on manipulating formulas with algebra and/or calculus. The other proof should be a “combinatorial proof,” using either a bijection or counting in two ways.
- Chapter 4# 28: *Hint:* Create a one-to-one function $f : P \rightarrow Q$ between particular sets of sizes $|P| = 2^n$, $|Q| = \binom{2n}{n}$. This will prove $|P| \leq |Q|$.
- Chapter 4# 46: *Hint:* Consider $(x + y)^n \pm (x - y)^n$ at suitable values of x, y .
- Chapter 5# 23 & 24: “even” (or “odd”) parts means every part is an even (or odd) number; it does not mean that the number of parts is even (or odd). E.g., $(4, 6, 4)$ is a composition of 14 into **even parts**, since 4, 6, and 4 are even, but $(4, 6, 4)$ has an **odd number of parts** since it's 3 parts.

Problem H-201. Let α be a real number and $k \geq 0$ be an integer. The falling factorial and binomial coefficient notations generalize to

$$(\alpha)_0 = 1, \quad (\alpha)_k = \alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1), \quad \binom{\alpha}{0} = 1, \quad \binom{\alpha}{k} = (\alpha)_k / k!$$

and the binomial series is $(1 + x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$.

The *double factorial* of a nonnegative integer n is

$$\begin{aligned} n!! &= n(n-2)(n-4) \cdots 2 && \text{if } n > 0 \text{ is even;} \\ &= n(n-2)(n-4) \cdots 1 && \text{if } n > 0 \text{ is odd;} \\ 0!! &= 1 \end{aligned}$$

E.g., $5!! = 5 \cdot 3 \cdot 1 = 15$ and $6!! = 6 \cdot 4 \cdot 2 = 48$.

- (a) Give simple formulas for $n!!$ in both the even and odd cases, in terms of factorials and powers of 2.

$$\text{Hint: } 1 \cdot 3 \cdot 5 \cdot 7 = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{2 \cdot 4 \cdot 6 \cdot 8}$$

- (b) Prove the following and state which values of k it holds for:

$$(1/2)_k = -2 \left(-\frac{1}{4} \right)^k \frac{(2k-2)!}{(k-1)!}$$

Also fill in the values of $(1/2)_k$ for any $k \geq 0$ for which it is not valid.

$$\text{Hint: } (1/2)_5 = \left(\frac{1}{2} \right) \left(-\frac{1}{2} \right) \left(-\frac{3}{2} \right) \left(-\frac{5}{2} \right) \left(-\frac{7}{2} \right)$$

Use part (a) to evaluate the numerator, but be careful about how the values of k correspond between (a) and (b). Also collect signs and powers of 2.

- (c) Use the binomial series formula to compute the Taylor series for $\sqrt{1+12x}$ and $1/(1+12x)$. Simplify the series coefficients as much as possible.
- (d) Prove this recursion, and state the values of α and k for which it is valid:

$$\binom{\alpha}{k} = \binom{\alpha-1}{k} + \binom{\alpha-1}{k-1}$$