Chapter 7
Planar graphs

In full: 7.1–7.3
Parts of: 7.4, 7.6–7.8
Skip: 7.5

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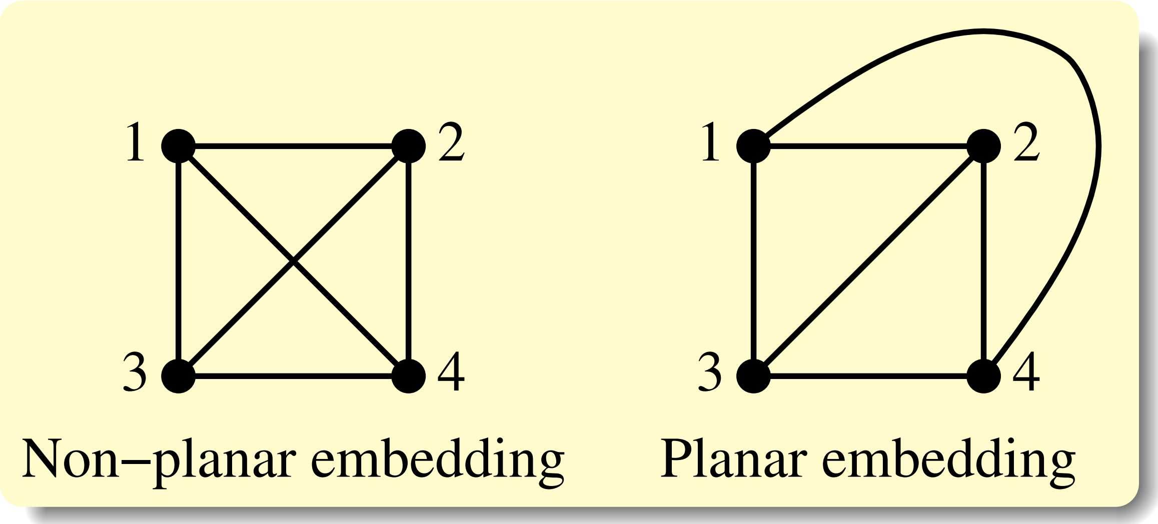
Planar graphs

Definition

- A *planar embedding* of a graph is a drawing of the graph in the plane without edges crossing.
- A graph is *planar* if a planar embedding of it exists.

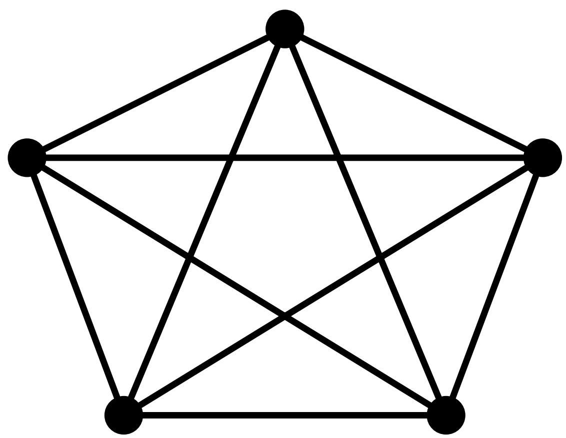
- Consider two drawings of the graph K_4:

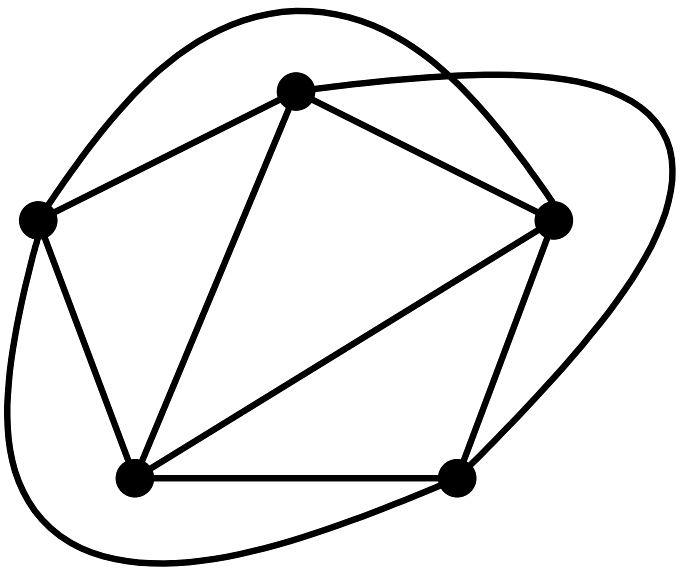
$$V = \{1,2,3,4\} \qquad E = \big\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\big\}$$



- The abstract graph K_4 is planar because it can be drawn in the plane without crossing edges.

How about K_5?





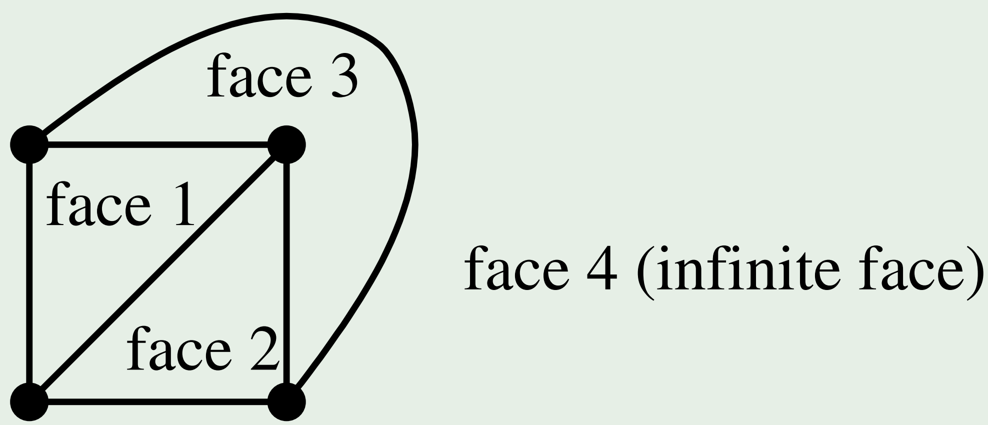
- Both of these drawings of K_5 have crossing edges.
- We will develop methods to prove that K_5 is not a planar graph, and to characterize what graphs are planar.

Euler's Theorem on Planar Graphs

- Let G be a connected planar graph (drawn w/o crossing edges).
- Define V = number of vertices
  E = number of edges
  F = number of faces, including the "infinite" face
- Then $V - E + F = 2$.
- **Note:** This notation conflicts with standard graph theory notation V and E for the sets of vertices and edges. Alternately, use

$$|V(G)| - |E(G)| + |F(G)| = 2.$$

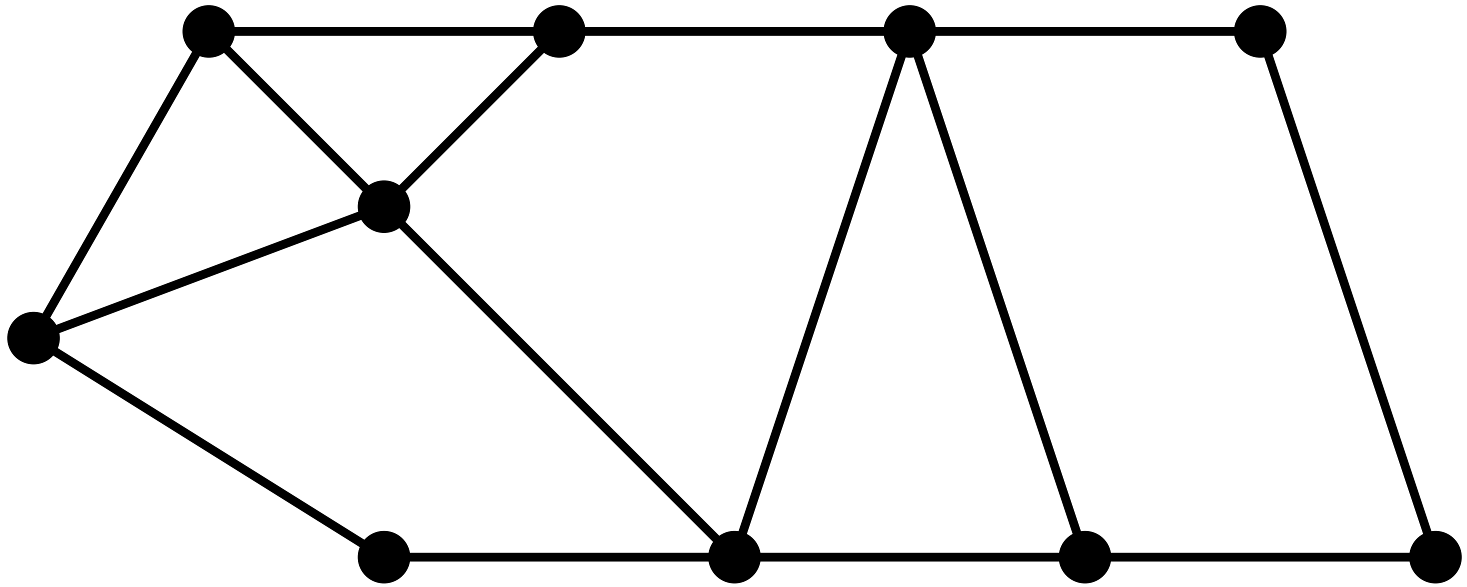
Example



$$V = 4$$
$$E = 6$$
$$F = 4$$

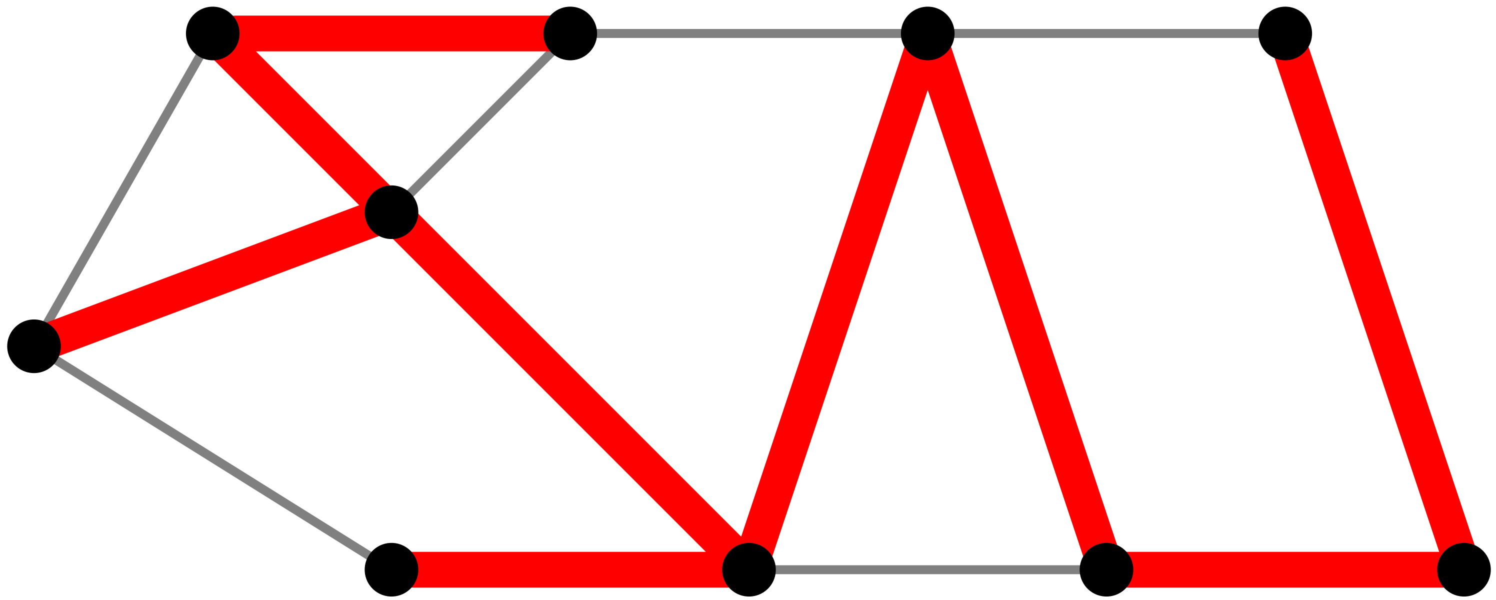
$$V - E + F = 4 - 6 + 4 = 2$$

Euler's formula for planar graphs



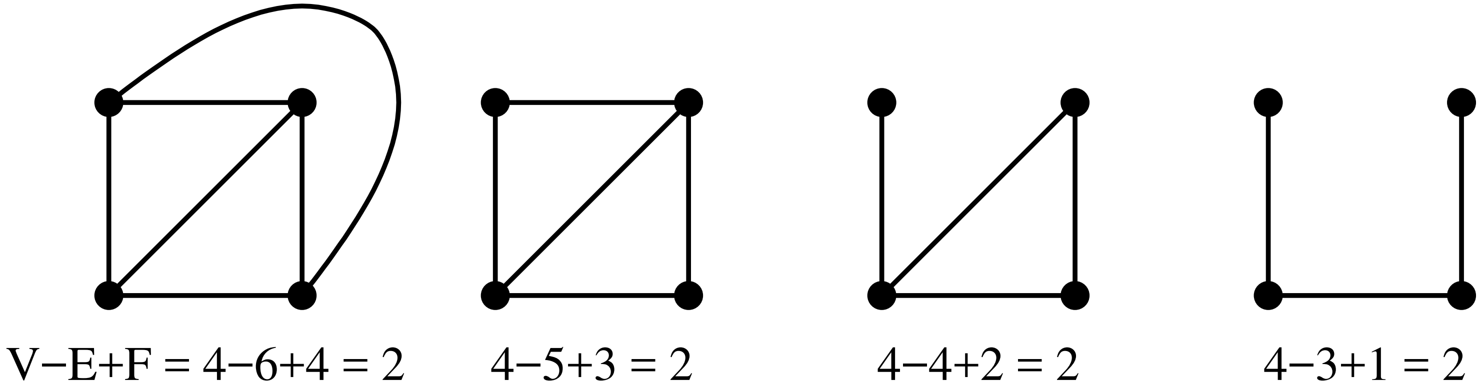
- $V = 10$
- $E = 15$
- $F = 7$
- $V - E + F = 10 - 15 + 7 = 2$

Spanning tree



- A *spanning tree* of a connected graph is a subgraph that's a tree reaching all vertices. An example is highlighted in red.
- **Algorithm to get a spanning tree of any connected graph:** Repeatedly pick a cycle and remove an edge, until there aren't any cycles.
- We also had other algorithms (DFS and BFS), but the one we need now is removing one edge at a time.

Proof of Euler's formula for planar graphs



We will do a proof by induction on the number of edges.

Motivation for the proof:

- Keep removing one edge at a time from the graph while keeping it connected, until we obtain a spanning tree.
- When we delete an edge:
  - V is unchanged.
  - E goes down by 1.
  - F also goes down by 1 since two faces are joined into one.
  - $V - E + F$ is unchanged.
- When we end at a tree, $E = V - 1$ and $F = 1$, so $V - E + F = 2$.

Proof of Euler's formula for planar graphs

Let G be a connected graph on n vertices, drawn without crossing edges. We will induct on the number of edges.

Base case: The smallest possible number of edges in a connected graph on n vertices is $n-1$, in which case the graph is a tree:

$$V = n$$

$$E = n - 1$$

$$F = 1 \quad \text{(no cycles, so the only face is the infinite face)}$$

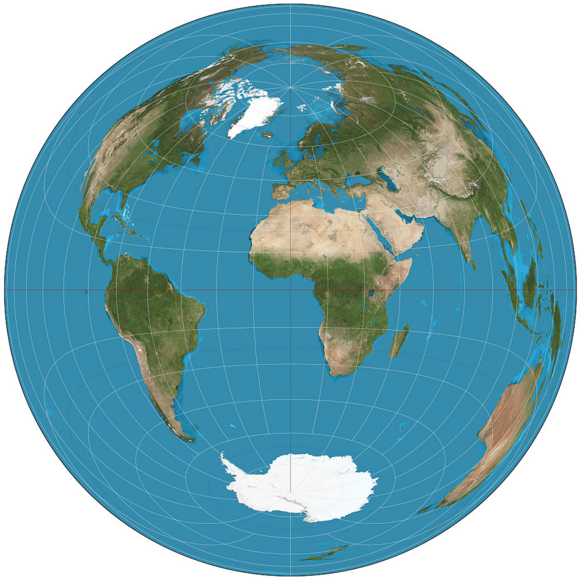
$$V - E + F = n - (n-1) + 1 = 2$$

Proof of Euler's formula for planar graphs

Induction step:

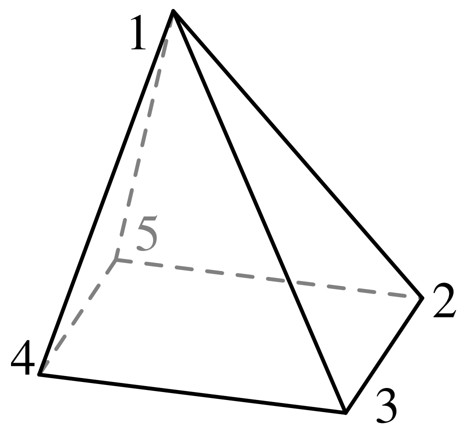
- Let G be a connected planar graph on n vertices and k edges, drawn without edge crossings.
- The base case was $k = n - 1$. Now consider $k \geqslant n$.
- **Induction hypothesis:** Assume Euler's formula holds for connected graphs with n vertices and $k - 1$ edges.
- Remove an edge from any cycle to get a connected subgraph G'.
- G' has V' vertices, E' edges, and F' faces:
  - $V' = V = n$
  - $E' = E - 1 = k - 1$ since we removed one edge.
  - $F' = F - 1$ since the faces on both sides of the removed edge were different but have been merged together.
- Since $E' = k - 1$, by induction, G' satisfies $V' - E' + F' = 2$.
- $V' - E' + F' = V - (E - 1) + (F - 1) = V - E + F$, so $V - E + F = 2$ also.

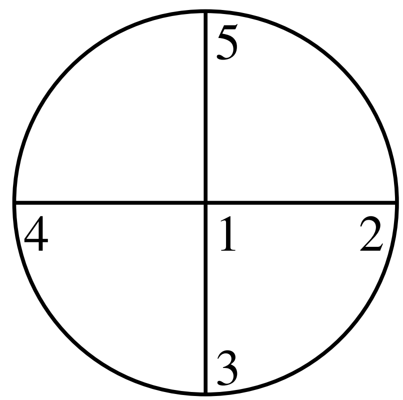
Graph on a sphere



http://en.wikipedia.org/wiki/File:Lambert_azimuthal_equal-area_projection_SW.jpg

- Consider a graph drawn on a sphere.
- Poke a hole inside a face, stretch it out from the hole, and flatten it onto the plane. *(Demo)*
- The face with the hole becomes the *outside* or *infinite* face.
  All other faces are distorted but remain finite.
- If a connected graph can be drawn on a sphere without edges crossing, it's a planar graph.
- The values of V, E, F are the same whether it's drawn on a plane or sphere, so $V - E + F = 2$ still applies.





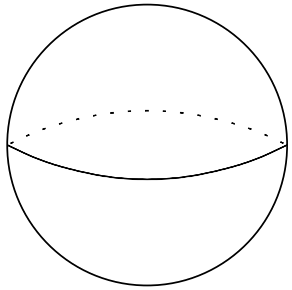
Pyramid with a square or rectangular base:

- Poke a pinhole in the base of the pyramid (left).
  Stretch it out and flatten it into a planar embedding (right).
  The pyramid base (left) corresponds to the infinite face (right).

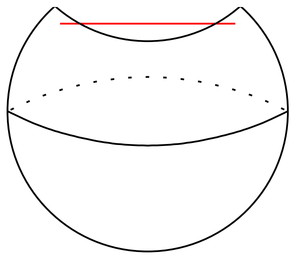
- Euler's formula (and other formulas we'll derive for planar embeddings) apply to polyhedra without holes.

- $V = 5, \quad E = 8, \quad F = 5, \qquad\qquad V - E + F = 5 - 8 + 5 = 2$

Convex polyhedra



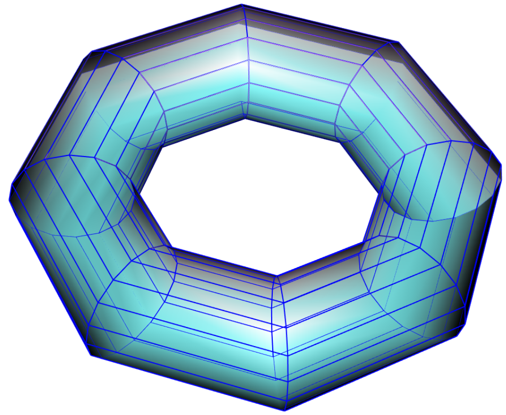
Sphere (convex)



Indented sphere (not convex)

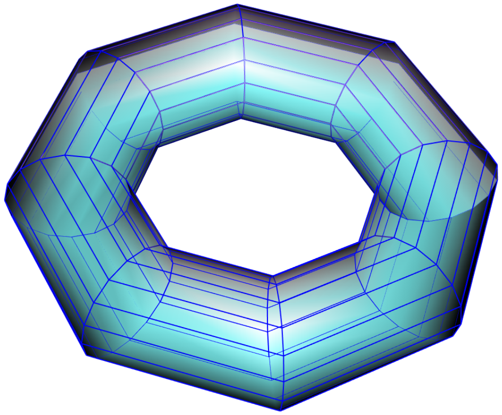
- A shape in 2D or 3D is *convex* when the line connecting any two points in it is completely contained in the shape.
- A sphere is convex. An indented sphere is not (red line above).
- But we can deform the indented sphere to an ordinary sphere, so the graphs that can be drawn on their surfaces are the same.
- Convex polyhedra are a special case of 3D polyhedra w/o holes.
- The book presents results about graphs on convex polyhedra; more generally, they also apply to 3D polyhedra without holes.

Beyond spheres – graphs on solids with holes



- A *torus* is a donut shape.
  It is not topologically equivalent to a sphere, due to a hole.
- Consider a graph drawn on a torus without crossing edges.
- Transforming a sphere to a torus requires cutting, stretching, and pasting. Edges on the torus through the cut can't be drawn there on the sphere. When redrawn on the sphere, they may cross.
- So, there are graphs that can be drawn on a torus w/o crossing edges, but which can't be drawn on a sphere w/o crossing edges.

Beyond spheres – graphs on solids with holes



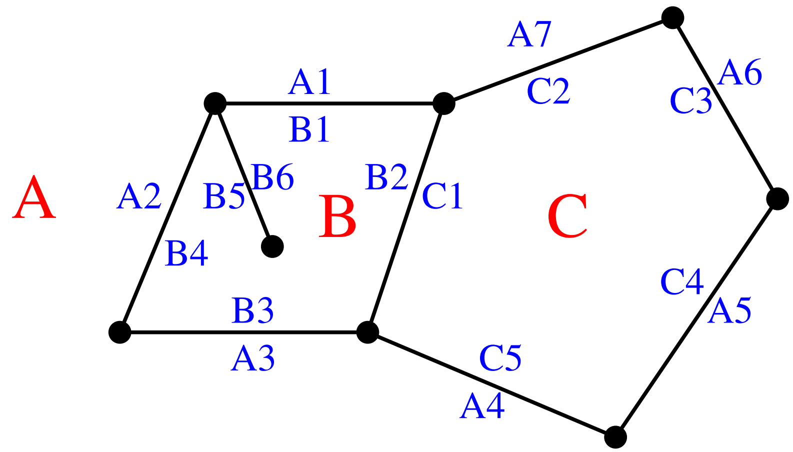
- An $m \times n$ grid on a torus has
$$V = mn, \quad E = 2mn, \quad F = mn$$
$$V - E + F = mn - 2mn + mn = 0$$

- **Theorem:** Let G be a connected graph drawn on a γ-holed torus without edge crossings, and with all faces homeomorphic to discs. ($\gamma = 0$ for sphere, 1 for donut, etc.) Then
$$V - E + F = 2(1 - \gamma).$$

- **Note:** The quantity $2(1-\gamma)$ is the *Euler characteristic*. It's usually denoted χ, which conflicts using $\chi(G)$ for chromatic number.

More relations on V, E, F in planar graphs

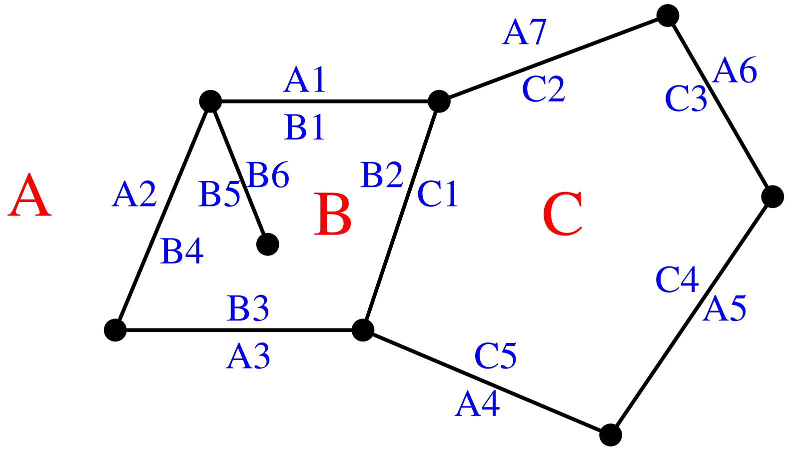
Face degrees



Face degrees

- Trace around a face, counting each encounter with an edge.
- Face A has edge encounters $A1$ through $A7$, giving deg$(A) = 7$.
- Face B has edge encounters $B1$ through $B6$, including two encounters with one edge ($B5$ and $B6$). So deg$(B) = 6$.
- deg$(C) = 5$.

Face degrees



Total degrees

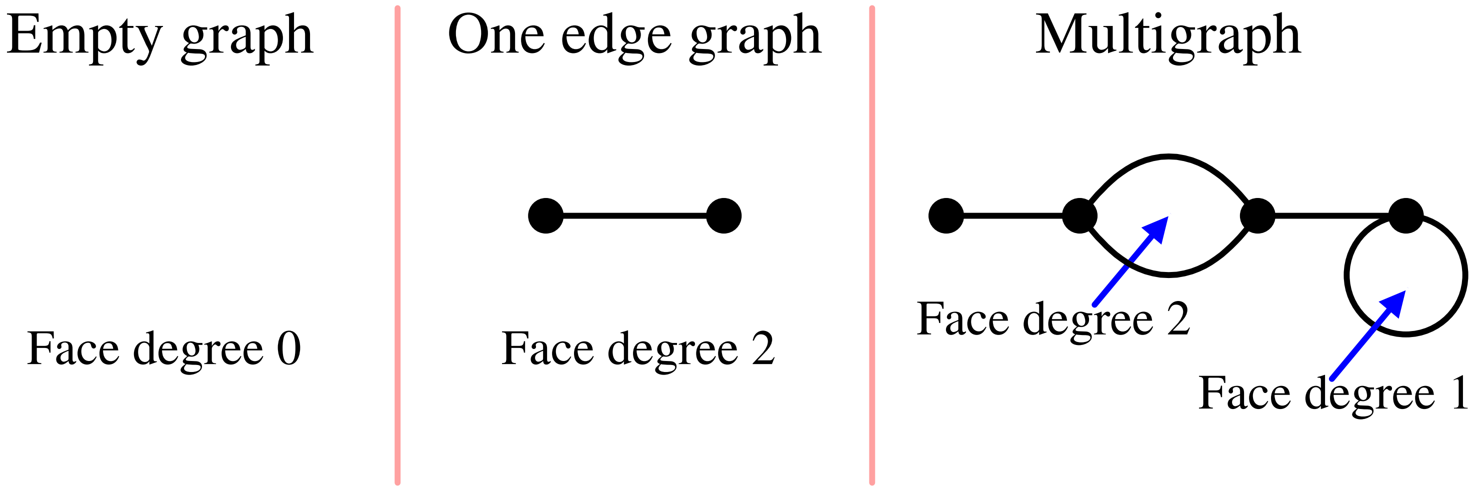
- The sum of the face degrees is $2E$, since each edge is used twice:

$$S = \text{deg}(A) + \text{deg}(B) + \text{deg}(C) = 7 + 6 + 5 = 18$$
$$2E = 2(9) = 18$$

- This is an analogue of the Handshaking Lemma.
- The sum of the vertex degrees is $2E$ for all graphs.
  Going clockwise from the upper left corner, we have

$$3 + 3 + 2 + 2 + 2 + 3 + 2 + 1 = 18.$$

Face degrees



- Faces usually have at least 3 sides, but it is possible to have fewer.
- In a simple (no loops, no multiedges) connected graph with at least three vertices, these cases don't arise, so all faces have face degree at least 3.
- Thus, the sum of the face degrees is $S \geqslant 3F$, so $2E \geqslant 3F$.
- In a bipartite graph, all cycles have even length, so all faces have even degree. Adding bipartite to the above conditions, each face has at least 4 sides. Thus, $2E \geqslant 4F$, which simplifies to $E \geqslant 2F$.

Inequalities between V, E, F

Theorem

In a connected graph drawn in the plane without crossing edges:

1. $V - E + F = 2$

2. *Additionally, if G is simple (no multiedges) and if $V \geqslant 3$, then*
   (a) $3F \leqslant 2E$ *(b)* $E \leqslant 3V - 6$ *(c)* $F \leqslant 2V - 4$

3. *If G is simple and bipartite, these bounds improve to*
   (a) $2F \leqslant E$ *(b)* $E \leqslant 2V - 4$ *(c)* $F \leqslant V - 2$

- Part 1 is Euler's formula. We just showed 2(a) and 3(a).
- We will prove the other parts, and use them to prove certain graphs are not planar.

Inequalities between V, E, F

(a) $3F \leqslant 2E$ (b) $E \leqslant 3V - 6$ (c) $F \leqslant 2V - 4$

Let G be a connected simple graph with $V \geqslant 3$, drawn in the plane without crossing edges.

(a) So far, we showed $V - E + F = 2$ and (a) $3F \leqslant 2E$.

(b) Thus, $F \leqslant 2E/3$ and

$$2 = V - E + F \leqslant V - E + (2E/3) = V - E/3$$

so $2 \leqslant V - E/3$, or $E \leqslant 3V - 6$, which is (b).

(c) $3F \leqslant 2E$ also gives $E \geqslant 3F/2$ and

$$2 = V - E + F \leqslant V - (3F/2) + F = V - F/2$$

so $2 \leqslant V - F/2$, or $F \leqslant 2V - 4$, which is (c).

Inequalities between V, E, F for a simple bipartite graph

(a) $2F \leqslant E$ (b) $E \leqslant 2V - 4$ (c) $F \leqslant V - 2$

Let G be a connected simple bipartite graph with $V \geqslant 3$, drawn in the plane without crossing edges.

(a) For this case, we showed $V - E + F = 2$ and (a) $2F \leqslant E$.

(b) Thus, $F \leqslant E/2$ and

$$2 = V - E + F \leqslant V - E + (E/2) = V - E/2$$

so $2 \leqslant V - E/2$, or $E \leqslant 2V - 4$, which is (b).

(c) $2F \leqslant E$ also gives

$$2 = V - E + F \leqslant V - 2F + F = V - F$$

so $2 \leqslant V - F$, or $F \leqslant V - 2$, which is (c).

Generalization: V, E, F inequalities when all cycle lengths $\geqslant g$

Let G be a connected graph
with $V \geqslant 3$, drawn in the plane without crossing edges.
Suppose all cycles have length $\geqslant g$, with $g \geqslant 3$.

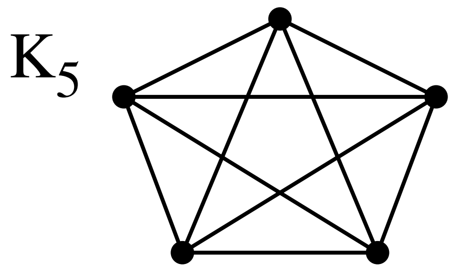
(a) Sum of face degrees: $S = 2E$ and $S \geqslant g \cdot F$, so $\boxed{\boldsymbol{F \leq \frac{2}{g}E}}$.

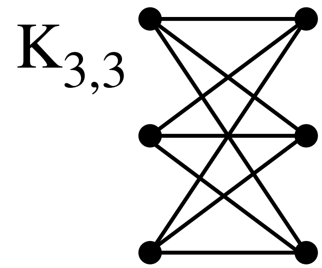
(b) Thus, $2 = V - E + F \leqslant V - E + \frac{2}{g}E$
$$= V - (1 - \tfrac{2}{g})E = V - \tfrac{g-2}{g}E$$
so $\boxed{\boldsymbol{E \leq \frac{g}{g-2}(V-2)}}$.

(c) $F \leqslant \frac{2}{g}E$ also gives $E \geqslant \frac{g}{2}F$ and
$$2 = V - E + F \leqslant V - \tfrac{g}{2}F + F = V - (\tfrac{g}{2} - 1)F$$
so $(\frac{g}{2} - 1)F \leqslant (V-2)$ so $\boxed{\boldsymbol{F \leq \frac{2}{g-2}(V-2)}}$.

Characterizing planar graphs

K_5 and $K_{3,3}$ are not planar





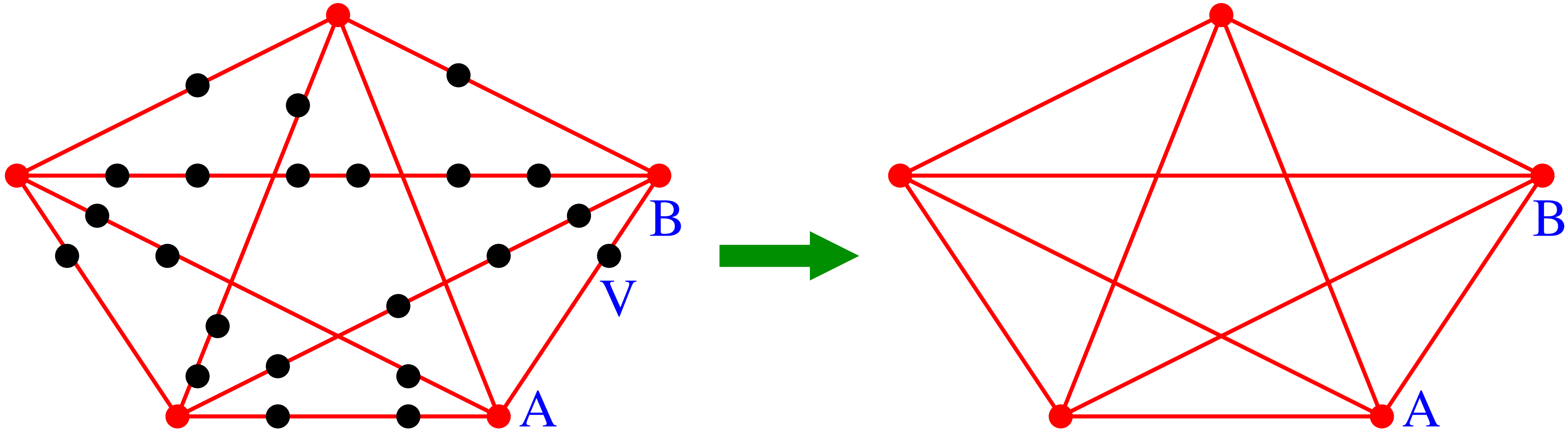
K_5 is not planar

- $V = 5$
- $E = \binom{5}{2} = 10$
- This violates $E \leqslant 3V - 6$ since $3V - 6 = 15 - 6 = 9$ and $10 \nleqslant 9$.

$K_{3,3}$ is not planar

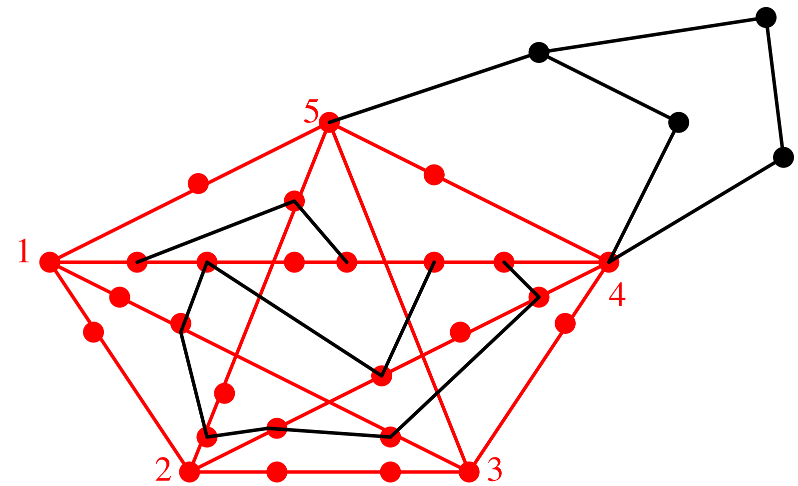
- $V = 6$
- $E = 3 \cdot 3 = 9$
- This is bipartite, so if it has a planar embedding, $E \leqslant 2V - 4$.
- However, $2V - 4 = 2(6) - 4 = 8$, and $9 \nleqslant 8$.

Homeomorphisms (a.k.a. edge equivalency)



- Suppose that we can turn graph G into graph H by repeatedly applying these two operations:
  - **Subdividing:** Split an edge AB into two edges AV and VB by adding a vertex V somewhere in the middle (not incident with any other edge).
  - **Smoothing:** Let V be a vertex of degree 2.
    Replace two edges AV and VB by one edge AB and delete vertex V.
- Then G and H are *homeomorphic* (a.k.a. *edge equivalent*).
- The left graph is homeomorphic to K_5 (on the right):
  - Smooth out every black vertex (left graph) to get K_5 (right graph).
  - Repeatedly subdivide edges of K_5 (right) to get the left graph.

Characterizing planar graphs



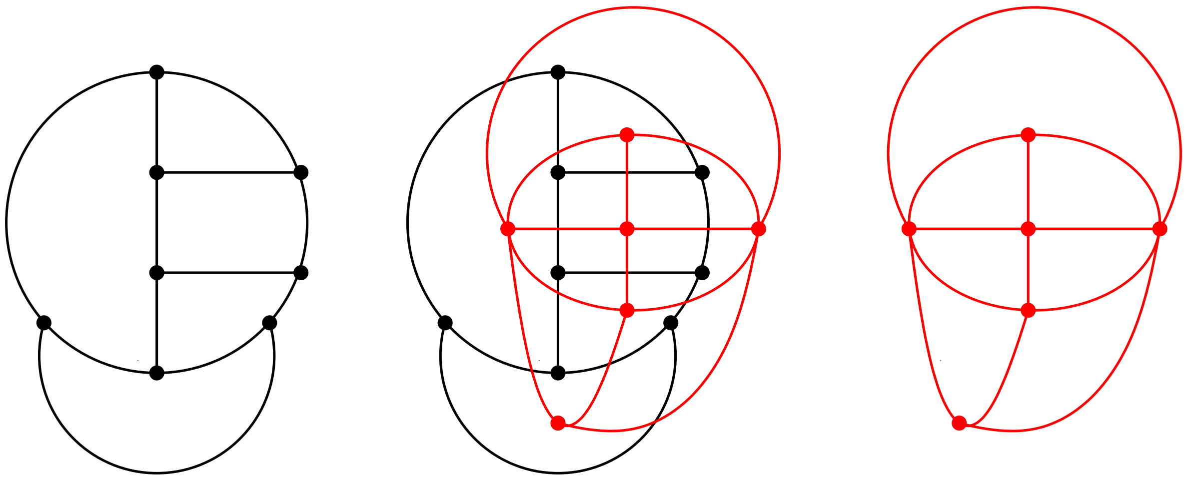
Theorem (Kuratowski's Theorem)

G is planar iff it does not have a subgraph homeomorphic to K_5 or $K_{3,3}$.

- **Necessity:** If G is planar, so is every subgraph. But if G has a subgraph homeomorphic to K_5 or $K_{3,3}$, the subgraph is not planar.
- **Sufficiency:** The proof is too advanced, but it's in the book.
- The graph shown above has a subgraph (shown in red) homeomorphic to K_5, and thus, it is not a planar graph.

Dual graphs

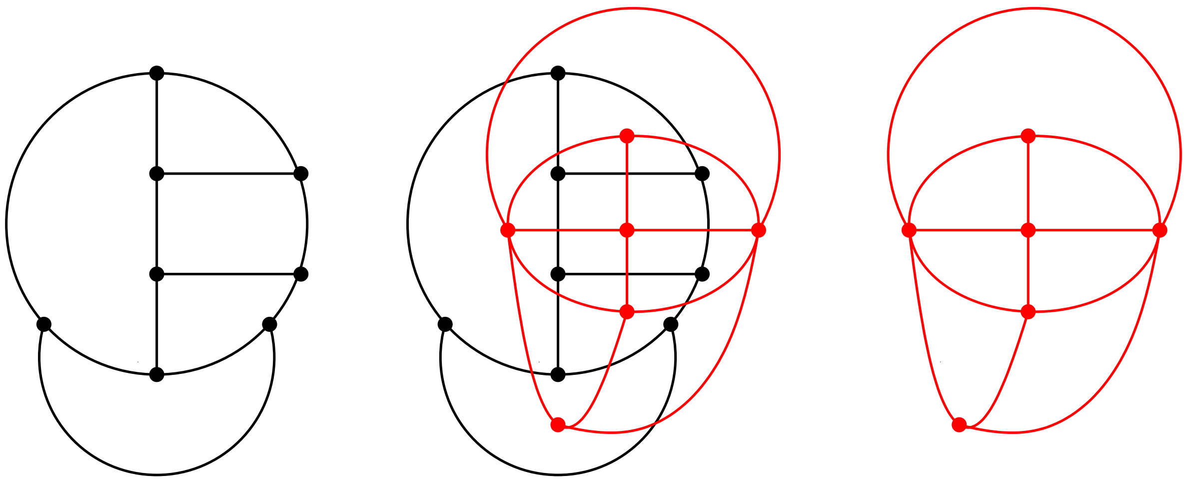
Dual graph



(a) Graph G (b) Constructing dual graph (c) Dual graph H

- Start with a planar embedding of a graph G (shown in black).
- Draw a red vertex inside each face, including the “infinite face.”
- For every edge e of G:
  - Let a, b be the red vertices in the faces on the two sides of e.
  - Draw a red edge $\{a, b\}$ crossing e.
- Remove the original graph G to obtain the red graph H.
- H is the *dual graph* of this drawing of G.
  (Also called *plane dual* or *combinatorial dual*.)
- The dual graph depends on how G is drawn.

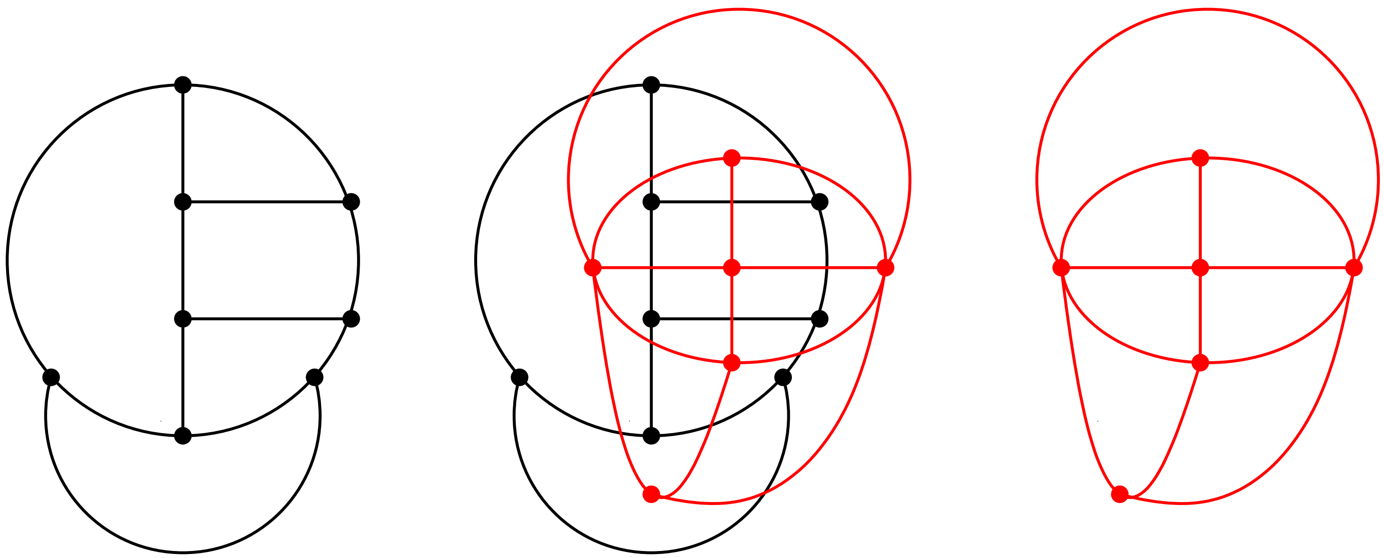
Dual graph



(a) Graph G (b) Constructing dual graph (c) Dual graph H

- **If G is connected,** then G is also a dual graph of H — just switch the roles of the colors!

Dual graph



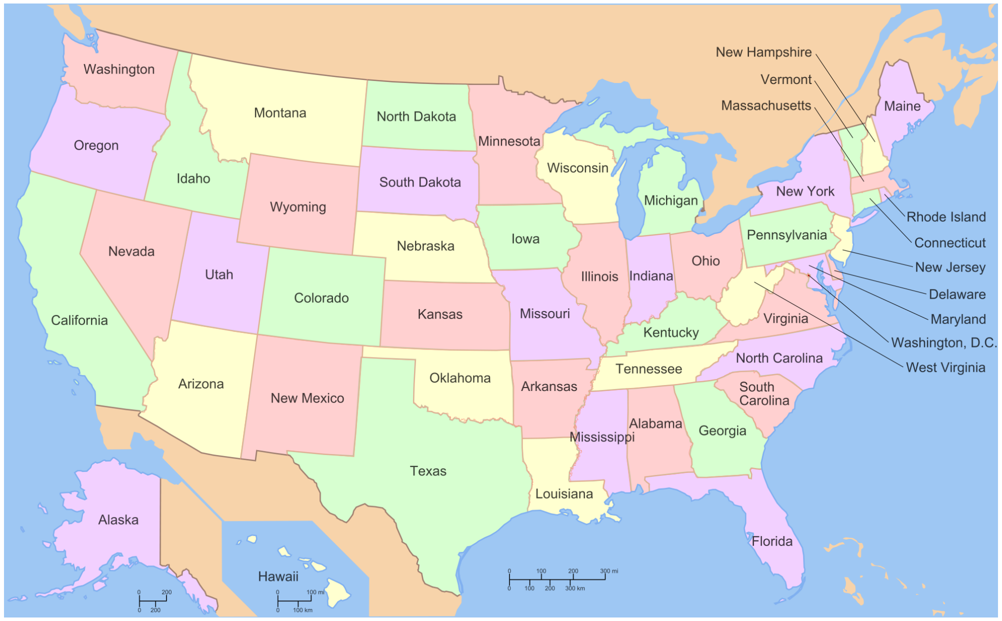
(a) Graph G (b) Constructing dual graph (c) Dual graph H

	G	H
V	8	6
E	12	12
F	6	8

- G and H have the same number of edges:
  - Each edge of G crosses exactly one edge of H and vice-versa.
- # faces of G = # vertices of H and
  # faces of H = # vertices of G:
  - Bijections: vertices of either graph $\leftrightarrow$ faces of the other.
- The fact that the sum of face degrees is $2E$ becomes the Handshaking Lemma applied to the dual graph!

Coloring maps

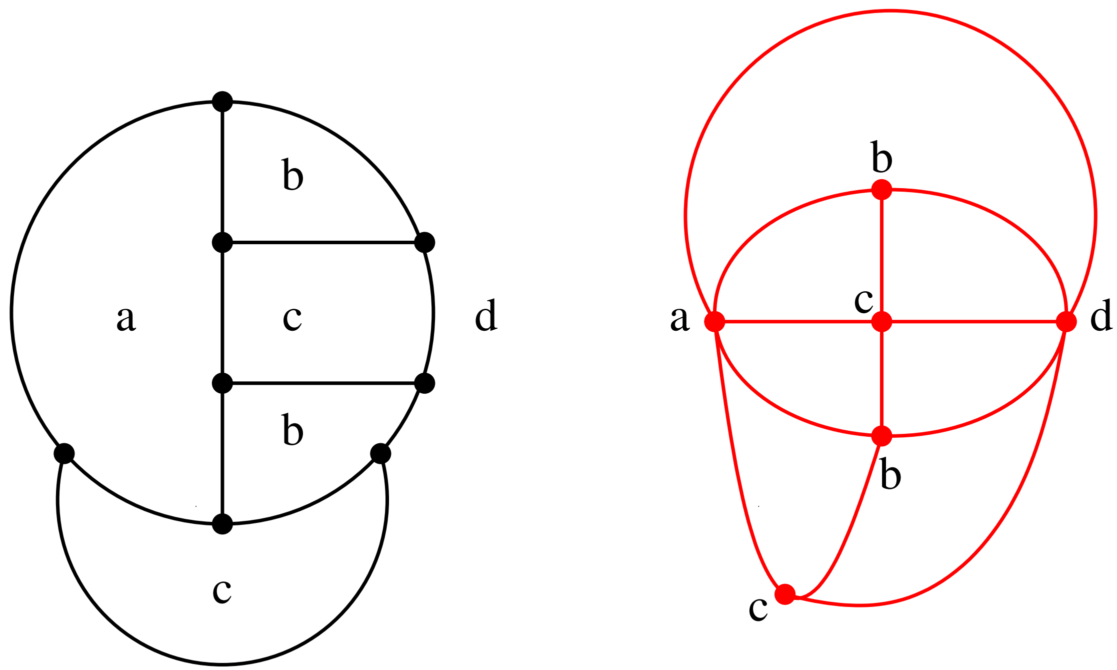
Coloring maps



http://en.wikipedia.org/wiki/File:Map_of_USA_with_state_names_2.svg

- Color states so that neighboring states have different colors. This map uses 4 colors for the states.
- Assume each state is a contiguous region.
  - Michigan isn't.
  - Its parts all have to be the same color, which could increase the # colors required. Artificially fill in Lake Michigan to make it contiguous.
- Also assume the states form a contiguous region.
  - Alaska and Hawaii are isolated, and just added on separately.

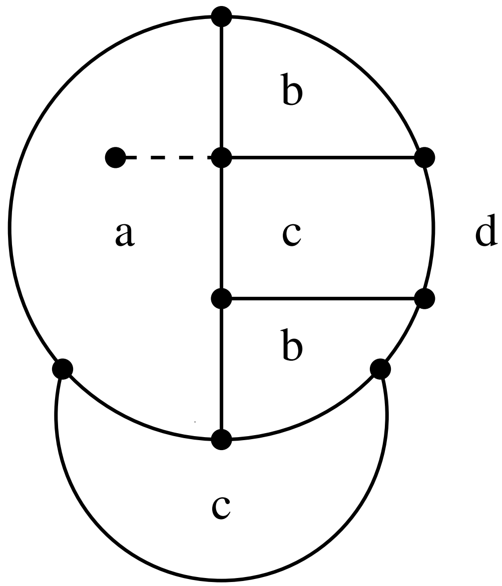
A proper coloring of the faces of a planar graph $\leftrightarrow$ a proper coloring of the vertices of its dual graph

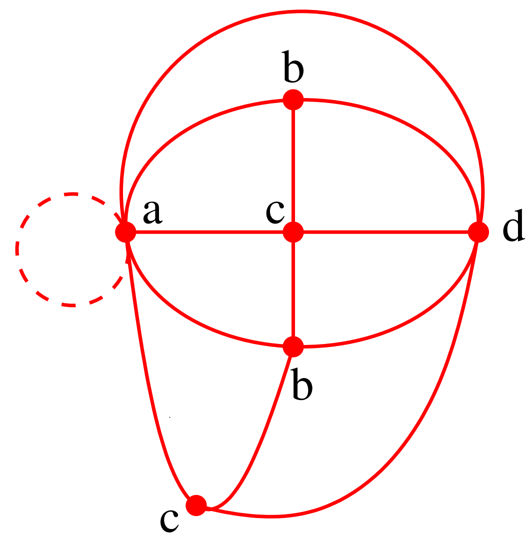


Coloring faces of G　　　Coloring vertices of H

- The regions/states/countries of the map are faces of a graph, G.
- Place a vertex inside each region and form the dual graph, H.
- A proper coloring of the vertices of H gives a proper coloring of the faces of G (aside from a technicality on the next slide).

A proper coloring of the faces of a planar graph $\leftrightarrow$ a proper coloring of the vertices of its dual graph





- **Technicality:** A vertex of degree 1 in G gives an edge sticking out into a face, resulting in a loop in H. (See dashed edges).
- A graph with a loop can't have a proper coloring!
- The edge sticking out in G doesn't separate faces of G.
- Delete vertices of degree 1 in G, and loops in H, to get an equivalent problem in terms of coloring the faces of G.

Coloring planar graphs

Possible degrees in a planar graph

Theorem

Every simple planar graph has a vertex with degree at most 5.

Proof:

- If the graph isn't connected, just restrict to one component of it.
- The sum of vertex degrees in any graph equals $2E$.
- Assume by way of contradiction that all vertices have degree $\geqslant 6$. Then the sum of vertex degrees is $2E \geqslant 6V$.
- So $2E \geqslant 6V$, so $E \geqslant 3V$.
- This contradicts $E \leqslant 3V - 6$ in any planar graph, so some vertex has degree $\leqslant 5$.

Possible degrees in a planar graph

Theorem

Every simple planar graph is 5-degenerate.

Proof:

- Recall that a graph is k-degenerate when all subgraphs have minimum degree $\leqslant k$.
- Every subgraph of a simple planar graph is also simple and planar, and thus has minimum degree $\leqslant 5$.
- So every simple planar graph is 5-degenerate.

Easy: Six Color Theorem

Theorem

Every simple planar graph is 6-colorable.

Proof:

- We showed that any k-degenerate graph is $(k+1)$-colorable.
- Every simple planar graph is 5-degenerate, and thus, 6-colorable.

Moderate difficulty: Five Color Theorem

Theorem

Every simple planar graph is 5-colorable.

Proof:

- We will induct on $|V(G)|$.
- **Base case:** If $|V(G)| \leqslant 5$, just assign all vertices different colors.

Five Color Theorem

Every simple planar graph is 5-colorable.

Proof, continued — Induction step:

- Assume the theorem holds for all graphs with fewer vertices.
- If G has a vertex v of degree $\leqslant 4$, then $G - \{v\}$ is 5-colorable by induction.
- When we add v back in, since it has $\leqslant 4$ neighbors, at least one of the 5 colors is available, so we can complete the 5-coloring.
- So, we will have to consider $\delta(G) \geqslant 5$. Since all simple planar graphs have $\delta(G) \leqslant 5$, this gives $\delta(G) = 5$.

Five Color Theorem

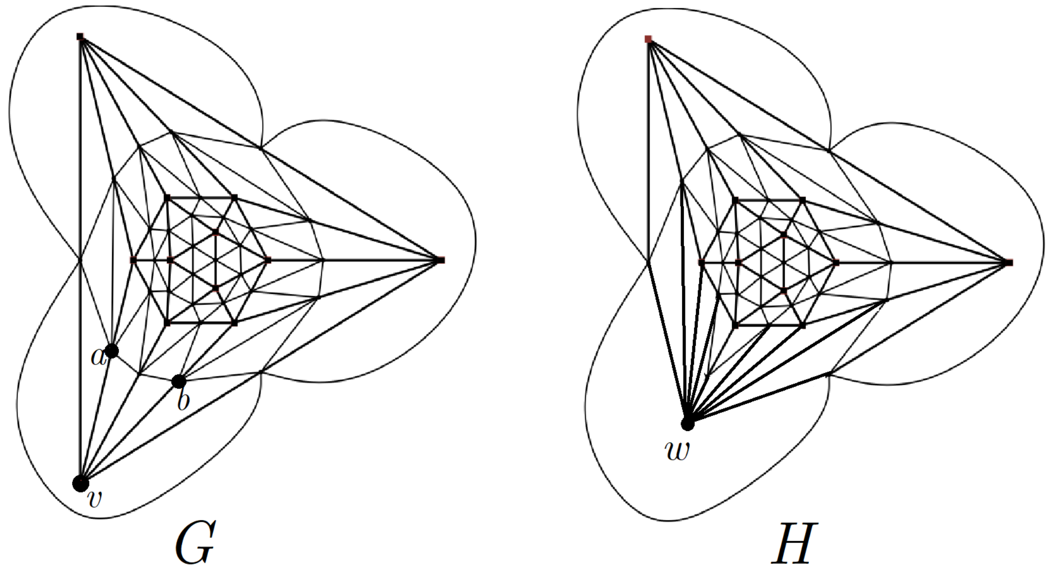
Every simple planar graph is 5-colorable.

Proof, continued — Induction step:

- Assume the theorem holds for all graphs with fewer vertices, and assume $\delta(G) = 5$.
- Let v be a vertex of degree 5.
- If all neighbors of v are adjacent to each other, they form a K_5.
  But then the graph isn't planar — a contradiction.
  So there are neighbors a and b of v with $ab \notin E(G)$.

Five Color Theorem

Every simple planar graph is 5-colorable.

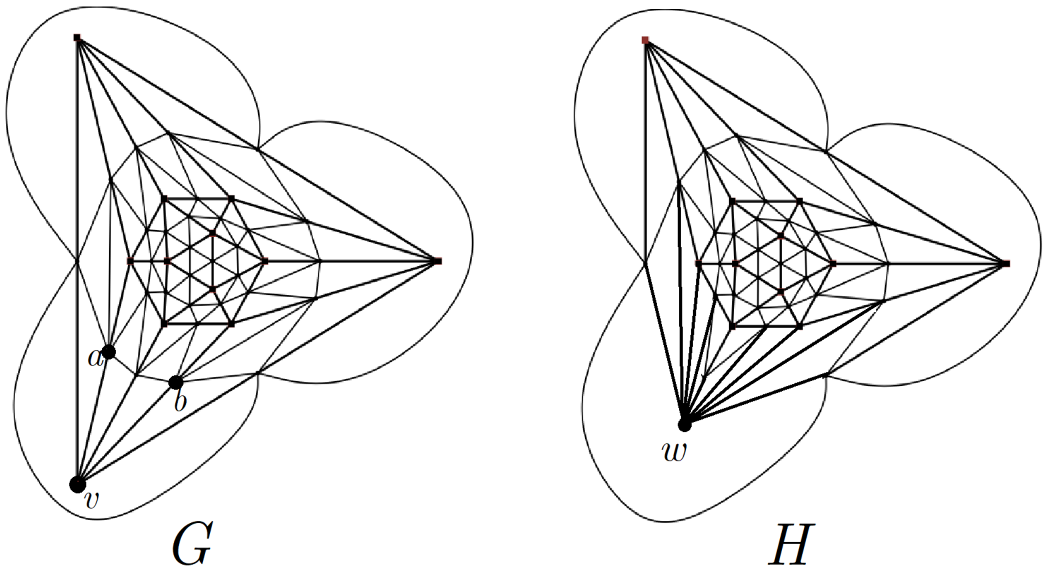


(figure from Verstraete textbook)

Proof, continued — Induction step:

- **Recall:** $d(v) = 5$, and a, b are neighbors of v with $ab \notin E(G)$
- Let $H = G/\{a, b, v\}$ (graph contraction).
  Vertices a, b, v are contracted to a new vertex w.
- H is still planar:
  - Slide a and v together along edge av. Same for b and v.
  - Merge a, b, v into one vertex w.
  - Remove edges av, bv, and reduce any multiedges just created.

Every simple planar graph is 5-colorable.



(figure from Verstraete textbook)

Proof, continued — Induction step:

- By induction, H has a 5-coloring $c_H : V(H) \to \{1, \ldots, 5\}$.
- Extend to a 5-coloring c_G of G:
  - For all vertices u except a, b, v, set $c_G(u) = c_H(u)$.
  - Set $c_G(a) = c_G(b) = c_H(w)$. This is fine since ab isn't an edge in G.
  - The 5 neighbors of v in G use at most 4 colors (since a and b use the same color). So there is a color available to assign to $c_G(v)$.

Extremely difficult: Four Color Theorem

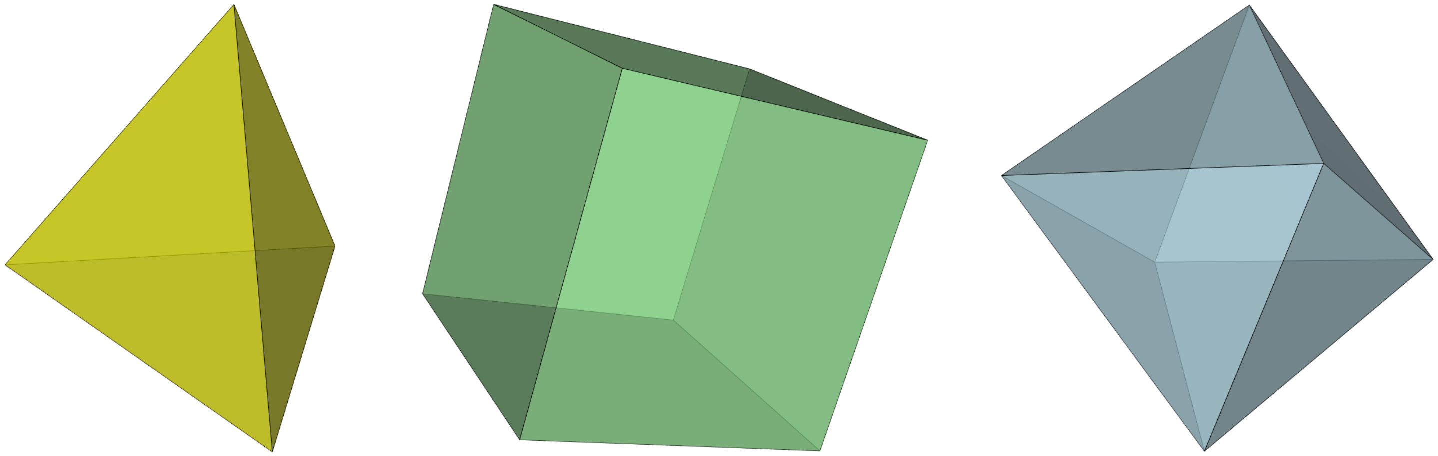
Theorem (Four Color Theorem)

Every simple planar graph is 4-colorable.

- Map makers have believed this for centuries empirically, but it wasn't proven mathematicallly.
- This was the first major theorem to be proved using a computer program (Kenneth Appel and Wolfgang Haken, 1976).
- The original proof had 1936 cases! Their program determined the cases and showed they are all 4-colorable.
- The proof was controversial because
  - It was the first proof that was impractical for any human to verify.
  - There could be bugs in the software, hardware, compiler, O/S, etc.
- Over the years, people have found errors in the proof, but they have been fixed, and the result still stands. The number of cases has been cut down to 633.

Classifying regular polyhedra

Classifying regular polyhedra



Tetrahedron **Cube** **Octahedron**

- A *polyhedron* is a 3D solid whose surface consists of polygons. As a graph, no loops and no multiple edges.
- All faces have $\geqslant 3$ edges and all vertices are in $\geqslant 3$ edges.
- To be 3D, there must be $\geqslant 4$ vertices, $\geqslant 4$ faces, and $\geqslant 6$ edges.
- A *regular polyhedron* has these symmetries:
  - All faces are regular ℓ-gons for the same $\ell \geqslant 3$.
  - All vertices have the same degree ($r \geqslant 3$).
  - All edges have the same length.
  - All pairs of adjacent faces have the same angle between them.

Classifying regular polyhedra

- Suppose all vertices have the same degree $r \geqslant 3$ and all faces are ℓ-gons (same $\ell \geqslant 3$ for all faces).
- The sum of vertex degrees is $r \cdot V = 2E$, so $V = 2E/r$.
- The sum of face degrees is $\ell \cdot F = 2E$, so $F = 2E/\ell$.
- Plug these into $V - E + F = 2$:

$$\frac{2E}{r} - E + \frac{2E}{\ell} = 2 \qquad E \cdot \left(\frac{2}{r} - 1 + \frac{2}{\ell}\right) = 2 \qquad E = \frac{2}{\frac{2}{r} + \frac{2}{\ell} - 1}$$

- We have to find all integers $r, \ell \geqslant 3$ for which V, E, F are positive integers, and then check if polyhedra with those parameters exist.

Classifying regular polyhedra

- Suppose all vertices have the same degree $r \geqslant 3$ and all faces are ℓ-gons (same $\ell \geqslant 3$ for all faces).
- Compute (V, E, F) using $E = \frac{2}{\frac{2}{r}+\frac{2}{\ell}-1}$, $V = \frac{2E}{r}$, $F = \frac{2E}{\ell}$:
- E.g., $r = 3$ and $\ell = 4$ gives

$$E = \frac{2}{\frac{2}{3}+\frac{2}{4}-1} = \frac{2}{1/6} = 12$$
$$V = 2(12)/3 = 8$$
$$F = 2(12)/4 = 6$$

- What shape is it?

Classifying regular polyhedra

What range of vertex degree (r) and face degree (ℓ) are permitted?

First method

- We have $r \geqslant 3$.
  Since some vertex has degree $\leqslant 5$, all do, so r is 3, 4, or 5.
- Vertices and faces are swapped in the dual graph, so ℓ is $3, 4$, or 5.

Second method: Analyze formula $E = 2/(\frac{2}{r} + \frac{2}{\ell} - 1)$

- E is a positive integer, so its denominator must be positive:
  $$\frac{2}{r} + \frac{2}{\ell} - 1 > 0$$
- We have $r, \ell \geqslant 3$.
- If both $r, \ell \geqslant 4$, the denominator of E is $\leqslant \frac{2}{4} + \frac{2}{4} - 1 = 0$, which is invalid. So r and/or ℓ is 3.
- If $r = 3$, then the denominator of E is $\frac{2}{3} + \frac{2}{\ell} - 1 = \frac{2}{\ell} - \frac{1}{3}$.
  To be positive requires $\ell \leqslant 5$.
- Similarly, if $\ell = 3$ then $r \leqslant 5$.

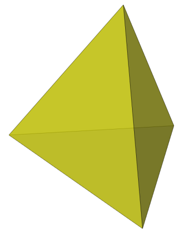
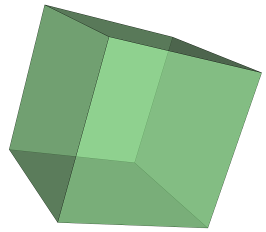
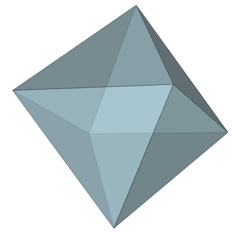
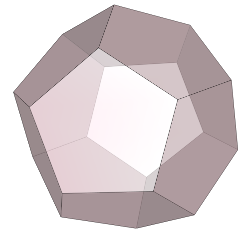
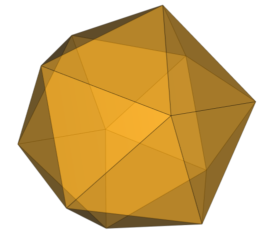
Classifying regular polyhedra

- Suppose all vertices have the same degree $r \in \{3,4,5\}$ and all faces are ℓ-gons (same $\ell \in \{3,4,5\}$ for all faces).
- Compute (V,E,F) using $E = \frac{2}{\frac{2}{r}+\frac{2}{\ell}-1}$, $V = \frac{2E}{r}$, $F = \frac{2E}{\ell}$:

(V,E,F)	$\ell = 3$	$\ell = 4$	$\ell = 5$
$r = 3$	$(4,6,4)$	$(8,12,6)$	$(20,30,12)$
$r = 4$	$(6,12,8)$	Division by 0	$(-10,-20,-8)$
$r = 5$	$(12,30,20)$	$(-8,-20,-10)$	$(-4,-10,-4)$

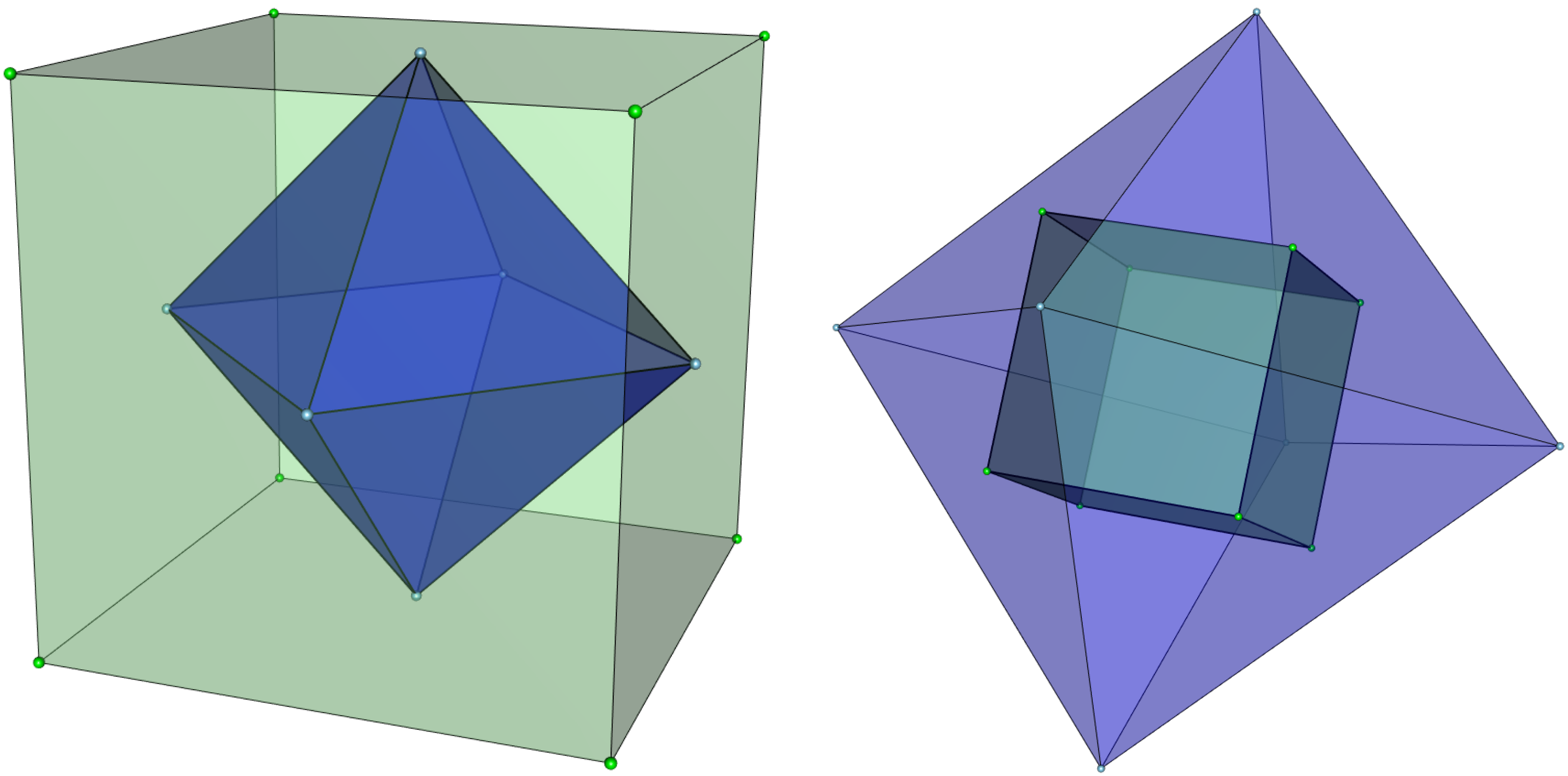
- If V, E, F are not all positive integers, it can't work (shown in pink).
- We found five possible values of (V,E,F) with graph theory. Use geometry to actually find the shapes (if they exist).

Classifying regular polyhedra

Shape	 Tetrahedron	 Cube	 Octahedron	 Dodecahedron	 Icosahedron
r = vertex degree	3	3	4	3	5
ℓ = face degree	3	4	3	5	3
V = # vertices	4	8	6	20	12
E = # edges	6	12	12	30	30
F = # faces	4	6	8	12	20

- These are known as the *Platonic solids*.
- The cube and octahedron are dual graphs.
  The dodecahedron and icosahedron are dual graphs.
  The tetrahedron is its own dual.

Octahedron and cube are dual



- Can draw either one inside the other.
  Place a dual vertex at the center of each face.
- In 3D, this construction shrinks the dual, vs. in 2D, it did not.