Chapter 1. Introduction to Graph Theory

(Chapters 1.1, 1.3–1.6, Appendices A.2–A.3)

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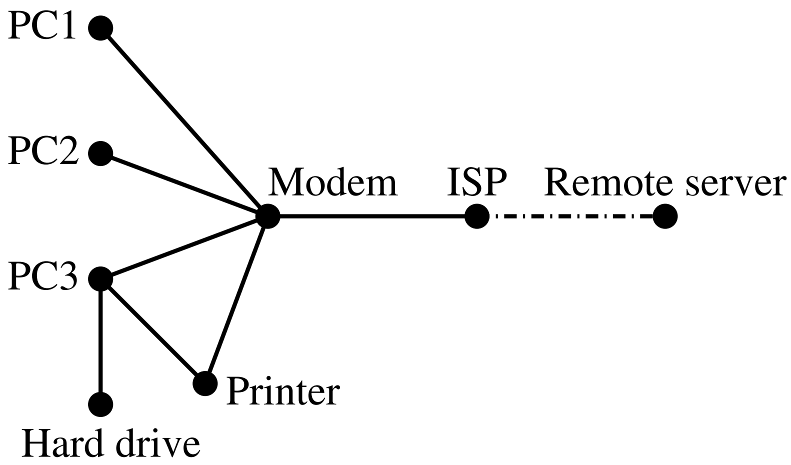
Math 154
Winter 2020

Related courses

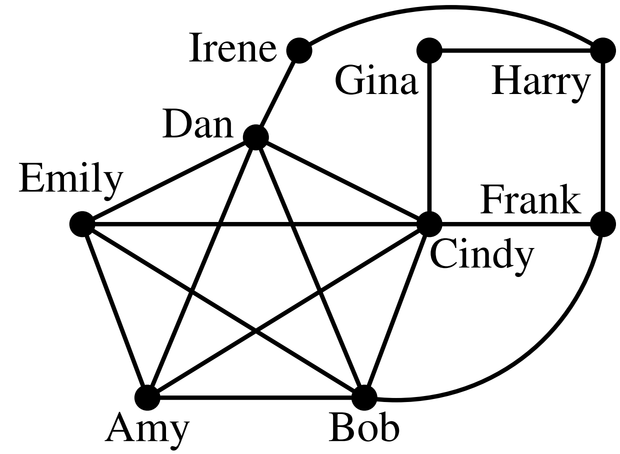
- **Math 184:** Enumerative combinatorics. For two quarters of Combinatorics, take Math 154 and 184 in either order.
- **Math 158 and 188:** More advanced/theoretical than Math 154 and 184. Recommended only for students with A/A+ in Math 109 or Math 31CH.
- **CSE 101:** Has some overlap with Math 154, but mostly different.

Graphs

Computer network



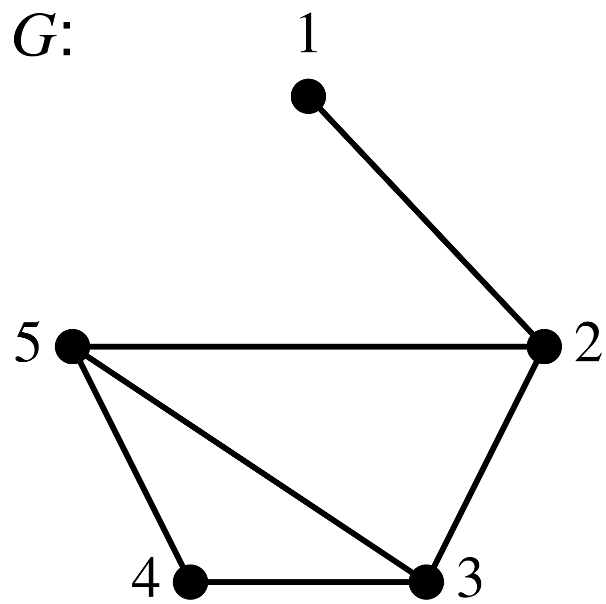
Friends



We have a network of items and connections between them. Examples:

- Telephone networks, computer networks
- Transportation networks (bus/subway/train/plane)
- Social networks
- Family trees, evolutionary trees
- Molecular graphs (atoms and chemical bonds)
- Various data structures in Computer Science

Graphs



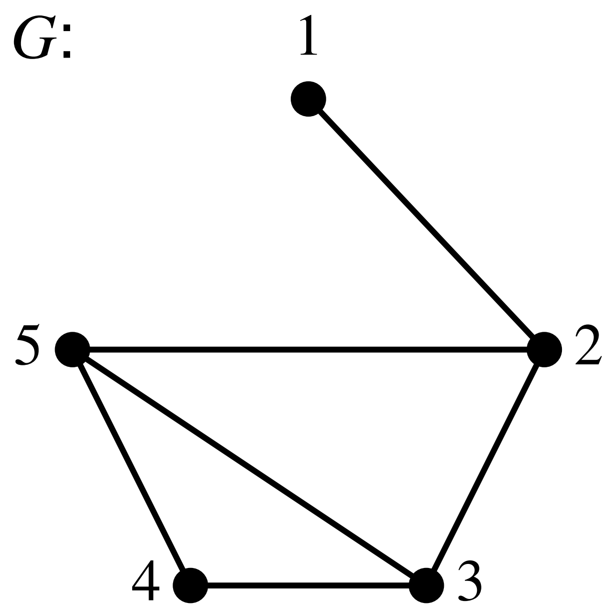
- The dots are called *vertices* or *nodes* (singular: vertex, node)

$$V = V(G) = \text{set of vertices} = \{1, 2, 3, 4, 5\}$$

- The connections between vertices are called *edges*.
- Represent an edge as a set $\{i, j\}$ of two vertices.
  E.g., the edge between 2 and 5 is $\{2, 5\} = \{5, 2\}$.

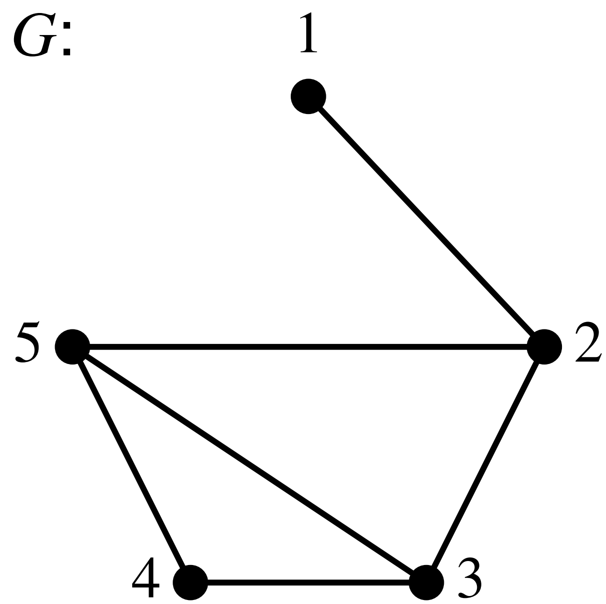
$$E = E(G) = \text{set of edges} = \big\{ \{1, 2\}, \{2, 3\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\} \big\}$$

Notation for edges



- Our book sometimes abbreviates edges as uv instead of $\{u, v\}$.
- In that notation, $\{2, 5\}$ becomes 25.
- Avoid that notation unless there is no chance of ambiguity. E.g., if there were 12 vertices, would 112 mean $\{1, 12\}$ or $\{11, 2\}$?

Adjacencies

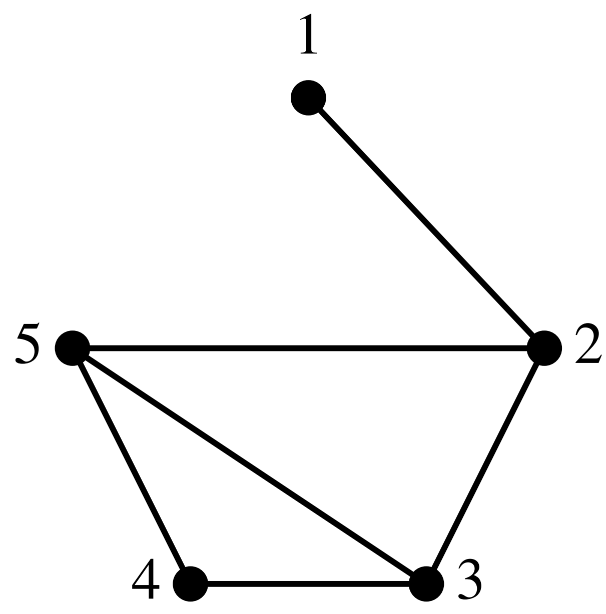


- Vertices connected by an edge are called *adjacent*.
  Vertices 1 and 2 are adjacent, but 1 and 5 are not.
- The *neighborhood* of a vertex v is the set of all vertices adjacent to v. It's denoted $N_G(v)$:

$$N_G(2) = \{1, 3, 5\}$$

- A vertex v is *incident* with an edge e when $v \in e$.
  Vertex 2 is incident with edges $\{1, 2\}$, $\{2, 5\}$, and $\{2, 3\}$.

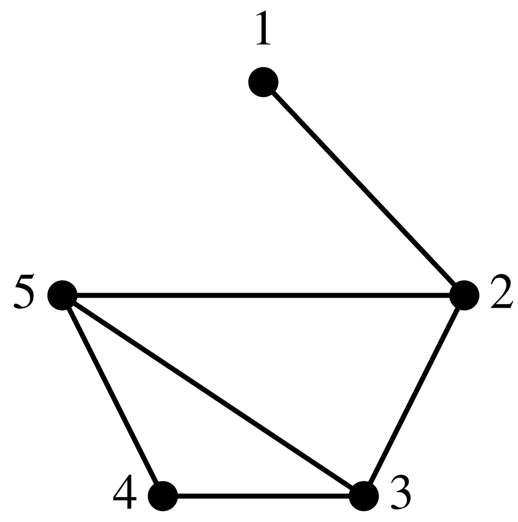
Simple graphs

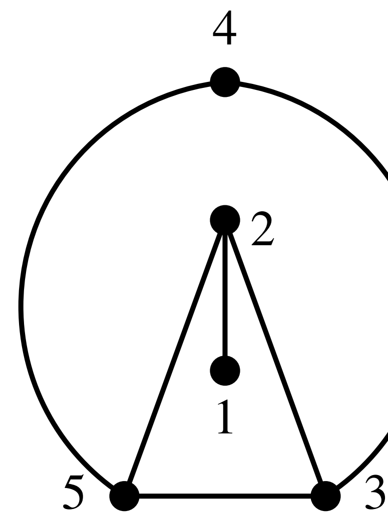


A *simple graph* is $G = (V, E)$:

- V is the set of vertices.
  It can be any set; $\{1, \ldots, n\}$ is just an example.
- E is the set of edges, of form $\{u, v\}$, where $u, v \in V$ and $u \neq v$.
  Every pair of vertices has either 0 or 1 edges between them.
- Usually, *graph* alone refers to *simple graph*, not to other kinds of graphs that we will consider.

Drawings of graphs



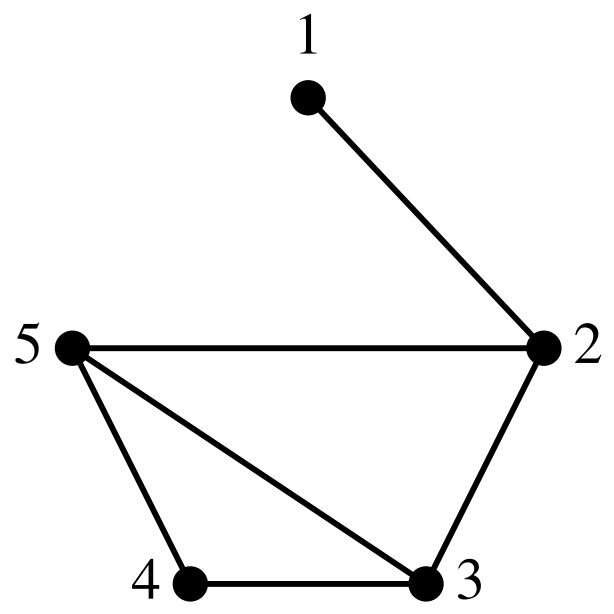


- Both graph drawings have

$$V = \{1, 2, 3, 4, 5\}$$
$$E = \big\{ \{1, 2\}, \{2, 3\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\} \big\}$$

- Both drawings represent the same graph (even though they look different) since they have the same vertices and edges in the abstract representation $G = (V, E)$.

Degrees



- The *degree* of a vertex is the number of edges on it:

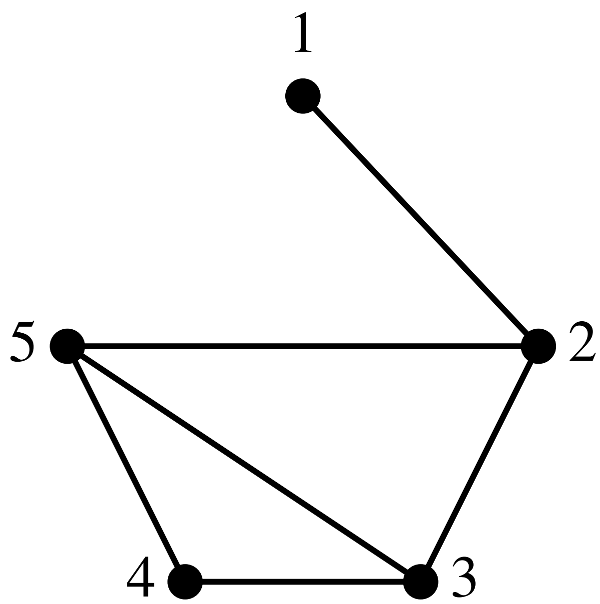
$$d(1) = 1 \qquad d(2) = 3 \qquad d(3) = 3 \qquad d(4) = 2 \qquad d(5) = 3$$

- The *degree sequence* is to list the degrees in descending order:

$$3, 3, 3, 2, 1$$

- The *minimum degree* is denoted $\delta(G)$. $\qquad \delta(G) = 1$
- The *maximum degree* is denoted $\Delta(G)$. $\qquad \Delta(G) = 3$

Degrees



$d(1) = 1 \qquad d(2) = 3 \qquad d(3) = 3 \qquad d(4) = 2 \qquad d(5) = 3$

Sum of degrees $= 1 + 3 + 3 + 2 + 3 = 12$
Number of edges $= 6$

The Handshaking Lemma

Lemma

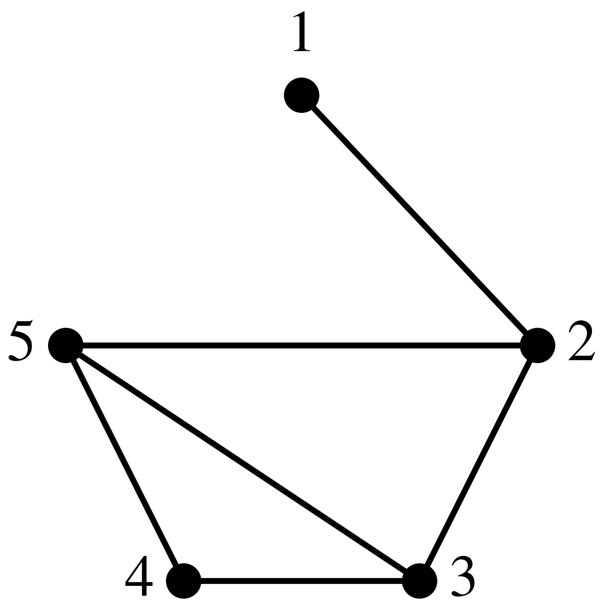
The sum of degrees of all vertices is twice the number of edges:

$$\sum_{v\in V} d(v) = 2\,|E|$$

Proof.

- Let $S = \{\,(v,e)\ :\ v \in V,\ \ e \in E,\ \ \text{vertex } v \text{ is in edge } e\,\}$
- **Count $|S|$ by vertices:** Each vertex v is contained in $d(v)$ edges, so
$$|S| = \sum_{v\in V} d(v).$$
- **Count $|S|$ by edges:** Each edge has two vertices, so
$$|S| = \sum_{e\in E} 2 = 2\,|E|\ .$$
- Equating the two formulas for $|S|$ gives the result. This is a common method in Combinatorics called *counting in two ways*.

Number of vertices of odd degree



$$d(1) = 1$$
$$d(2) = 3$$
$$d(3) = 3$$
$$d(4) = 2$$
$$d(5) = 3$$

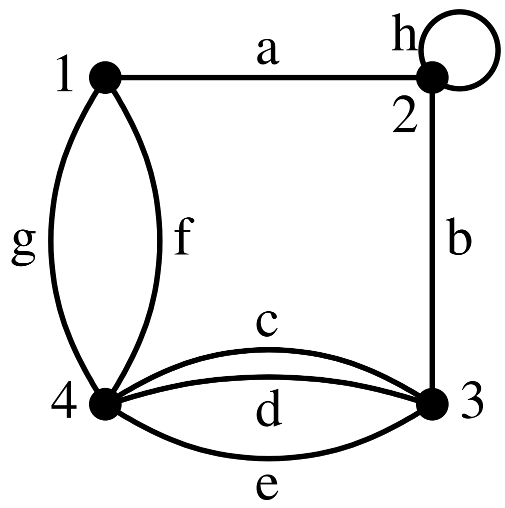
Lemma

For any graph, the number of vertices of odd degree is even.
E.g., this example has four vertices of odd degree.

Proof.

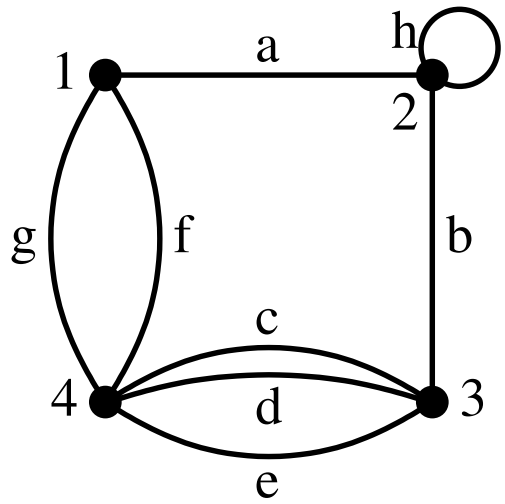
Since the degrees are integers and their sum is even ($2|E|$), the number of odd numbers in this sum is even. □

Multigraphs and pseudographs



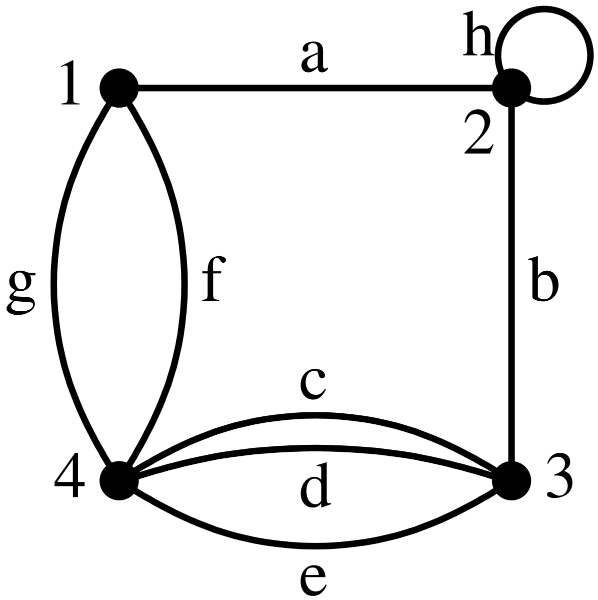
- Some networks have *multiple edges* between two vertices. Notation $\{3, 4\}$ is ambiguous, so write labels on the edges: c, d, e.
- There can be an edge from a vertex to itself, called a *loop* (such as h above). A loop has one vertex, so $\{2, 2\} = \{2\}$.
- A simple graph does not have multiple edges or loops.
- Our book uses *multigraph* if loops aren't allowed and *pseudograph* if loops are allowed (whether or not they actually occur). Other books call it a *multigraph [with / without] loops allowed*.

Multigraphs and pseudographs



- Computer network with multiple connections between machines.
- Transportation network with multiple routes between stations.
- **But:** A graph of Facebook friends is a simple graph. It does not have multiple edges, since you're either friends or you're not. Also, you cannot be your own Facebook friend, so no loops.

Multigraphs and pseudographs



$$V = \{1,2,3,4\}$$
$$E = \{a,b,c,d,e,f,g,h\}$$

$$\phi(a) = \{1,2\}$$
$$\phi(b) = \{2,3\}$$
$$\phi(c) = \phi(d) = \phi(e) = \{3,4\}$$
$$\phi(f) = \phi(g) = \{1,4\}$$
$$\phi(h) = \{2\}$$

Represent a *multigraph* or *pseudograph* as $G = (V, E, \phi)$, where:

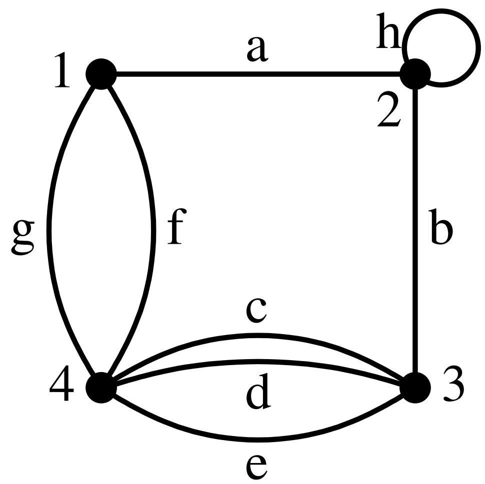
- V is the set of vertices. It can be any set.
- E is the set of edge labels (with a unique label for each edge).
- ϕ is a function from the edge labels to the pairs of vertices:

$$\phi : E \to \left\{ \{u,v\} \;:\; u,v \in V \right\}$$

$\phi(L) = \{u,v\}$ means the edge with label L connects u and v.

Adjacency matrix of a multigraph or pseudograph

- Let $n = |V|$
- The *adjacency matrix* is an $n \times n$ matrix $A = (a_{uv})$.
  Entry a_{uv} is the number of edges between vertices $u, v \in V$.



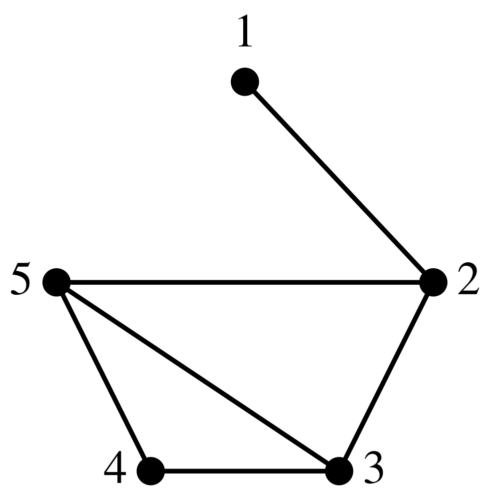
$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 3 & 0 \end{bmatrix} \end{array}$$

- $a_{uv} = a_{vu}$ for all vertices u, v. Thus, A is a *symmetric matrix* ($A = A^T$).
- The sum of entries in row u is the degree of u.
- **Technicality:** A loop on vertex v counts as
  - 1 edge in E,
  - degree 2 in $d(v)$ and in a_{vv} (it touches vertex v twice),

  With these rules, graphs with loops also satisfy $\sum_{v \in V} d(v) = 2|E|$.

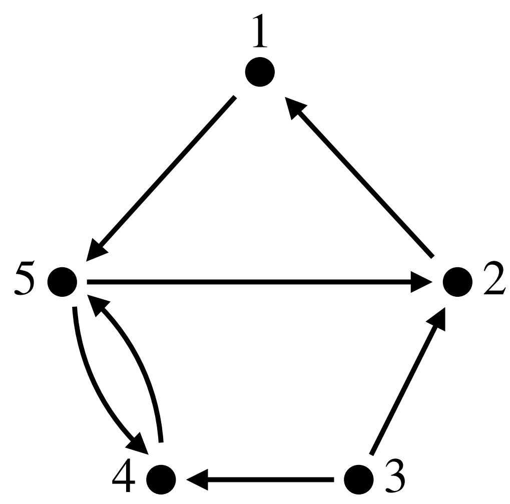
In a simple graph:

- All entries of the adjacency matrix are 0 or 1 (since there either is or is not an edge between each pair of vertices).
- The diagonal is all 0's (since there are no loops).



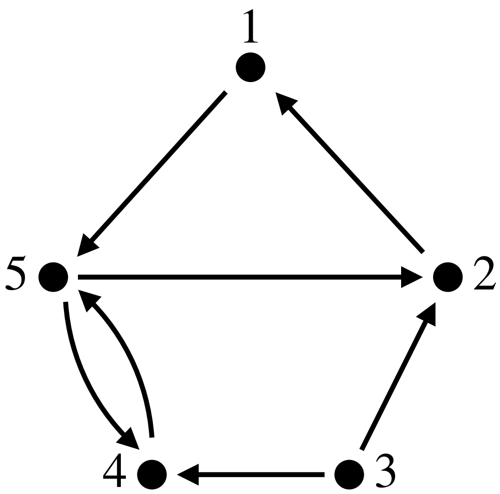
$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Directed graph (a.k.a. digraph)



- A *directed edge* (also called an *arc*) is a connection with a direction.
- One-way transportation routes.
- Broadcast and satellite TV / radio are one-way connections from the broadcaster to your antenna.
- Familiy tree: parent $\rightarrow$ child

Directed graph (a.k.a. digraph)

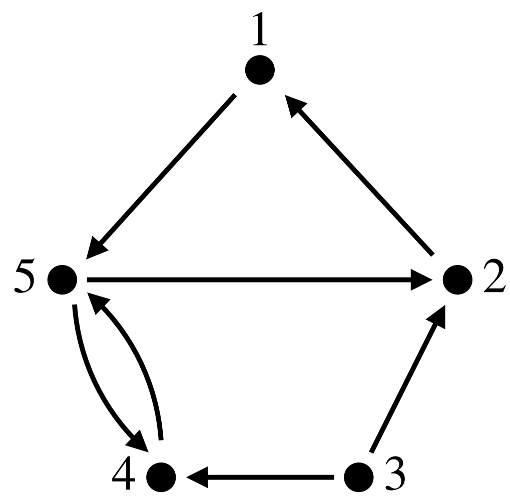


$V = \{1,2,3,4,5\}$

$E = \{(1,5),(2,1),(3,2),(3,4),(4,5),(5,2),(5,4)\}$

- Represent a directed edge $u \to v$ by an ordered pair (u,v).
  E.g., $3 \to 2$ is $(3,2)$, but we do not have $2 \to 3$, which is $(2,3)$.
- A directed graph is *simple* if each (u,v) occurs at most once, and there are no loops.
  - Represent it as $G = (V,E)$ or $\vec{G} = (V,\vec{E})$.
  - V is a set of vertices. It can be any set.
  - E is the set of edges. Each edge has form (u,v) with $u, v \in V$, $u \neq v$.
  - It is permissible to have both $(4,5)$ and $(5,4)$, since they are distinct.

Degrees in a directed graph

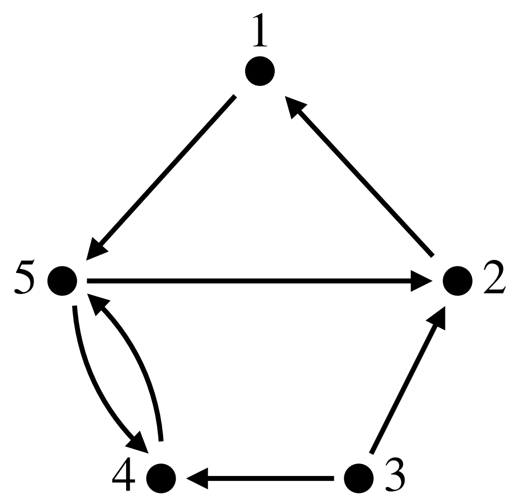


- For a vertex v, the *indegree* $d^{-}(v)$ is the # edges going into v, and the *outdegree* $d^{+}(v)$ is the # edges going out from v.

v	indegree(v)	outdegree(v)
1	1	1
2	2	1
3	0	2
4	2	1
5	2	2
Total	7	7

- Sum of indegrees $=$ sum of outdegrees $=$ total # edges $= |E|$

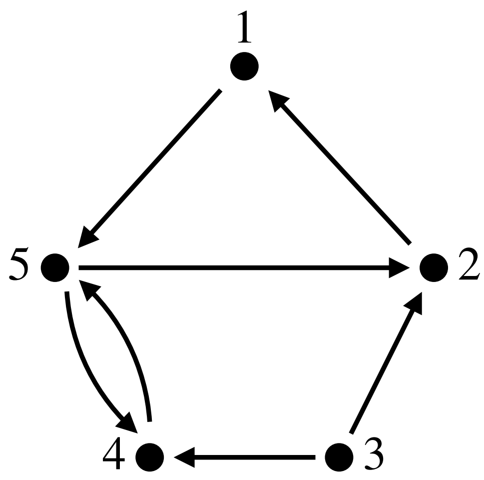
Neighborhoods in a directed graph



Out-neighborhood $N^+(v) = \{u : (v,u) \in E\}$
In-neighborhood $N^-(v) = \{u : (u,v) \in E\}$

- **Example:** $N^+(2) = \{1\}$ $N^-(2) = \{3,5\}$.
- For a simple directed graph:
  outdegree $d^+(v) = |N^+(v)|$
  indegree $d^-(v) = |N^-(v)|$

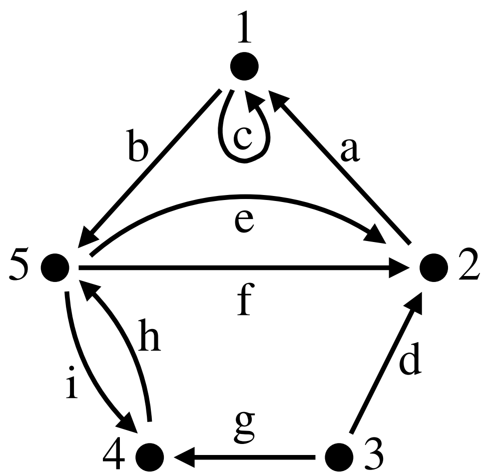
Adjacency matrix of a directed graph



$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

- Let $n = |V|$
- The *adjacency matrix* of a directed graph is an $n \times n$ matrix $A = (a_{uv})$ with $u, v \in V$.
- Entry a_{uv} is the number of edges directed from u to v.
- a_{uv} and a_{vu} are not necessarily equal, so A is usually not symmetric.
- The sum of entries in row u is the outdegree of u.
  The sum of entries in column v is the indegree of v.

Directed multigraph



$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

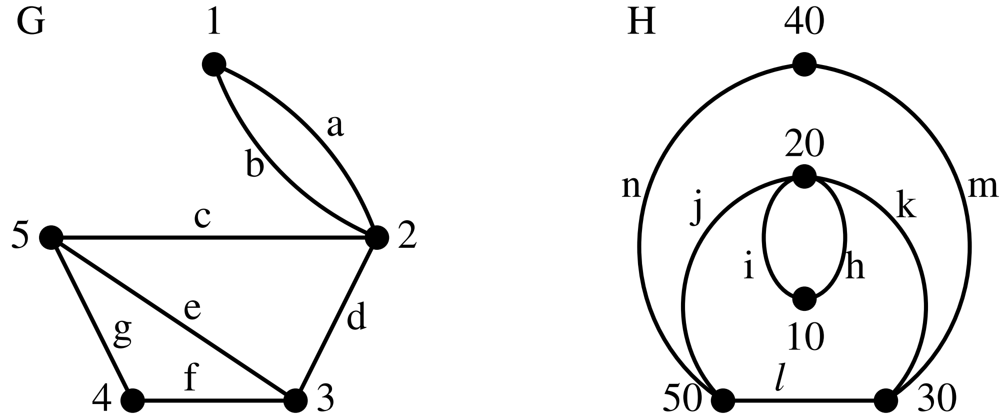
$V = \{1, \ldots, 5\}$ $\quad \phi(a) = (2, 1)$ $\quad \phi(d) = (3, 2)$ $\quad \phi(g) = (3, 4)$

$E = \{a, \ldots, i\}$ $\quad \phi(b) = (1, 5)$ $\quad \phi(e) = (5, 2)$ $\quad \phi(h) = (4, 5)$

$\phi(c) = (1, 1)$ $\quad \phi(f) = (5, 2)$ $\quad \phi(i) = (5, 4)$

- A directed multigraph may have loops and multiple edges.
  - Represent it as $G = (V, E, \phi)$.
  - Name the edges with labels. Let E be the set of the labels.
  - $\phi(L) = (u, v)$ means the edge with label L goes from u to v.
- **Technicality:** A loop counts once in indegree, outdegree, and a_{vv}.

Isomorphic graphs



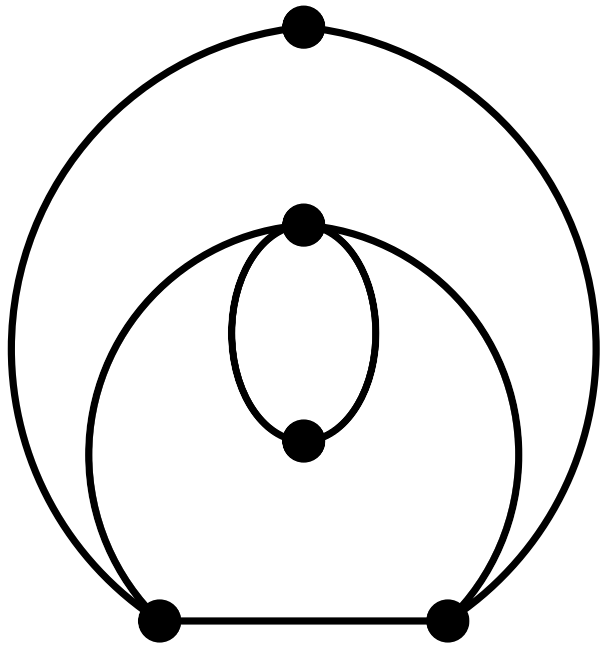
- Graphs G and H are *isomorphic* if there are bijections $\nu : V(G) \to V(H)$ and $\epsilon : E(G) \to E(H)$ that are compatible:
  - Undirected: Every edge $e = \{x, y\}$ in G has $\epsilon(e) = \{\nu(x), \nu(y)\}$ in H
  - Directed: Every edge $e = (x, y)$ in G has $\epsilon(e) = (\nu(x), \nu(y))$ in H
- The graphs are equivalent up to renaming the vertices and edges. One solution (there are others):

Vertices: $\nu(1) = 10$ $\nu(2) = 20$ $\nu(3) = 30$ $\nu(4) = 40$ $\nu(5) = 50$

Edges: $\epsilon(a) = h$ $\epsilon(b) = i$ $\epsilon(c) = j$ $\epsilon(d) = k$ $\epsilon(e) = l$
$\epsilon(f) = m$ $\epsilon(g) = n$

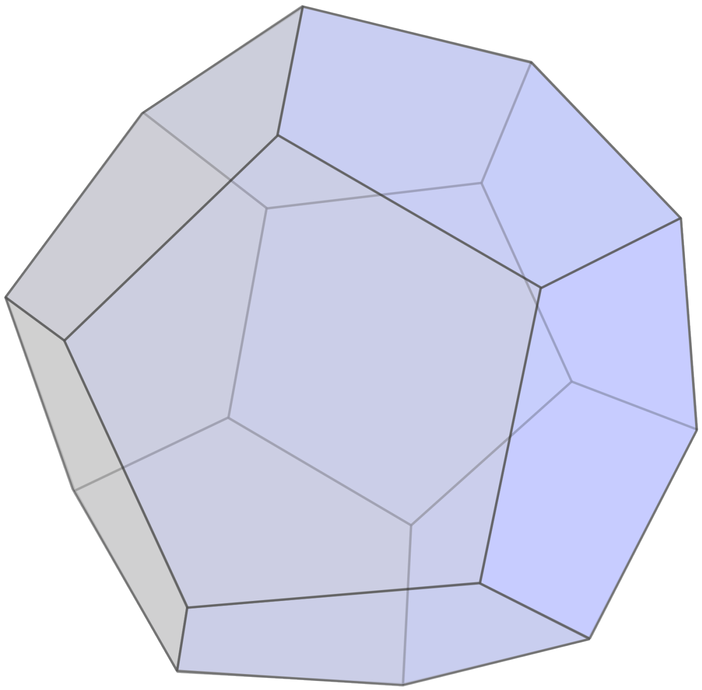
Compatibility: $a = \{1, 2\}$ and $\epsilon(a) = h = \{10, 20\} = \{\nu(1), \nu(2)\}$
... (Need to check all edges) ...

Unlabeled graphs



- In an *unlabeled graph*, omit the labels on the vertices and edges.
- If labeled graphs are isomorphic, then removing the labels gives equivalent unlabeled graphs.
- This simplifies some problems by reducing the number of graphs (e.g., 1044 unlabeled simple graphs on 7 vertices vs. 2^{21} labeled).

Application: Polyhedra



http://commons.wikimedia.org/wiki/File:Dodecahedron.svg

- A dodecahedron is a 3D shape with 20 vertices, 30 edges, and 12 pentagonal faces.
- Unlabeled graphs are used in studying other polyhedra, polygons and tilings in 2D, and other geometric configurations. We can treat them as unlabeled, or pick one labeling if needed.

Basic combinatorial counting methods

See appendix. Covered in more detail in Math 184.

Multiplication rule

Example

- How many outcomes (x, y, z) are possible, where

$$\begin{aligned} x &= \text{roll of a 6-sided die;} \\ y &= \text{value of a coin flip;} \\ z &= \text{card drawn from a 52 card deck?} \end{aligned}$$

- (6 choices of x) $\times$ (2 choices of y) $\times$ (52 choices of z) $= \boxed{\mathbf{624}}$

Multiplication rule

The number of sequences $(x_1, x_2, \ldots, x_k)$ where there are

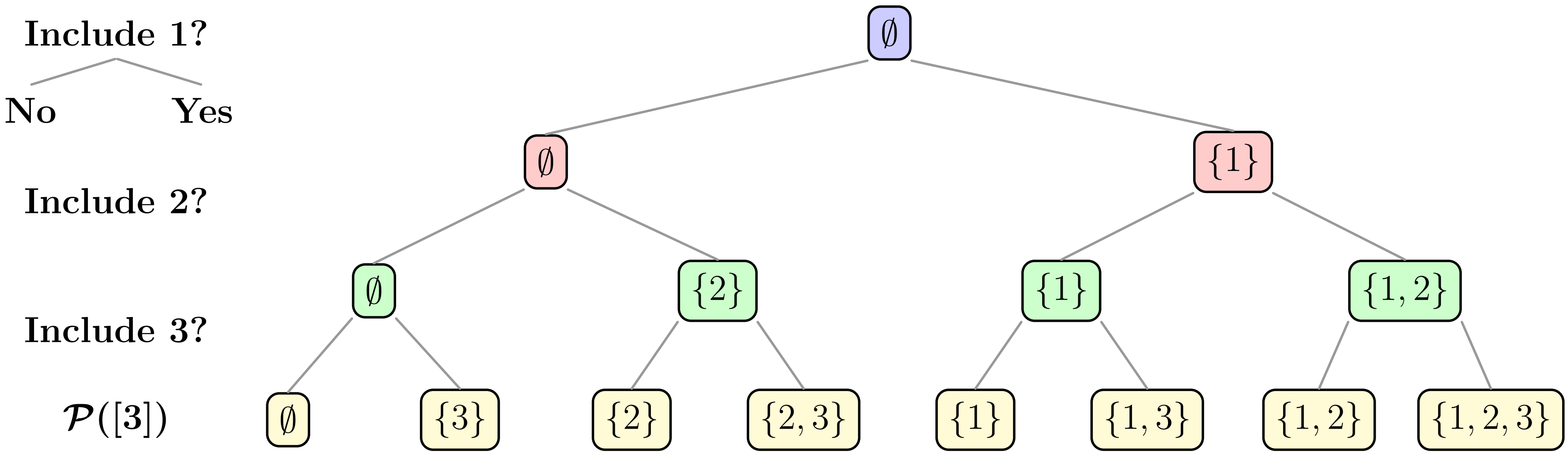
n_1 choices of x_1, $\quad n_2$ choices of x_2, $\quad \ldots, \quad n_k$ choices of x_k

is $n_1 \cdot n_2 \cdots n_k$.

This assumes the number of choices of x_i is a constant n_i that doesn't depend on the other choices.

Number of subsets of an n-element set

- How many subsets does an n element set have?
- We'll use $\{1, 2, \ldots, n\}$. Make a sequence of decisions:
  - Include 1 or not? 2? 3? $\cdots$ n?
  - Total: $(2 \text{ choices})(2 \text{ choices}) \cdots (2 \text{ choices}) = \boxed{2^n}$
- It's also 2^n for any other n element set.



Set partitions

- How many pairs (m, d) are there where
$$m = \text{month } 1, \ldots, 12;$$
$$d = \text{day of the month?}$$
- Assume it's not a leap year.
- The # days/month varies, so can't use multiplication rule $12 \times \_\_$.
- Split dates into $A_m = \{\, (m, d) \; : \; d$ is a valid day in month $m\,\}$:
$$A = A_1 \cup \cdots \cup A_{12} = \text{whole year}$$
$$|A| = |A_1| + \cdots + |A_{12}|$$
$$= 31 + 28 + \cdots + 31 = 365$$

Set partition

Let A be a set. A *partition* of A into blocks $A_1, \ldots, A_n$ means:

- $A_1, \ldots, A_n$ are nonempty sets.
- $A = A_1 \cup \cdots \cup A_n$.
- The blocks are *pairwise disjoint*: $A_i \cap A_j = \emptyset$ when $i \neq j$.

Addition rule

Addition rule

For pairwise disjoint sets $A_1, \ldots, A_n$:

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i|$$

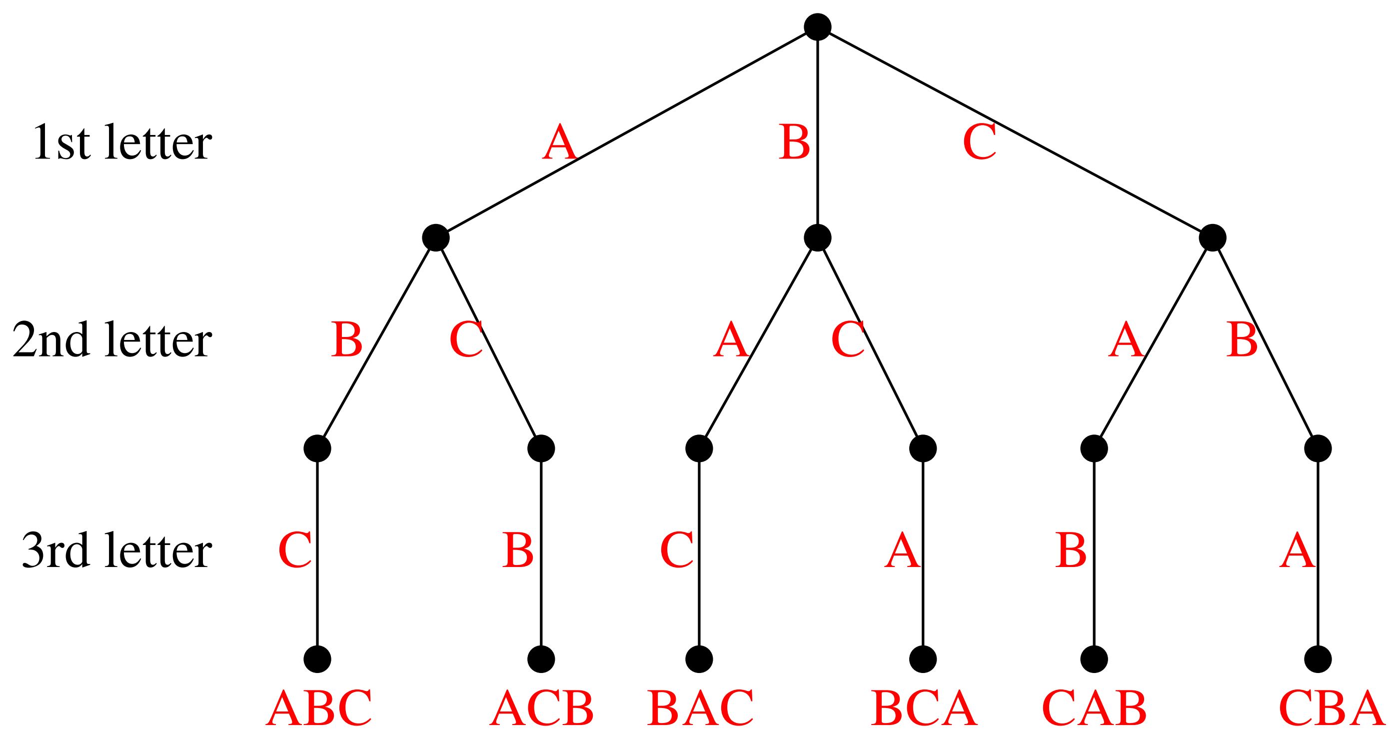
This only applies to pairwise disjoint sets.
If any sets overlap, the right side will be bigger than the left side.

Permutations of distinct objects

Here are all the permutations of A, B, C:

$$ABC \quad ACB \quad BAC \quad BCA \quad CAB \quad CBA$$

- There are 3 items: A, B, C.
- There are 3 choices for which item to put first.
- There are 2 choices remaining to put second.
- There is 1 choice remaining to put third.
- Thus, the total number of permutations is $3 \cdot 2 \cdot 1 = 6$.



Permutations of distinct objects

- In the example on the previous slide, the specific choices available at each step depend on the previous steps, but the number of choices does not, so the multiplication rule applies.
- The number of permutations of n distinct items is "n-factorial": $n! = n(n-1)(n-2)\cdots 1$ for integers $n = 1, 2, \ldots$

Convention: $0! = 1$

- For integer $n > 1$, $$\begin{aligned} n! &= n \cdot (n-1) \cdot (n-2) \cdots 1 \\ &= n \cdot (n-1)! \end{aligned}$$ so $(n-1)! = n!/n$.
- E.g., $2! = 3!/3 = 6/3 = 2$.
- Extend it to $0! = 1!/1 = 1/1 = 1$.
- Doesn't extend to negative integers: $(-1)! = \frac{0!}{0} = \frac{1}{0} =$ undefined.

Partial permutations of distinct objects

- How many ways can you deal out 3 cards from a 52 card deck, where the order in which the cards are dealt matters?
  E.g., dealing the cards in order $(A\clubsuit, 9\heartsuit, 2\diamondsuit)$ is counted differently than the order $(2\diamondsuit, A\clubsuit, 9\heartsuit)$.
- $52 \cdot 51 \cdot 50 = 132600$. This is also $52!/49!$.
- This is called an *ordered* 3-card hand, because we keep track of the order in which the cards are dealt.

- How many ordered k-card hands can be dealt from an n-card deck?

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!} = {}_nP_k$$

  Above example is ${}_{52}P_3 = 52 \cdot 51 \cdot 50 = 132600$.
- This is also called permutations of length k taken from n objects.

Combinations

- In an *unordered* hand, the order in which the cards are dealt does not matter; only the set of cards matters. E.g., dealing in order $(A\clubsuit, 9\heartsuit, 2\diamondsuit)$ or $(2\diamondsuit, A\clubsuit, 9\heartsuit)$ both give the same hand. This is usually represented by a set: $\{A\clubsuit, 9\heartsuit, 2\diamondsuit\}$.
- How many 3 card hands can be dealt from a 52-card deck if the order in which the cards are dealt does not matter?
- The 3-card hand $\{A\clubsuit, 9\heartsuit, 2\diamondsuit\}$ can be dealt in $3! = 6$ different orders:

$$\begin{array}{ccc} (A\clubsuit, 9\heartsuit, 2\diamondsuit) & (9\heartsuit, A\clubsuit, 2\diamondsuit) & (2\diamondsuit, 9\heartsuit, A\clubsuit) \\ (A\clubsuit, 2\diamondsuit, 9\heartsuit) & (9\heartsuit, 2\diamondsuit, A\clubsuit) & (2\diamondsuit, A\clubsuit, 9\heartsuit) \end{array}$$

- Every unordered 3-card hand arises from 6 different orders.
So $52 \cdot 51 \cdot 50$ counts each unordered hand $3!$ times; thus there are

$$\frac{52 \cdot 51 \cdot 50}{3 \cdot 2 \cdot 1} = \frac{52!/49!}{3!} = \frac{{}_{52}P_3}{3!}$$

unordered hands.

Combinations

- The # of unordered k-card hands taken from an n-card deck is
$$\frac{n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)}{k\cdot(k-1)\cdots 2\cdot 1}=\frac{(n)_k}{k!}=\frac{n!}{k!\,(n-k)!}$$
- This is denoted $\binom{n}{k}=\frac{n!}{k!\,(n-k)!}$ (or ${}_nC_k$, mostly on calculators).
- $\binom{n}{k}$ is the “binomial coefficient” and is pronounced “n choose k.”
- The number of unordered 3-card hands is
$$\binom{52}{3}={}_{52}C_3=\text{“52 choose 3”}=\frac{52\cdot 51\cdot 50}{3\cdot 2\cdot 1}=\frac{52!}{3!\,49!}=22100$$
- **General problem:** Let S be a set with n elements. The number of k-element subsets of S is $\binom{n}{k}$.
- **Special cases:** $\binom{n}{0}=\binom{n}{n}=1$ $\qquad$ $\binom{n}{k}=\binom{n}{n-k}$ $\qquad$ $\binom{n}{1}=\binom{n}{n-1}=n$

How many simple graphs are there on n vertices?

How many simple undirected graphs on vertices $1, \ldots, n$?

- There are $\binom{n}{2}$ unordered pairs $\{u, v\}$ with $u \neq v$.
- The edges are a subset of those pairs, so $\boxed{\mathbf{2}^{\binom{n}{2}}}$.
- For $n = 5$: $2^{5 \cdot 4/2} = 2^{10} = 1024$

How many simple undirected graphs on $1, \ldots, 5$ have 3 edges?

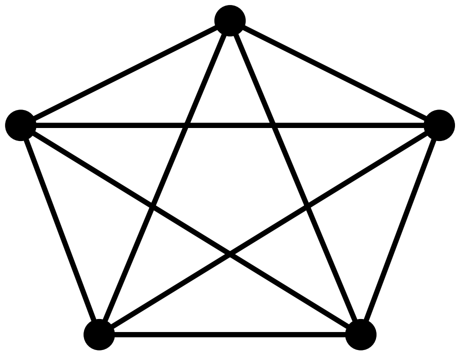
- There are $\binom{5}{2} = 10$ possible edges.
- Select 3 of them in one of $\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = \boxed{\mathbf{120}}$ ways.

How many simple directed graphs on vertices $1, \ldots, n$?

- There are $n(n-1)$ ordered pairs (u, v) with $u \neq v$.
- The edges are a subset of those pairs, so $\boxed{\mathbf{2}^{\boldsymbol{n(n-1)}}}$.
- For $n = 5$: $2^{5 \cdot 4} = 2^{20} = 1048576$

Some classes of graphs

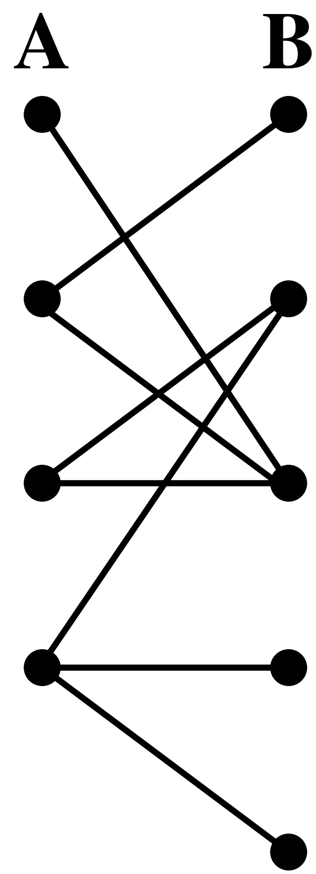
Complete graph K_n



- The *complete graph* on n vertices, denoted K_n, is a graph with n vertices and an edge for all pairs of distinct vertices.

- How many edges are in K_n? $\boxed{\binom{n}{2}}$

Bipartite graph

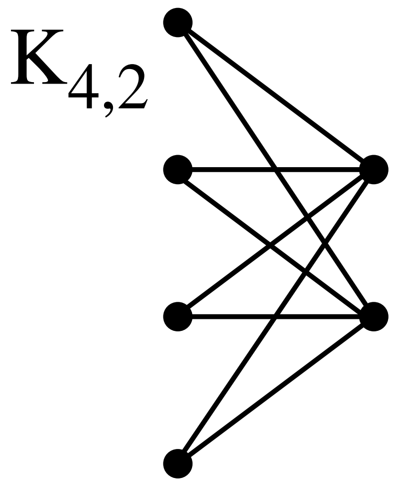


A *bipartite graph* is a graph in which:

- The set of vertices can be split as $V = A \cup B$, where $A \cap B = \emptyset$.
- Every edge has the form $\{a, b\}$ where $a \in A$ and $b \in B$.

Note that there may be vertices $a \in A$, $b \in B$ that do not have an edge.

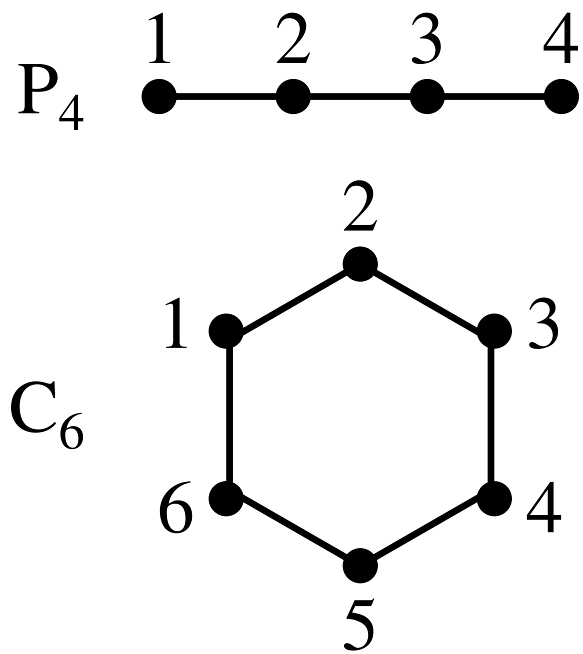
Complete bipartite graph $K_{m,n}$



The *complete bipartite graph* $K_{m,n}$ has

- Vertices $V = A \cup B$ where $|A| = m$ and $|B| = n$, and $A \cap B = \emptyset$.
- Edges $E = \left\{ \{a, b\} \;:\; a \in A \text{ and } b \in B \right\}$
  All possible edges with one vertex in A and the other in B.
- In total, $m + n$ vertices and mn edges.

Path graph and cycle graph



- P_k (k-path, for $k \geqslant 1$): vertices $1, \ldots, k$ and edges

$$\big\{\{1,2\},\{2,3\},\ldots,\{k-1,k\}\big\}$$

- C_k (k-cycle, for $k \geqslant 3$): vertices $1, \ldots, k$ and edges

$$\big\{\{1,2\},\{2,3\},\ldots,\{k-1,k\},\{k,1\}\big\}$$

- These are specific examples of paths and cycles.
Paths and cycles will be discussed in more generality soon.