Chapter 9
Introduction to Extremal Graph Theory

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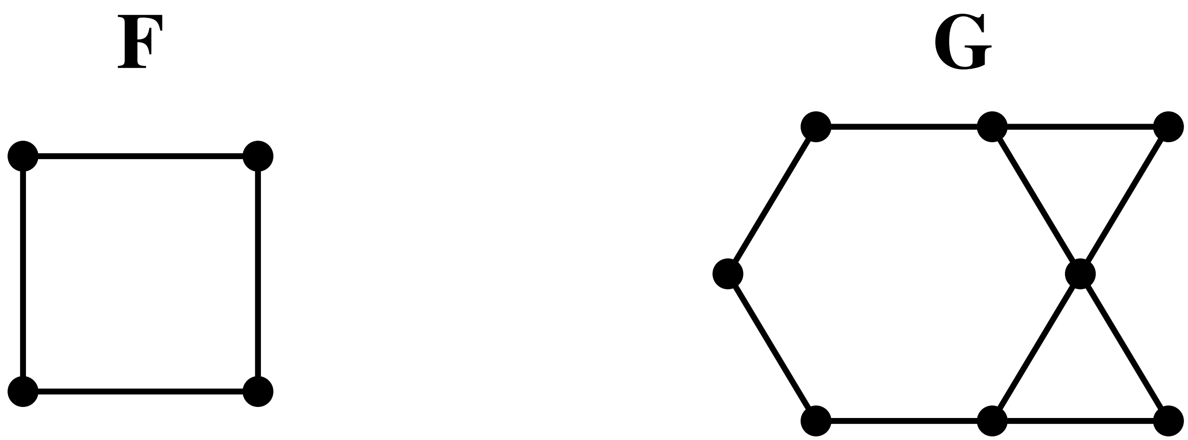
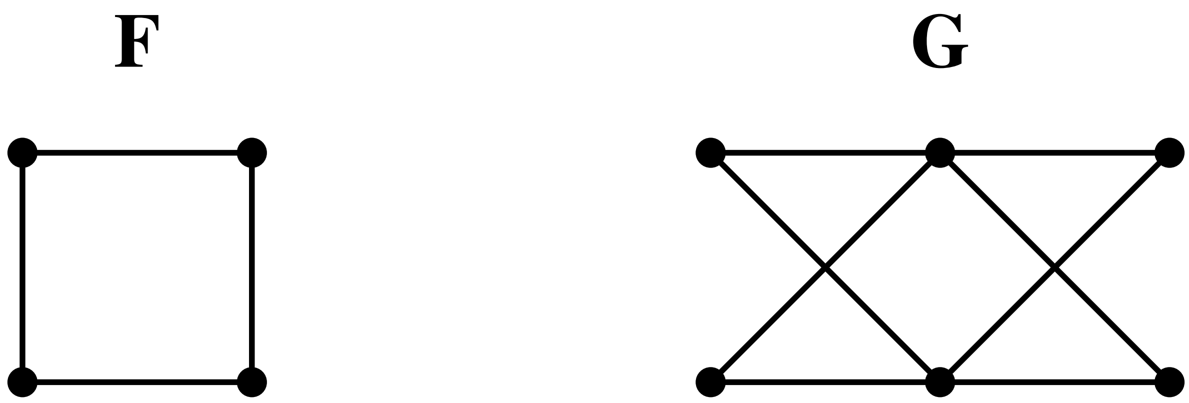
Math 154
Winter 2020

Avoiding a subgraph



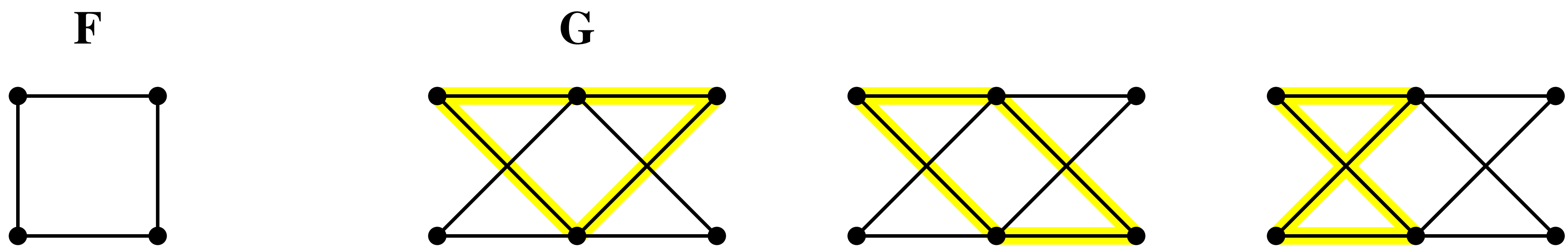
- Let F and G be graphs.
- G is called *F-free* if there's no subgraph isomorphic to F.
- An example is above.

Avoiding a subgraph



- Is the graph on the right F-free?

Avoiding a subgraph



- No. There are subgraphs isomorphic to F, even though they're drawn differently than F.

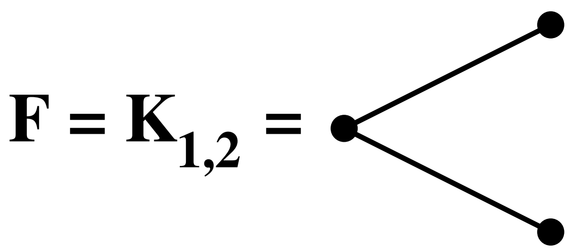
Extremal Number

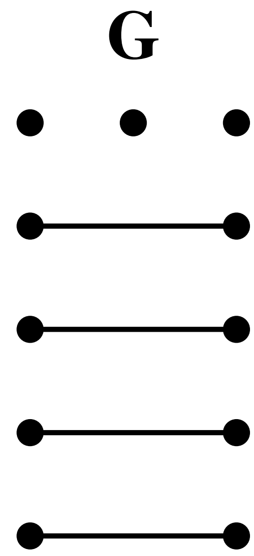
Question

Given a graph F (to avoid), and
a positive integer n,
what's the largest # of edges an F-free graph on n vertices can have?

- This number is denoted $\text{ex}(n, F)$.
- This number is called the *extremal number* or *Turán number* of F.
- An F-free graph with n vertices and $\text{ex}(n, F)$ edges is called an *extremal graph*.

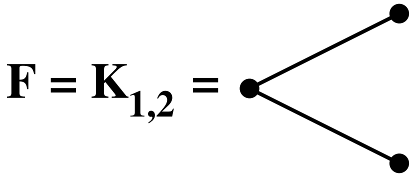
Extremal Number for $K_{1,2}$

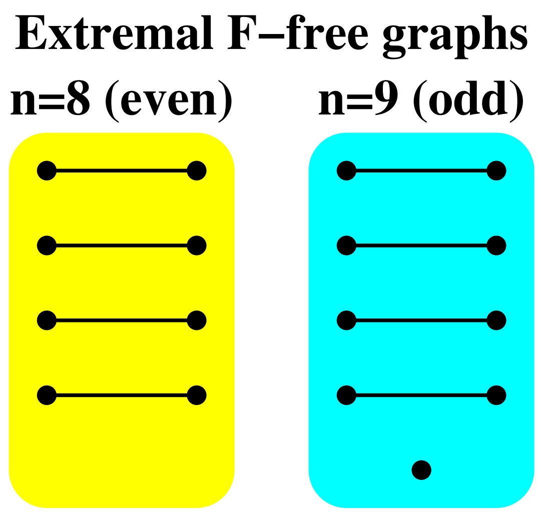




- Let $F = P_2 = K_{1,2}$.   (A two edge path and $K_{1,2}$ are the same.)
- For this F, being F-free means no vertex can be in $\geqslant 2$ edges.
- So, an F-free graph G must consist of vertex-disjoint edges (a matching) and/or isolated vertices.

Extremal Number for $K_{1,2}$





- For each positive integer n, what is the extremal number and the extremal graph(s) for $F = P_2 = K_{1,2}$?
- The extremal graph is a matching with $\lfloor n/2 \rfloor$ edges, plus an isolated vertex if n is odd. So $\text{ex}(n, K_{1,2}) = \lfloor n/2 \rfloor$.
- The book also studies $\text{ex}(n,\ K_{r,s})$ and $\text{ex}(n,\ P_k)$, but to-date, these only have partial solutions.

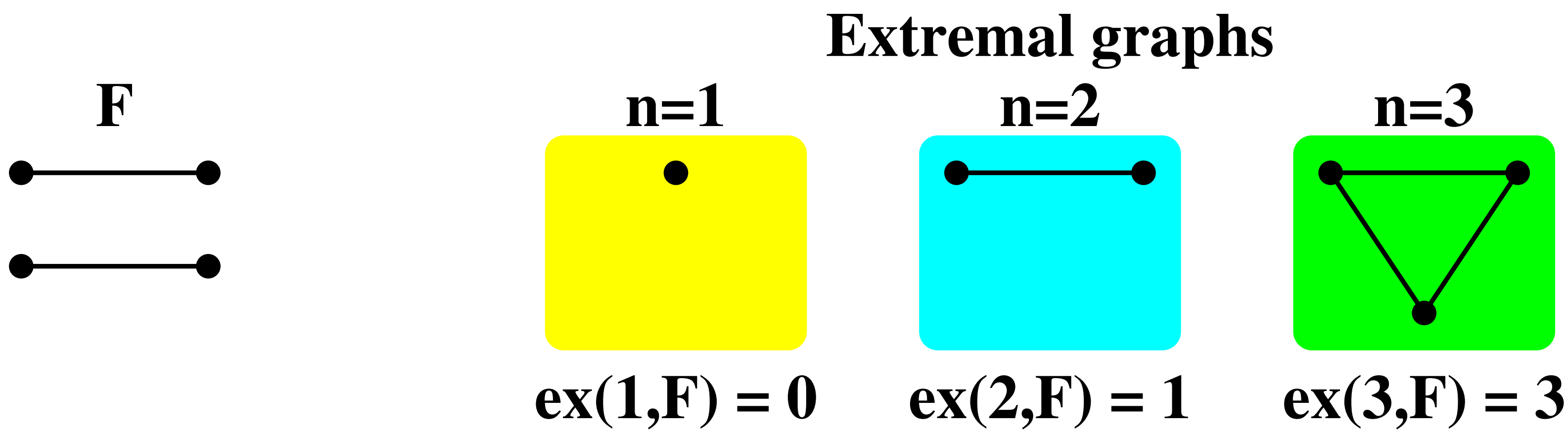
Avoiding 2 disjoint edges

Avoiding 2 disjoint edges

$\mathbf{F} =$ 

- Now we consider avoiding a matching of size two (two disjoint edges).

Avoiding 2 disjoint edges: $n = 1, 2, 3$



- Let F be a matching of size two (two disjoint edges).
- For $n = 1, 2, 3$, we can put in all possible edges, giving extremal graph K_n and ex$(n, F) = \binom{n}{2}$.

ex(n, F) for small n

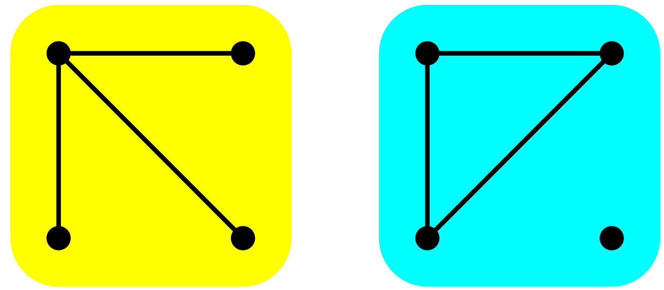
For *any graph* F (not just the example above), if $n < |V(F)|$ then the extremal graph is K_n and ex$(n, F) = \binom{n}{2}$.

- This is because any graph with fewer than $|V(F)|$ vertices can't have F as a subgraph.

Avoiding 2 disjoint edges: $n = 4$

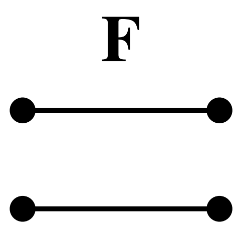
F

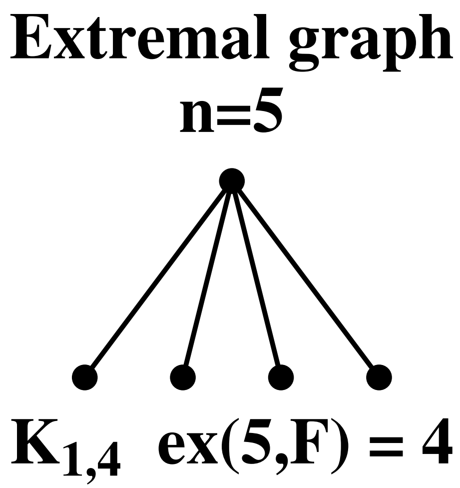
Extremal graphs for n=4



- For $n = 4$, there are two F-free graphs with 3 edges.
  Either one implies $\mathrm{ex}(4, F) \geqslant 3$.
- **Easy to check:** all graphs with 4 vertices and $\geqslant 4$ edges have F as a subgraph.
- So these are both extremal graphs, and $\mathrm{ex}(4, F) = 3$.
- These graphs aren't isomorphic, so there may be more than one extremal graph. It does not have to be unique!

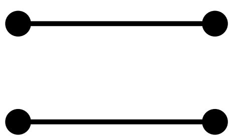
Avoiding 2 disjoint edges: $n = 5$



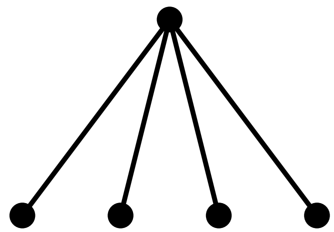


Avoiding 2 disjoint edges: $n \geqslant 4$

F



Extremal graph
n=5



$\mathbf{K_{1,4}}$ **ex(5,F) = 4**

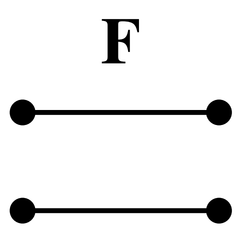
Theorem

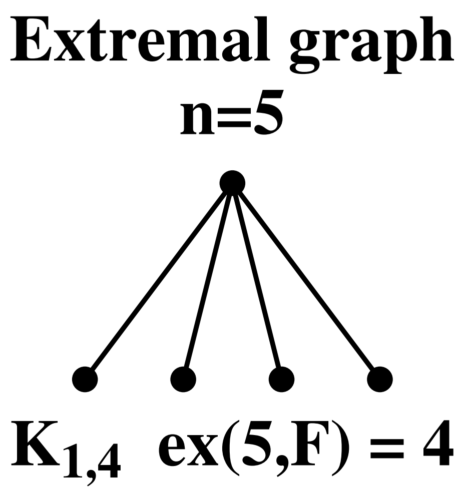
Let F be two disjoint edges as shown above.

- If $n \geqslant 4$, then ex$(n, F) = n - 1$.
- If $n \geqslant 5$, the unique extremal F-free graph is $K_{1,n-1}$.

Avoiding 2 disjoint edges: $n \geqslant 4$

Proving: If $n \geqslant 4$, then ex$(n, F) = n - 1$



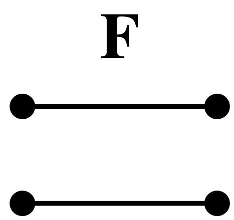


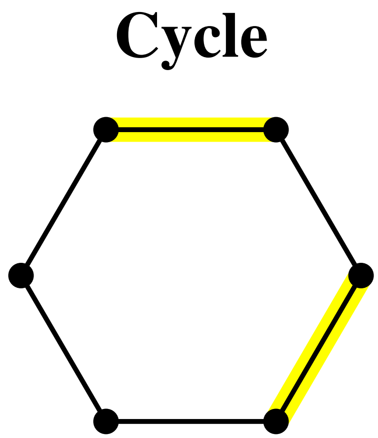
Proof:

- $K_{1,n-1}$ is F-free and has $n - 1$ edges, so ex$(n, F) \geqslant n - 1$.

Avoiding 2 disjoint edges: $n \geqslant 4$

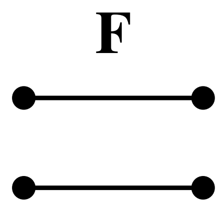
Proving: If $n \geqslant 4$, then $\text{ex}(n, F) = n - 1$



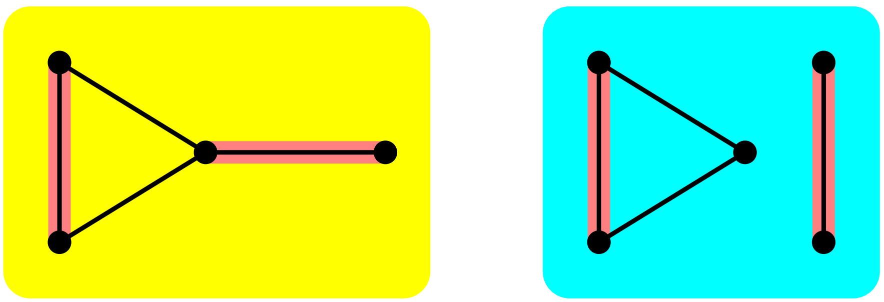


Proof, continued:

- Assume by way of contradiction that there is an F-free graph G on n vertices with $\geqslant n$ edges.
- Then G must have a cycle, C.
- If C has $\geqslant 4$ edges, then it contains two vertex-disjoint edges, so it's not F-free. So C must be a 3-cycle.



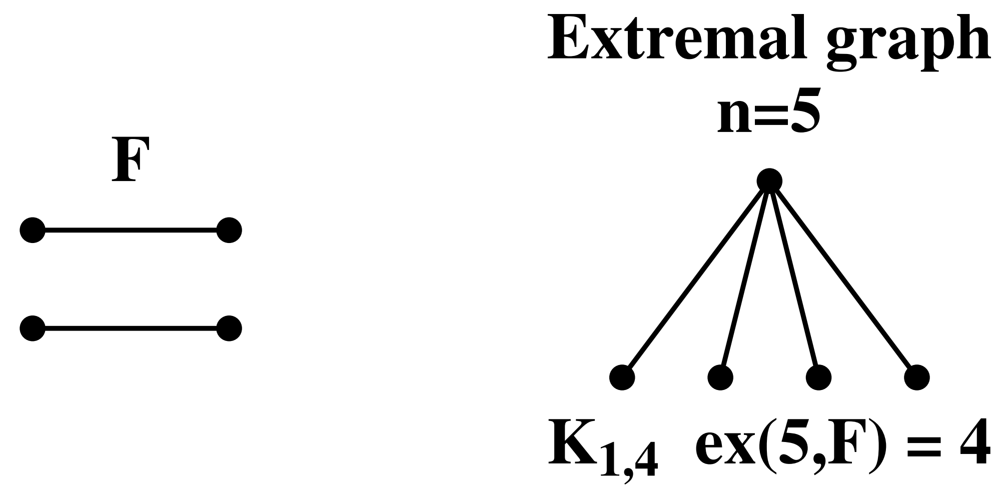
3–cycle + an edge



Proof, continued:

- We assumed that there is an F-free graph on n vertices with $\geqslant n$ edges, and showed there must be a 3-cycle C.
- Since C has 3 edges while G has $\geqslant 4$ edges, G has at least one more edge, e, not in C.
- Edge e is vertex disjoint with at least one edge of C, so G contains F, a contradiction.
- Thus, $\mathrm{ex}(n, F) \leqslant n - 1$. We already showed $\geqslant$, so $\mathrm{ex}(n, F) = n - 1$.

Avoiding 2 disjoint edges: $n \geqslant 4$



Theorem

Let F be two disjoint edges as shown above.

- If $n \geqslant 4$, then $\text{ex}(n, F) = n - 1$. ✓
- If $n \geqslant 5$, the unique extremal F-free graph is $K_{1,n-1}$.

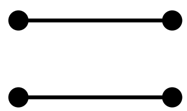
Avoiding 2 disjoint edges: $n \geqslant 4$

Proving that if $n \geqslant 5$, the unique extremal F-free graph is $K_{1,n-1}$.

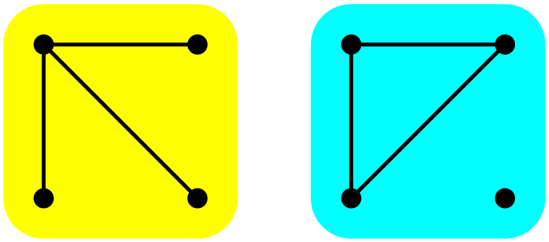
- All edges of G are in one component:
  - If G has edges in two or more components, it's not F-free.
  - However, it can have multiple components, where all edges are in one component, and the other components are isolated vertices.
- If G has exactly one vertex of degree $\geqslant 2$, then G is $K_{1,n-1-m}$ plus m isolated vertices.
  - For this case, $G = K_{1,n-1}$ has the most edges.
- If G has two or more vertices of degree $\geqslant 2$:
  - G can't have a path of length $\geqslant 3$ or a cycle of length $\geqslant 4$, since it's F-free.
  - So G must be a triangle plus $n-3$ isolated vertices.

Avoiding 2 disjoint edges: $n \geqslant 4$

Proving that if $n \geqslant 5$, the unique extremal F-free graph is $K_{1,n-1}$.



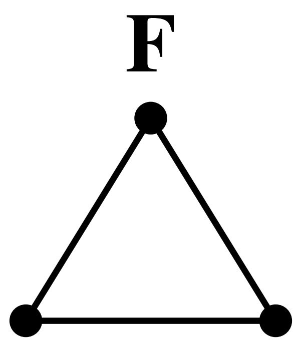
Extremal graphs for n=4



- We've narrowed down the candidates for extremal graphs to
  - (a) $K_{1,n-1}$ — $n-1$ edges
  - (b) A triangle plus $n-3$ isolated vertices. — 3 edges
- For $n = 4$, these are tied at 3 edges, so $\text{ex}(4, F) = 3$ and there are two extremal graphs, as we showed before.
- But for $n \geqslant 5$, the unique solution is $K_{1,n-1}$.

Triangle-free graphs and Mantel's Theorem

Avoiding triangles

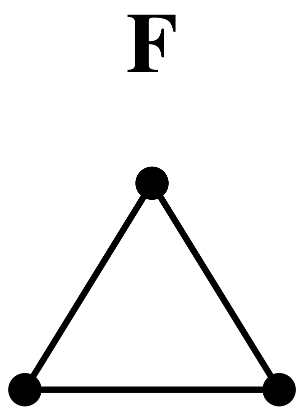


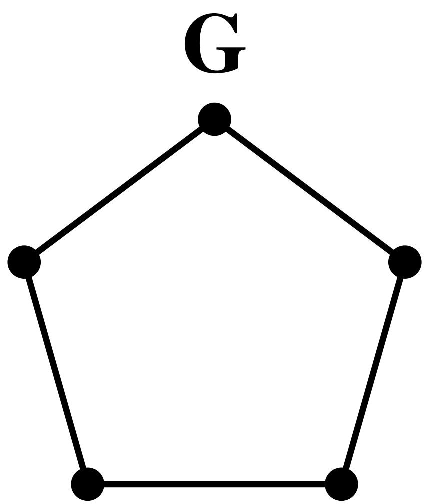
- Next we consider avoiding triangles ($F = K_3$).
- Instead of literally saying "F-free", you can plug in what F is: "triangle-free."

Avoiding triangles

- This graph is triangle-free, so ex$(5, F) \geqslant 4$.
- You can't add more edges without making a triangle, so it's a *maximal* triangle-free graph.
- Can you make a graph on 5 vertices with more edges?

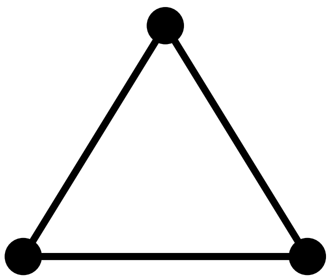
Avoiding triangles



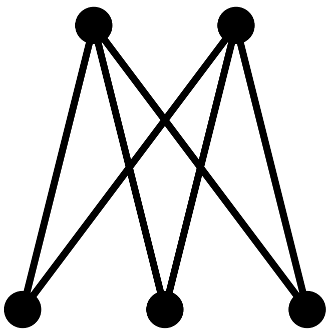


- A pentagon is triangle-free, so ex$(5, F) \geqslant 5$.
- You can't add more edges without making a triangle, so it's also a *maximal* triangle-free graph.
- Can you make a graph on 5 vertices with more edges?

F



G



- $K_{2,3}$ shows $\text{ex}(5, F) \geqslant 6$.
- This turns out to be the extremal graph! So $\text{ex}(5, F) = 6$.

Maxim*al* vs. Maxim*um*

- A *maximal* F-free graph means there is no F-free graph H extending G (by adding edges to G, keeping it at n vertices).
- A *maximum* F-free graph means the size (in edges) is maximum.
- $K_{1,4}$ and a pentagon are not subgraphs of $K_{2,3}$.
  They are *maximal* but not *maximum*.
- The distinction between *maximal* and *maximum* arises in problems concerning *partially ordered sets*.
  - For real numbers, $\leqslant$ is a *total order*: for any real numbers x, y, either $x = y$, $x < y$, or $y < x$.
  - For sets, $\subseteq$ is a *partial order*: sometimes neither set is contained in the other. E.g., $\{1, 3\}$ and $\{2, 3\}$ are not comparable.
  - Subgraph is a partial order.

Mantel's Theorem

Mantel's Theorem (1907)

Let $n \geqslant 2$ and G be an n-vertex triangle-free graph. Then

(a) $|E(G)| \leqslant \lfloor n^2/4 \rfloor$.

(b) $|E(G)| = \lfloor n^2/4 \rfloor$ iff $G = K_{k,n-k}$ for $k = \lfloor n/2 \rfloor$.

(c) $\text{ex}(n, K_3) = \lfloor n^2/4 \rfloor$.

That is, the unique extremal graph is $K_{k,n-k}$, and it has $\lfloor n^2/4 \rfloor$ edges.

Mantel's Theorem

- Consider the complete bipartite graph $K_{\ell,n-\ell}$ with $\ell = 1, \ldots, n-1$.
- It's triangle free, and adding any edge would form a triangle (since it would be between two vertices in the same part, both connected to each vertex in the other part).
- It has $\ell(n-\ell)$ edges. This is maximum at $\ell = \lfloor n/2 \rfloor$ (or $\lceil n/2 \rceil$, but that's equivalent; for example, $K_{2,3}$ and $K_{3,2}$ are isomorphic).
- The max value is $k(n-k) = \lfloor n^2/4 \rfloor$ (where $k = \lfloor n/2 \rfloor$):
  - For even n, $\quad k(n-k) = \frac{n}{2} \cdot \frac{n}{2} = \frac{n^2}{4} \quad$ is an integer.
  - For odd n, $\quad k(n-k) = \frac{n-1}{2} \cdot \frac{n+1}{2} = \frac{n^2-1}{4} = \lfloor n^2/4 \rfloor$.
  - Further odd/even verifications are listed at the end / left to you.
- Thus, $\mathrm{ex}(n, K_3) \geqslant \lfloor n^2/4 \rfloor$.

Mantel's Theorem

- We showed $K_{k,n-k}$ is triangle-free and has the most edges among bipartite graphs.
- Could there be a different triangle-free graph with more edges? We'll prove not.

Mantel's Theorem

Claim

For $n \geqslant 2$, if G is a triangle-free n vertex graph with at least $\lfloor n^2/4 \rfloor$ edges, then $G = K_{k,n-k}$, where $k = \lfloor n/2 \rfloor$.

Proof (base case):

- We will induct on n.
- **Base case:** For $n = 2$, since $n < |V(F)| = 3$, the extremal graph is K_2, which is equivalent to $K_{1,1}$:

$$K_2 = K_{1,1} =$$

Claim: If G is a triangle-free graph with $\geqslant \lfloor n^2/4 \rfloor$ edges, then $G = K_{k,n-k}$ $\quad (k = \lfloor n/2 \rfloor)$.

Proof (induction step):

- For $n \geqslant 3$, assume the claim holds for smaller n.
- Let H be a subgraph of G with all n vertices and $\lfloor n^2/4 \rfloor$ edges.
- We'll prove $H = K_{k,n-k}$.
- Since adding any edge to H would give a triangle, and H is a subgraph of G, we must have $G = H = K_{k,n-k}$.

Mantel's Theorem

Claim: If G is a triangle-free graph with $\geqslant \lfloor n^2/4 \rfloor$ edges, then $G = K_{k,n-k}$ $\quad (k = \lfloor n/2 \rfloor)$.

Proof (induction step), continued:

- Let H be a subgraph of G with all n vertices and $\lfloor n^2/4 \rfloor$ edges. We'll prove $H = K_{k,n-k}$.
- By the Handshaking Lemma, the sum of degrees in H is

$$\sum_{v \in H} d_H(v) = 2\,|E(H)| = 2\left\lfloor n^2/4 \right\rfloor .$$

- Thus, the average degree in H is

$$\frac{\text{sum of degrees}}{\text{\# vertices}} = \frac{2\left\lfloor n^2/4 \right\rfloor}{n}.$$

Mantel's Theorem

Claim: If G is a triangle-free graph with $\geqslant \lfloor n^2/4 \rfloor$ edges, then $G = K_{k,n-k}$ $\quad (k = \lfloor n/2 \rfloor)$.

Proof (induction step), continued:

- Let H be a subgraph of G with all n vertices and $\lfloor n^2/4 \rfloor$ edges.
- We've shown the average degree in H is $\frac{2\lfloor n^2/4 \rfloor}{n}$.
- Let v be a vertex in H of minimum degree, $d_H(v) = \delta_H(v)$.
- The min degree is $\leqslant$ the average degree, and is an integer, so

$$\delta_H(v) \leqslant \underbrace{\left\lfloor \frac{2\lfloor n^2/4 \rfloor}{n} \right\rfloor = \lfloor n/2 \rfloor}_{\text{prove this on your own}} = k\ .$$

Mantel's Theorem

Claim: If G is a triangle-free graph with $\geqslant \lfloor n^2/4 \rfloor$ edges, then $G = K_{k,n-k}$ $\quad (k = \lfloor n/2 \rfloor)$.

Proof (induction step), continued:

- Let H be a subgraph of G with all n vertices and $\lfloor n^2/4 \rfloor$ edges.
- Let v be a vertex in H of minimum degree: $d_H(v) = \delta(H) \leqslant k$.
- Let $H' = H - \{v\}$. This is a subgraph of H on $n-1$ vertices. It's triangle-free and the number of edges is:

$$|E(H')| = |E(H)| - \delta_H(v) \geqslant \left\lfloor \frac{n^2}{4} \right\rfloor - k = \underbrace{\left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor}_{\text{prove this on your own}}$$

- Since the claim holds for $n-1$ vertices, $H' = K_{\ell,n-1-\ell}$ where

$$\ell = \left\lfloor \tfrac{n-1}{2} \right\rfloor = \begin{cases} k & \text{if } n \text{ odd;} \\ k-1 & \text{if } n \text{ even.} \end{cases}$$

and $n - 1 - \ell = k$.

Mantel's Theorem

Claim: If G is a triangle-free graph with $\geqslant \lfloor n^2/4 \rfloor$ edges, then $G = K_{k,n-k}$ $\quad (k = \lfloor n/2 \rfloor)$.

Proof (induction step), continued:

- Let H be a subgraph of G with all n vertices and $\lfloor n^2/4 \rfloor$ edges;
  v be a vertex in H of minimum degree: $d_H(v) = \delta(H) \leqslant k$;
  $H' = H - \{v\} = K_{\ell,n-1-\ell}$, where $\ell = \left\lfloor \frac{n-1}{2} \right\rfloor$.
- We have $|E(H')| = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$, so $d_H(v) = \left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{(n-1)^2}{4} \right\rfloor = k$.
- Add v back in to H' to get H.
  - H' is bipartite with two parts, A' and B', of sizes ℓ and $n-1-\ell$.
  - If v has neighbors in both parts, there would be a triangle.
    So all neighbors of v are in A', or all are in B'.
  - Putting v back in gives $H = K_{k,n-k}$ (have to check n even/odd).
- Adding any more edges to H would form a triangle, but G is triangle-free, so $G = H = K_{k,n-k}$.

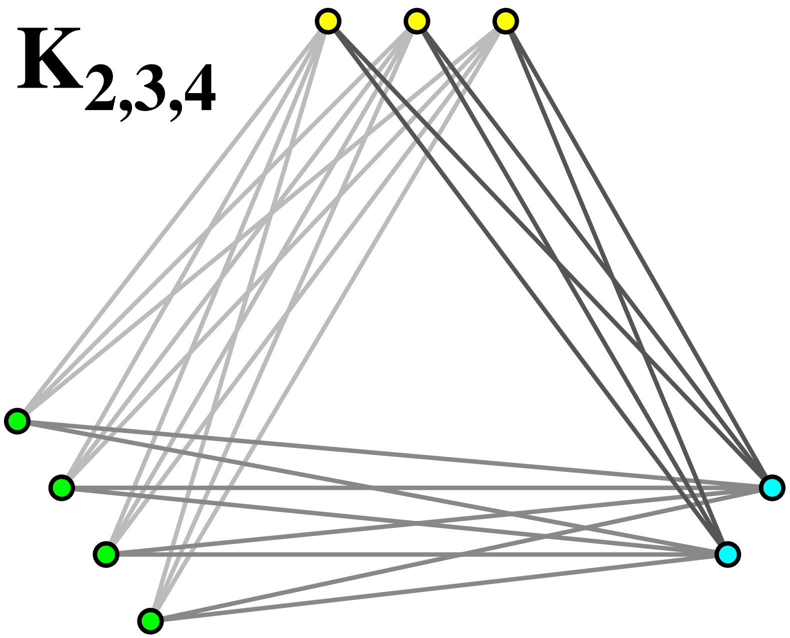
Mantel's Theorem

Odd/even n details — These are all straightforward to verify

Quantity	n even	n odd
$k = \lfloor n/2 \rfloor$	$\frac{n}{2}$	$\frac{n-1}{2}$
$\left\lfloor \frac{2\lfloor n^2/4 \rfloor}{n} \right\rfloor$	$\frac{n}{2} = k$	$\frac{n-1}{2} = k$
$\ell = \left\lfloor \frac{n-1}{2} \right\rfloor$	$\frac{n}{2} - 1 = k - 1$	$\frac{n-1}{2} = k$
$n - 1 - \ell$	$\frac{n}{2} = k$	$\frac{n-1}{2} = k$
$d_H(v) = \left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$	$\frac{n}{2} = k$	$\frac{n-1}{2} = k$

Complete multipartite graph

Complete multipartite graph

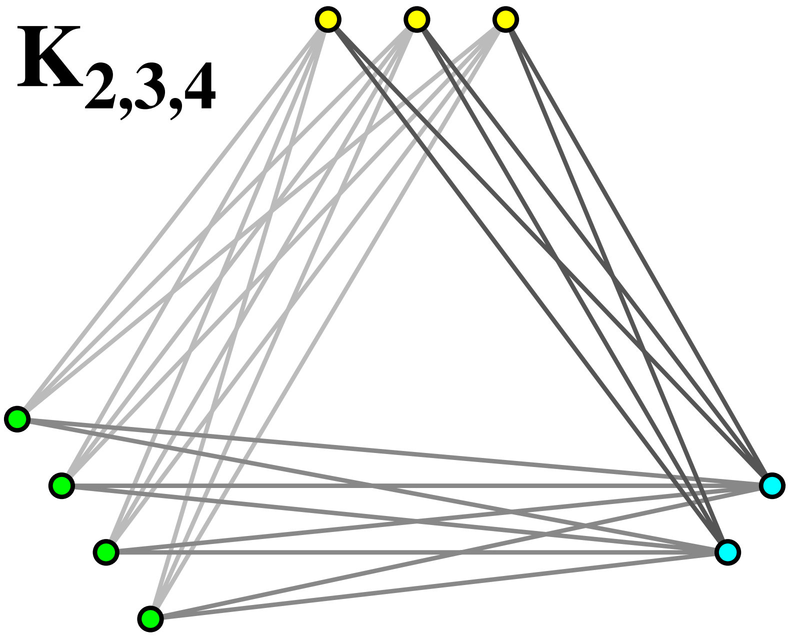


The *complete multipartite graph* $K_{q_1,q_2,\ldots,q_m}$ has:

- Vertices split into disjoint parts $V_1, \ldots, V_m$ with $|V_i| = q_i$
  Total vertices: $n = q_1 + \cdots + q_m$
- Edges between all pairs of vertices in different parts:

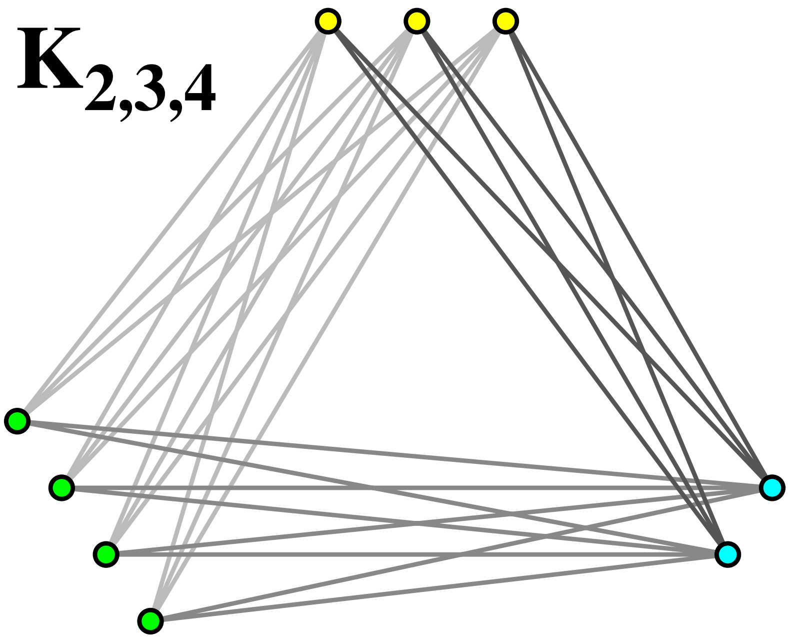
$$E = \left\{ \{x,y\} \;:\; x \in V_i,\ y \in V_j \quad \text{where } i \neq j \text{ are between } 1 \text{ and } m \right\}$$

Complete multipartite graph



- $K_{q_1,q_2,\ldots,q_m}$ is m-colorable, so it cannot contain K_{m+1}.
- This example has 3-parts, so it's 3-colorable, so it can't contain K_4.

Complete multipartite graph

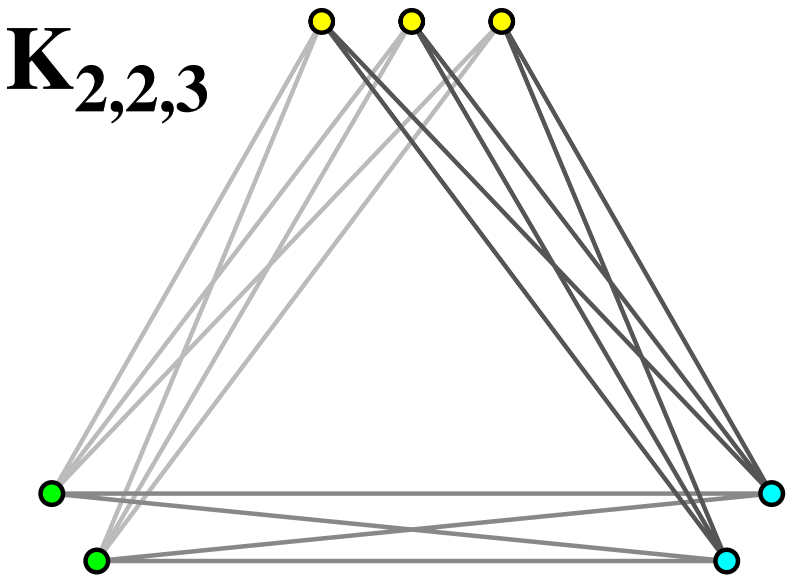


- The number of edges in $K_{q_1,q_2,\ldots,q_m}$ is

$$\sum_{1 \leqslant i < j \leqslant m} q_i q_j$$

- For $K_{2,3,4}$: $2 \cdot 3 + 2 \cdot 4 + 3 \cdot 4 = 6 + 8 + 12 = 26$

Complete multipartite graph



- For n vertices and m parts, the # edges is maximized when all parts are as close as possible; so all parts are $\lfloor n/m \rfloor$ or $\lceil n/m \rceil$.
  - The graph with these parameters is called the *Turán graph*.
  - The graph is denoted by $T_m(n)$.
  - The number of edges is denoted $t_m(n)$. It's roughly $\frac{1}{2}(1-\frac{1}{m})n^2$.
- E.g., for 7 vertices and 3 parts:
  - The Turán graph is $T_3(7) = K_{2,2,3}$.
  - It has $t_3(7) = 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 3 = 16$ edges.

Turán's Theorem: Avoiding cliques of a certain size

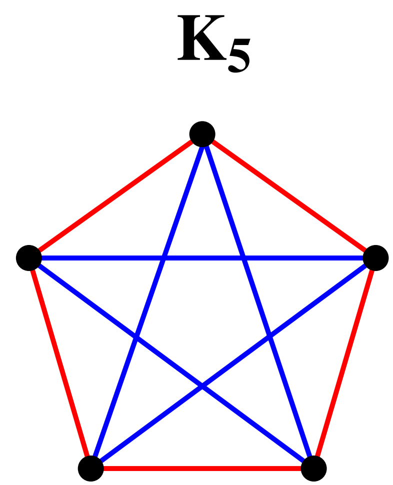
Turán's Theorem (1941)

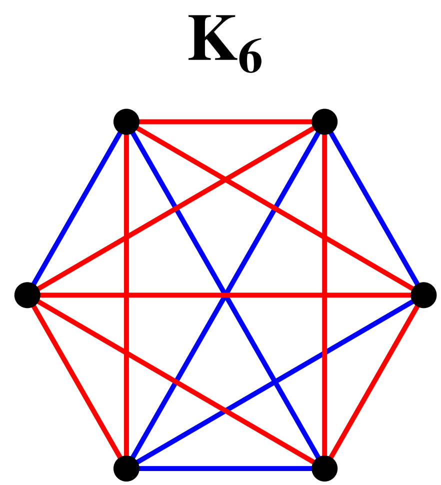
Let $n \geqslant 1$ and G be an n-vertex graph with no K_{m+1}.
Then $|E(G)| \leqslant t_m(n)$, with equality iff $G = T_m(n)$.

- Mantel's Theorem is the $m = 2$ case of this.
- The proof is similar to Mantel's Theorem, but the graph has m parts instead of two, and the formulas are a bit messier.
  See the proof in the book.
- Turán's Theorem is considered the start of the field of extremal graph theory.

Ramsey Numbers

Monochromatic triangles



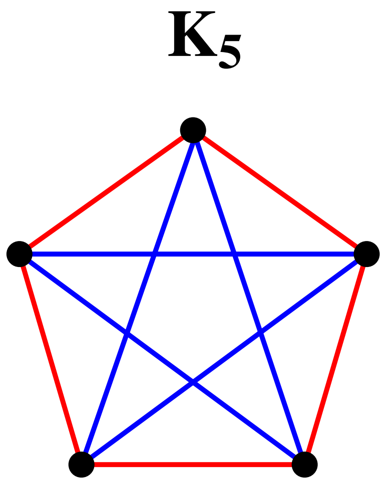


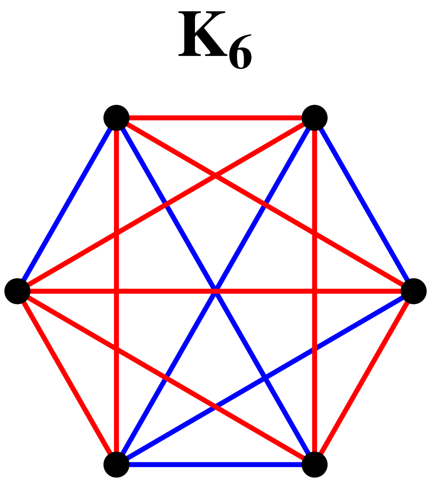
- Assign every edge of K_n a color: red or blue.
  Note: This is not **proper** *edge colorings; this is a different topic. Edges that share a vertex are allowed to be the same color for this application.*

- A *monochromatic triangle* is a 3-cycle where all the edges are the same color (all red or all blue).

- Do you see any monochromatic triangles in either example above?

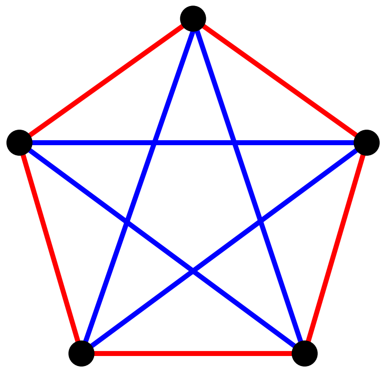
Monochromatic triangles

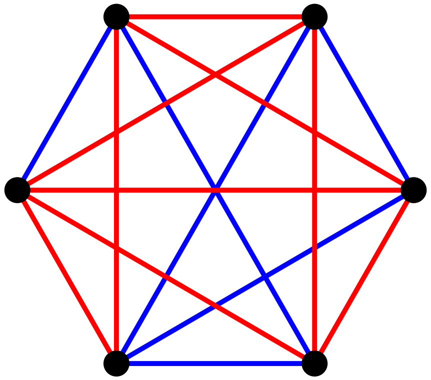




- It turns out that every red/blue coloring of the edges of K_6 has at least one red triangle or blue triangle!
  - This holds for K_n with $n \geqslant 6$, too, since K_n contains K_6.
- But some colorings of K_5 don't have a monochromatic triangle.
  - Thus, K_n for $n \leqslant 5$ does not have to have a monochromatic triangle. E.g., if K_4 must have a monochromatic triangle, then K_5 must too since it contains a K_4.

Proving there are monochromatic triangles for $n \geqslant 6$



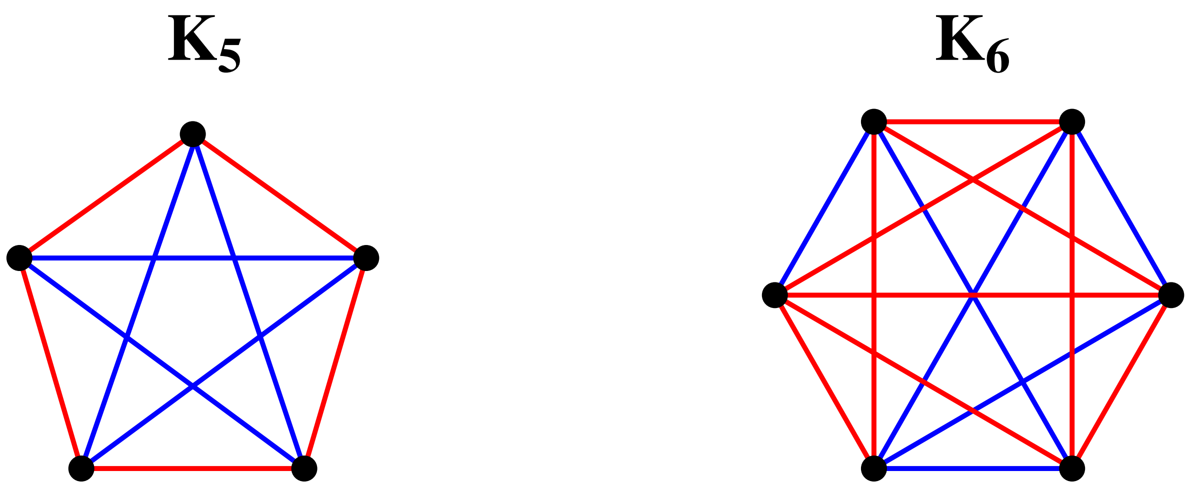


- Color the edges of K_n red/blue.
- Let r_i be the number of red edges on vertex i so $n-1-r_i$ is the number of blue edges.
- Each triangle that isn't monochromatic has two vertices with one red and one blue edge, so

$$\text{\# non-monochromatic triangles} = \frac{1}{2}\sum_{i=1}^{n} r_i(n-1-r_i)$$

(the sum counts each triangle twice, so divide by 2).

Proving there are monochromatic triangles for $n \geqslant 6$



- The number of monochromatic triangles is
$$\binom{n}{3} - \frac{1}{2}\sum_{i=1}^{n} r_i(n-1-r_i)$$
- This is minimized by
  - n odd: $r_i = \frac{n-1}{2}$
  - n even: each $r_i = \frac{n}{2}$ or $\frac{n}{2} - 1$

which leads to:
$$\text{\# monochromatic triangles} \geqslant \binom{n}{3} - \left\lfloor \frac{n}{2} \left\lfloor \frac{(n-1)^2}{4} \right\rfloor \right\rfloor$$

Monochromatic triangles

$$\#\text{ monochromatic triangles} \geqslant \binom{n}{3} - \left\lfloor \frac{n}{2} \left\lfloor \frac{(n-1)^2}{4} \right\rfloor \right\rfloor$$

n	**# monochr. triangles $\geqslant$**
$1, \ldots, 5$	0
6	2
7	4

So for $n = 6$, there are actually at least two monochromatic triangles (and this increases as n increases past 6).

Ramsey Numbers

- Color the edges of K_n with c colors, $\{1, \ldots, c\}$.
  Again, this isn't **proper** *edge colorings; it's any function from edges to* $\{1, \ldots, c\}$*.*
- Let $m_1, \ldots, m_c$ be positive integers.
- It turns out that for sufficiently large n, every such edge coloring must have a monochromatic clique K_{m_i} of some color i.

Ramsey's Theorem (1930) — Version for graphs

There is a number $R = R(m_1, \ldots, m_c)$ (the *Ramsey Number*) such that if $n \geqslant R$, then all edge colorings of K_n with c colors must have a monochromatic clique K_{m_i} of some color i.

- Monochromatic red/blue triangles is $R(3, 3) = 6$:
  for $n \geqslant 6$, every K_n has a red $K_{m_1} = K_3$ and/or a blue $K_{m_2} = K_3$.

Ramsey Numbers

- Trivial cases:
  - $R(a,b) = R(b,a)$
  - $R(1,b) = 1$
  - $R(2,b) = b$
- Very few non-trivial Ramsey numbers have been determined, but people have studied bounds and also asymptotic results.

Ramsey Numbers

Graphs are a special case of Ramsey's Theorem.
Ramsey actually proved a more general result for *hypergraphs*:

- Let $n, c, r \geqslant 1$: $n =$ # vertices
  $c =$ # colors
  $r =$ hyperedge size
- A *hyperedge* is an r-element subset of the vertices, generalizing $r = 2$ for ordinary edges.
- Assign every r-element subset of $\{1, \ldots, n\}$ a color in $\{1, \ldots, c\}$.

Ramsey's Theorem

There is a number $R = R(m_1, \ldots, m_c; r)$ such that if $n \geqslant R$, then in all such colorings, there is a color i and an m_i-element set $S \subseteq \{1, \ldots, n\}$, where all r-element subsets of S have color i.

- The monochromatic red/blue triangles case is $R(3, 3; 2) = 6$.