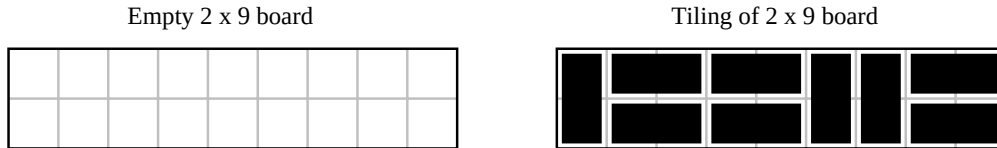


Math 154, Winter 2020, Prof. Tesler
Homework #5, Due Friday February 14, 2020

Chapter 4# 4.1(c,d,e)
Chapter 5# 5.1, 5.3*, 5.9, 5.15(a)
and Problem H-501 below.

*In 5.3, $\text{odd}(G)$ means the number of components of G with an odd number of vertices.

Problem H-501. A *tiling* of an $m \times n$ chessboard is a configuration of dominoes which cover all squares on the board exactly once. Each domino covers two adjacent squares, either horizontally or vertically adjacent, and dominoes cannot stick out of the board. Here is a 2×9 example:



This is actually a perfect matching in a graph: the squares of the board are the vertices of a graph; adjacent squares are the edges; nonoverlapping dominoes specify a matching; and if all squares are covered, it's a perfect matching. An example is shown on the next page.

(a) For which integers $m \geq 1$ and $n \geq 1$ does an $m \times n$ chessboard have a tiling?

(b) We will consider $2 \times n$ boards, such as the 2×9 example shown above.

Let $T(n)$ be the number of tilings of a $2 \times n$ board, for integers $n \geq 0$. The first two values are:

$T(0) = 1$: There are no squares, so the empty 2×0 board is trivially covered.

$T(1) = 1$: The only solution is one vertical domino.

Determine the values $T(2)$, $T(3)$, and $T(4)$.

(c) Find a recursive formula for $T(n)$.

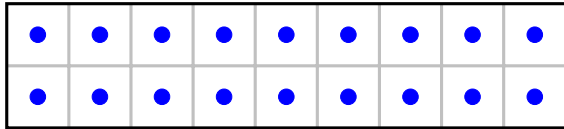
Hint: Consider the ways the first (or last) column of the board can be covered by domino(es), and what that implies about the rest of the board.

Reminder: A recursion is a formula expressing $T(n)$ in terms of previous values; in this case, some or all of $T(0), T(1), \dots, T(n-1)$. But the formula only holds for $n \geq n_0$ (for some specific constant value n_0). A recursion also requires *initial values* $T(0), T(1), \dots, T(n_0-1)$ to get the recursion started. For example, if you get $T(n) = 10T(n-3)$ (that's not the actual answer in this problem), then it would not be valid to compute $T(1)$ in terms of $T(1-3) = T(-2)$; you would need to work out initial values $T(0), T(1), T(2)$ separately to get the recursion started, and then use the formula $T(n) = 10T(n-3)$ for $n \geq 3$.

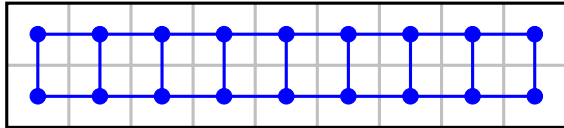
(d) Use the recursion to compute $T(5)$, $T(6)$, and $T(7)$.

Illustration of how a domino tiling corresponds to a perfect matching

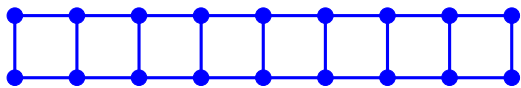
Place blue vertex in each cell of 2 x 9 board



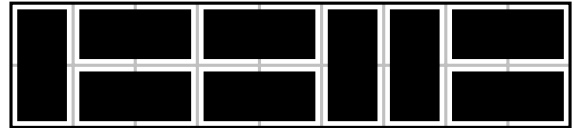
Connect vertices in adjacent cells by blue edges



Remove original board / keep blue graph



Tiling of 2 x 9 board



Perfect matching representing above tiling

