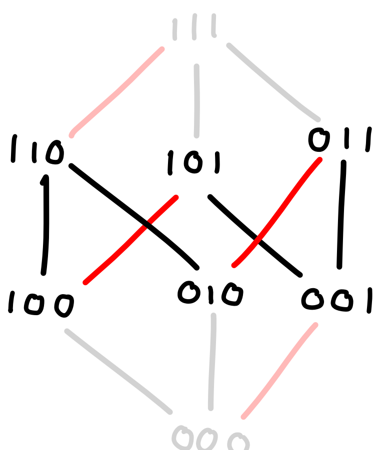
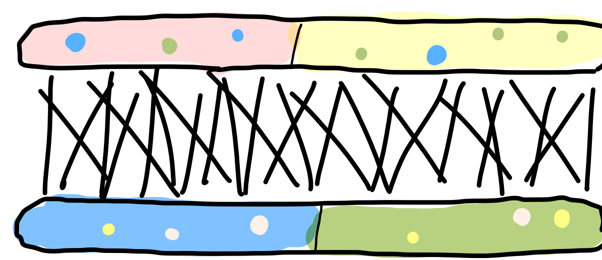
Coloring the Middle Layers of the Hamming Cube

Lina Li, Gwen McKinley, Jinyoung Park

me

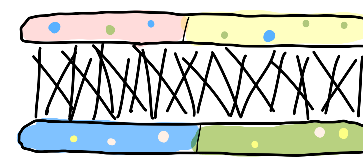




Outline

- Background & Definitions
- Our results

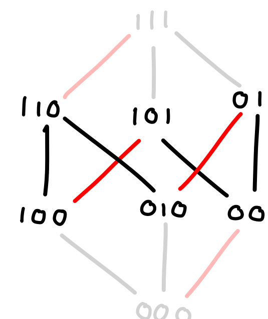
explain this picture:



- Techniques

just a bit, time permitting!

explain this picture:



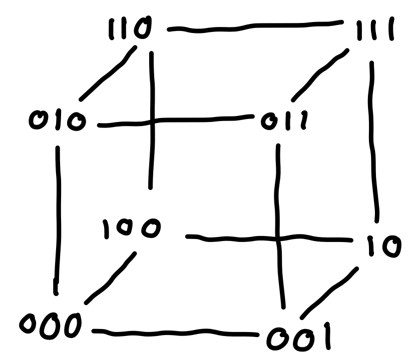
Hamming cube Q_n:

vertices = binary strings of length n

edges = flip 1 bit

applications in CS:
e.g., error-correcting codes

Q_3



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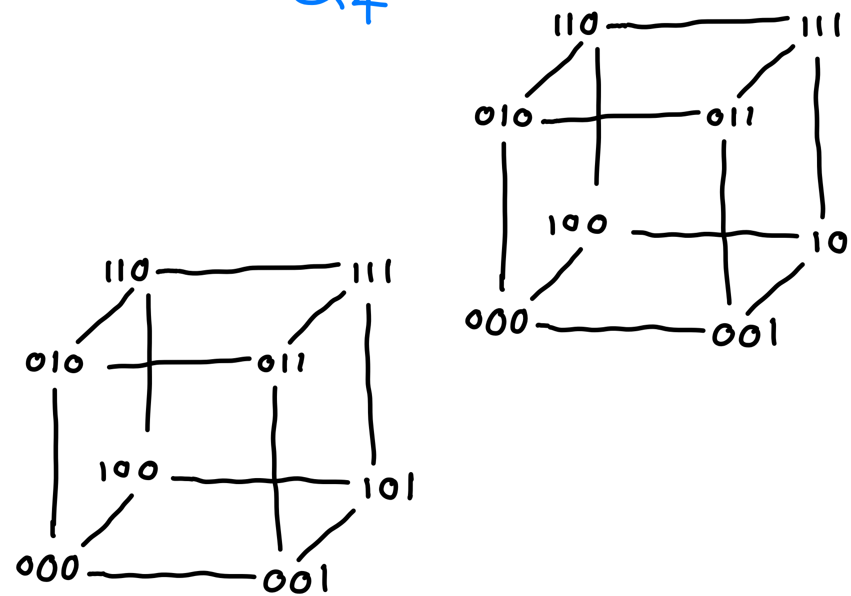
Q_4



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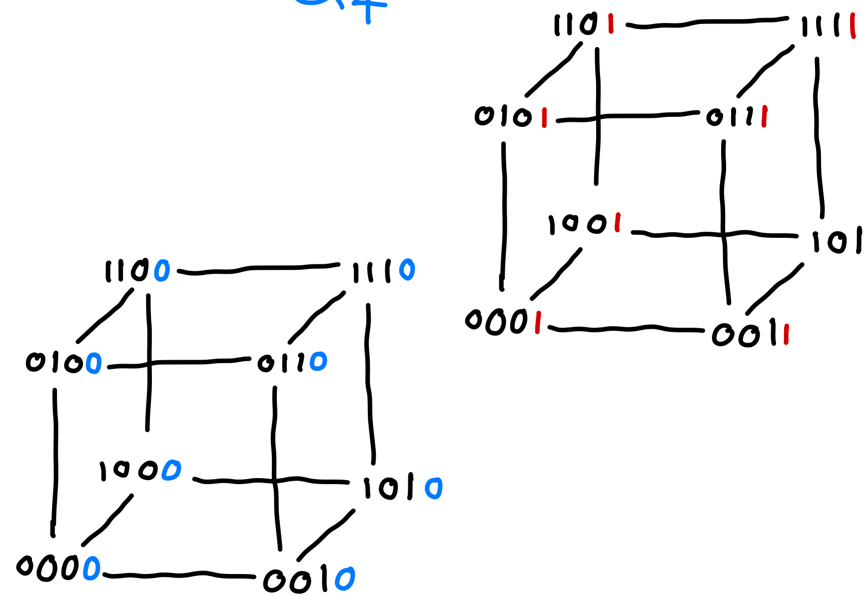

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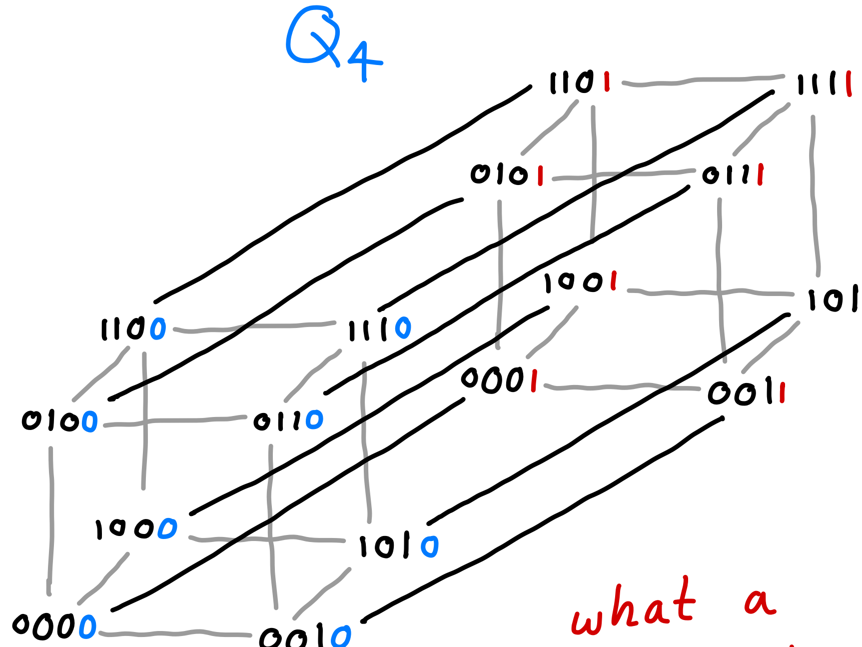
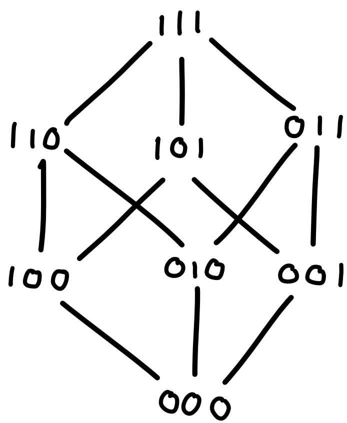


what a mess!

$\underline{Q}$ What is $\underbrace{\chi(Q_n)}_{\text{\# colors needed to color } Q_n}$?

$\underline{A}$ 2

Q_n is <u>bipartite</u>



Q What is $\chi(Q_n)$?

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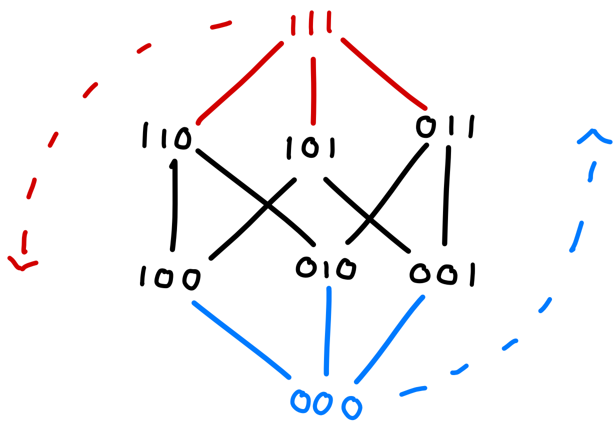

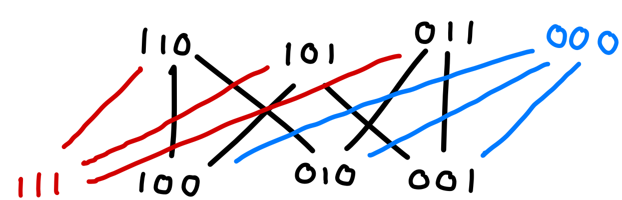

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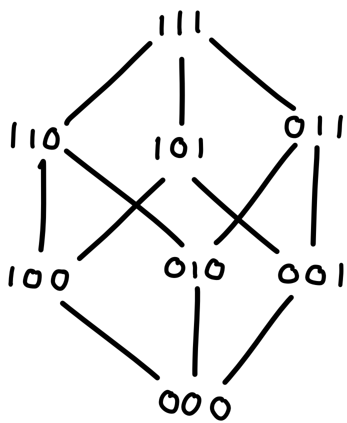
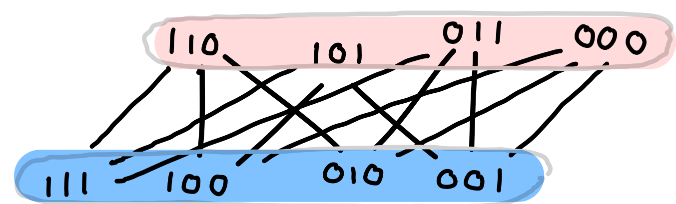
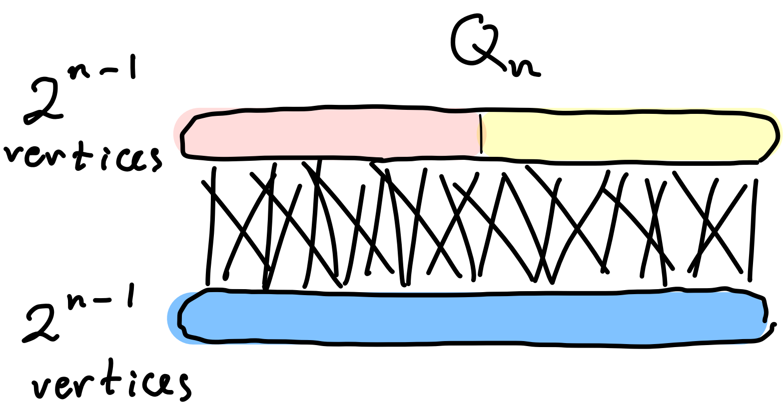




Q_n can be colored with 2 colors.

Question How many ways can Q_n be colored with 3 colors?



Lower bound:

$$\underbrace{2}_{\text{swap top vs bottom}} \cdot \underbrace{3}_{\text{swap ``special color''}} \cdot 2^{\underbrace{(2^{n-1})}_{\text{2 choices for each vtx on top}}}$$

Question How many ways can Q_n be colored with 3 colors?

(**Observation**: at least $6 \cdot 2^{2^{n-1}}$ ways)

Thm (Galvin, 2003) The # of 3-colorings of Q_n is $(1+o(1))e\left(6 \cdot 2^{2^{n-1}}\right)$.

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<u>Up the ante:</u>

- How many <u>q-colorings</u> of Q_n are there? ← $q \geq 3$
- <u>Structure</u> of a typical q-coloring?
- <u>Other</u> bipartite graphs?

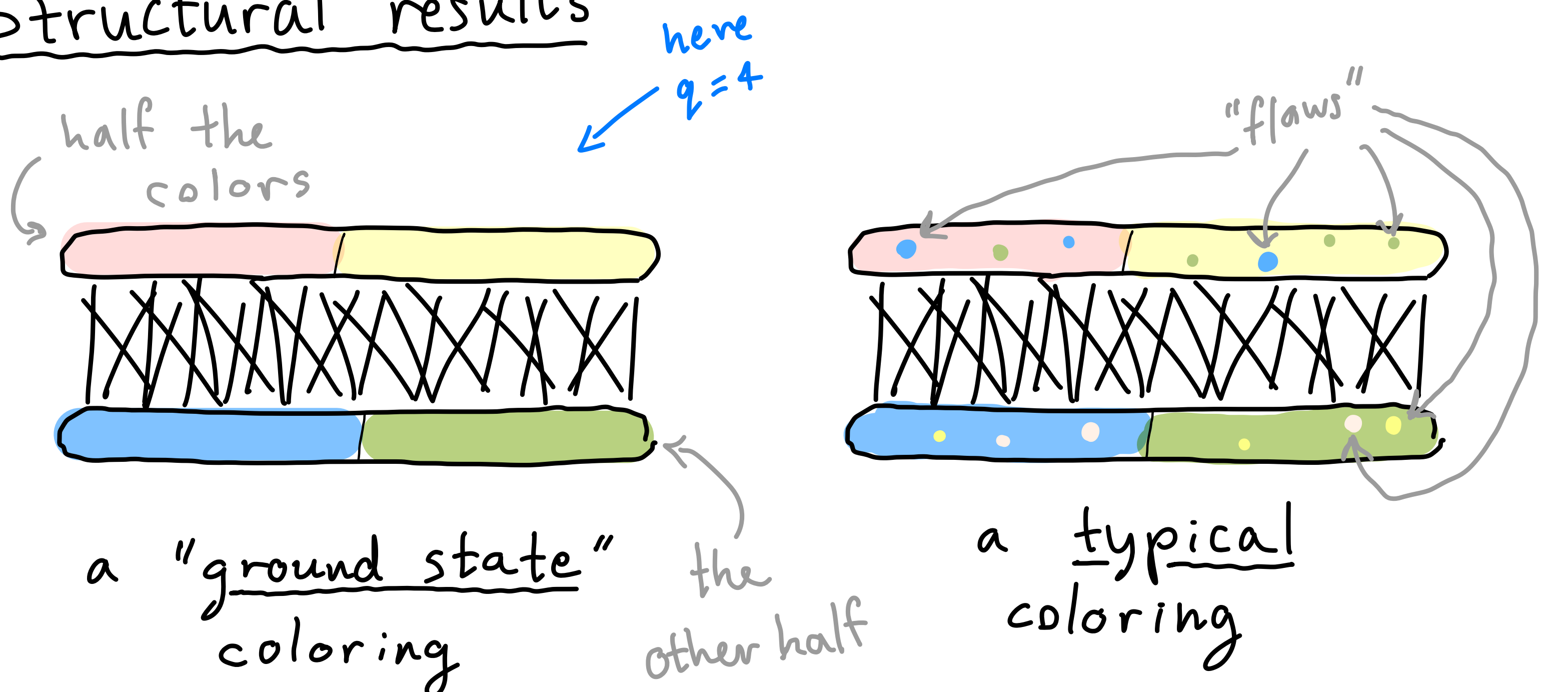
Q How many q-colorings of Q_n are there?

- $q = 3$ → Galvin, 2003
- typical structure for $q \geq 4$ → Engbers + Galvin, 2012

  counting harder: gave general conjecture for $q \geq 4$
- $q = 4$ → Park + Kahn, 2020
- $q \geq 5$ → Jenssen + Keevash, 2020+

  asymptotics, very fine structure + much, much more!

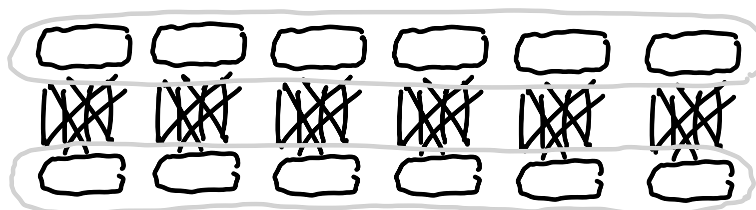
Structural results



Flaws are typically very small and "not too close together" in q-colorings of Q_n.

Q What about other bipartite graphs?

Should depend on structure:



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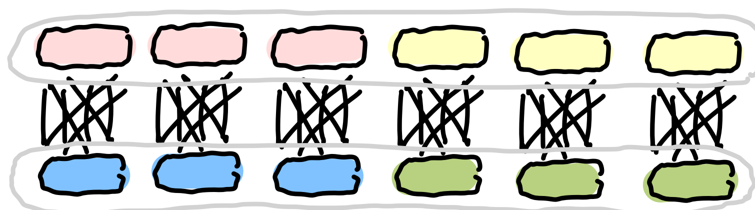

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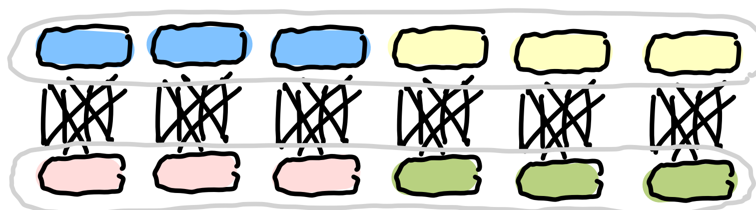

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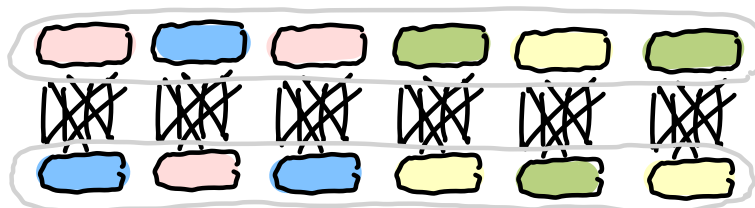

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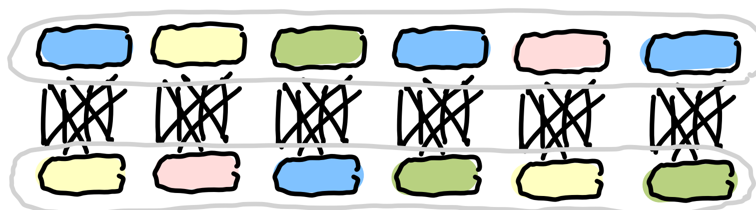
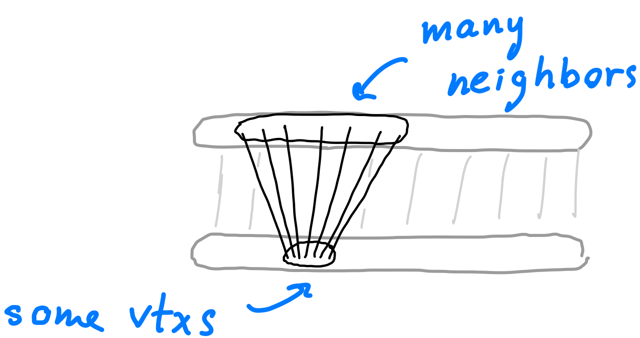
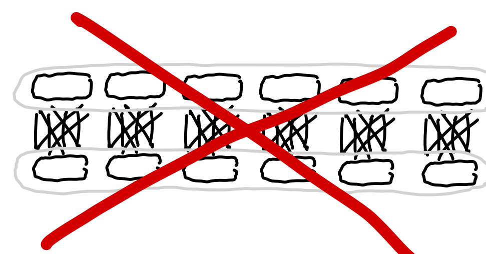


Q What about other bipartite graphs?

Conjecture (folklore?) A regular bipartite graph G has typical colorings with this structure ("small flaws") if it is a "good enough expander".

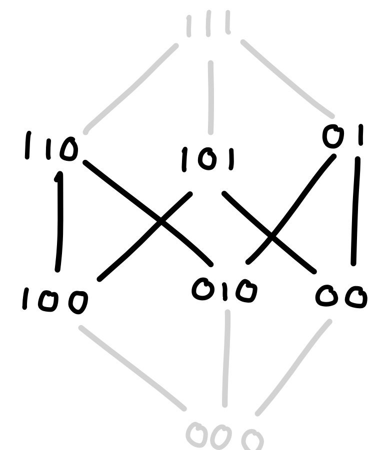




Our problem

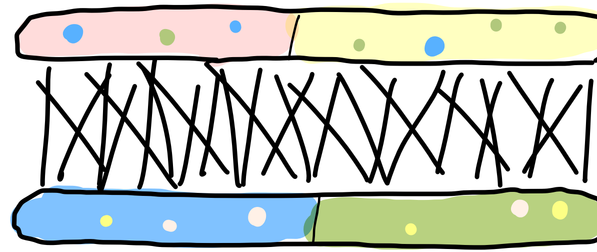
Look at just the middle layers of Q_n.

Still good expansion, but less structure than Q_n.



Our results

- Typical q-coloring of middle layers has "small flaws" if $q \geq 4$ is even.



const. size, depending on q

- Counting:

Theorem 1.1. *Let $q \geq 4$ be even. Then we have*

$$c_q(\mathcal{B}_d) = (q/2)^N \binom{q}{q/2} \exp\left((1+o(1))f(q,d)\right)$$

as $d \to \infty$, where

$$f(q,d) = N(1-2/q)^d + N(1-2/q)^{2d} \cdot \frac{1}{2}\left(d(1-2/q)^{-2} - d - 1\right).$$

In particular, for $q = 4$,

\# ways to (evenly) partition 4 colors

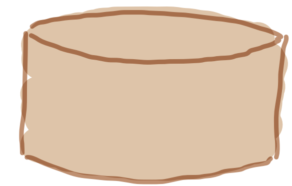
\# colorings in one ground state

$$c_4(\mathcal{B}_d) \sim 6 \cdot 2^N \exp\left(f(4,d)\right) = 6 \cdot 2^N \exp\left(N2^{-d} + N2^{-2d}(3d/2 - 1/2)\right).$$

Proof strategy

- Step 1 No large flaws
  entropy approach: Kahn/Engbers-Galvin
  ▷ where it breaks for odd q

"cake"



- Step 2 No medium flaws
  graph containers + entropy

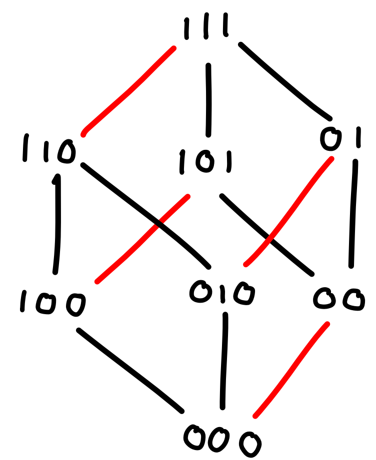
"frosting"

- Step 3 Counting
  polymer model + cluster expansion

"assembly"

Step 1 Original argument from Kahn relies on a matching between half cubes:

In middle layers, only half of the matching survives:



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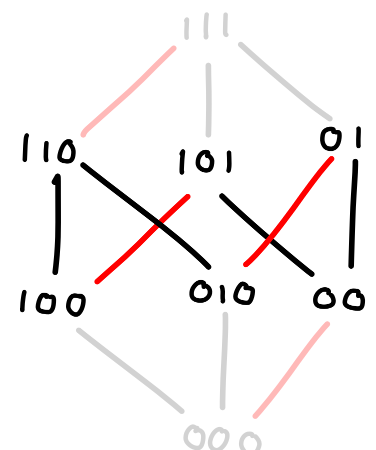

This is good enough!

(for even q...)

Remaining questions

- Odd q?
- Other bipartite expanders - Step 1? (large flaws)

Thank you!

What breaks for odd q?