Math 20A: Calculus for Science and Engineering I
Midterm Exam 1

Winter 2022

- You will have **50 minutes** to complete this exam.
- Please have your student ID easily accessible to show to a proctor when asked.
- You may use one 8.5 x 11 inch sheet of handwritten notes, but no calculators, phones, or other study aids.
- Simplify all answers, unless specified otherwise. (**Note: it is okay for your answers to contain fractions in lowest terms, as well as numbers like e and π and $\sqrt{2}$.**)
- Please show your work and explain your answers for each problem unless otherwise specified– we will not award full credit for the correct numerical answer without proper explanation.
- Please write your final answer for each problem in the indicated area. If you do any work on the backs of the pages or on additional scratch paper that you would like to have graded, **please indicate that clearly; otherwise it will not be graded**.
- Don't forget to write your name on the top of every page.
- Good luck!

Name: Solutions

PID:________________________

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Problem 1: (4 points) The following limit represents a derivative $f'(a)$. Identify $f(x)$ and a. *(You do not need to show your work or justify your answers for this problem.)*

$$\lim_{h\to 0} \frac{\cos(\frac{\pi}{3}+h)-\frac{1}{2}}{h}$$

Recall definition:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$f(x) =$ $\cos(x)$ $\qquad a =$ $\pi/3$

Problem 2: (5 points) Which of the following is an equation of the line tangent to the graph of $y = x + e^x$ at $x = 0$?
(You do not need to show your work or justify your answers for this problem.)

Choose one:

- ○ $y = x$
- ○ $y = x + 1$
- ○ $y = x + 2$
- ○ $y = 2x$
- ● $y = 2x + 1$

If $f(x) = x + e^x$, then equation of tangent line at $x = 0$ is: $y = \underset{\text{point}}{f(0)} + \underbrace{f'(0)}_{\text{slope}}(x-0)$

Compute

- $f(0) = 0 + e^0 = \boxed{1}$
- $f'(x) = 1 + e^x \Rightarrow f'(0) = 1 + e^0 = 1 + 1 = \boxed{2}$

$\Rightarrow$ tangent line is $y = 1 + 2(x-0) = \underline{\underline{2x+1}}$

Problem 3: (5 points) What is the slope of the curve $y = 10x^2 - 3x - 1$ at the point where it crosses the positive part of the x-axis?
(You do not need to show your work or justify your answers for this problem. Also, note/clarification: "slope of the curve" is another way of saying "slope of the line tangent to the curve.")

Choose one:

- ○ 3/20
- ○ 1/5
- ○ 1/3
- ● 7

Strategy

1. Find the point where the curve crosses the positive part of the x-axis. (i.e., $y=0$ and $x>0$)
2. Find the slope of the tangent line at that point.

① Crosses the x-axis when $y = 0$. Solve:

$0 = 10x^2 - 3x - 1$

$0 = (5x+1)(2x-1)$ (factor)

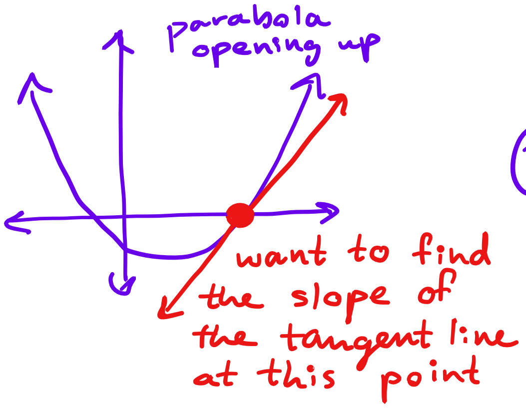
$\Rightarrow x = -\frac{1}{5}$ or $\boxed{x = \frac{1}{2}}$ (positive)

② Take derivative:

$y' = 20x - 3$

Plug in $x = \frac{1}{2}$ to get slope at that point:

$y'\big|_{x=1/2} = 20\left(\frac{1}{2}\right) - 3$
$= 10 - 3 = \boxed{7}$



Problem 4: (11 points) Let $f(x) = (1+x)e^x$.

(a) (5 points) Find $f'(x)$. (You may use any rules you know.)

Product rule

$f'(x) = (1+x) \cdot \frac{d}{dx} e^x + e^x \frac{d}{dx}(1+x)$

$f'(x) = (1+x)e^x + e^x(1)$

$f'(x) = e^x + xe^x + e^x$

$f'(x) = e^x(1+x+1)$

$f'(x) = e^x(x+2)$

Answer:

$\boxed{f'(x) = e^x(x+2)}$

(Notice that the answer for 4(a) can be written as either $e^x(x+2)$ or $xe^x + 2e^x$).

(b) (6 points) Find $f^{(3)}(-1)$. (You may use any rules you know.)

2nd $\Rightarrow \frac{d}{dx}(2e^x + xe^x) = \frac{d}{dx} e^x(2+x)$

prod rule $= 1 \cdot e^x + (2+x)e^x$

$= e^x + 2e^x + xe^x$

$= e^x(3+x)$

3rd $\Rightarrow \frac{d}{dx} e^x(3+x)$

prd. rule $\Rightarrow 1 \cdot e^x + (3+x)e^x = e^x + 3e^x + xe^x$

$4e^x + xe^x$

$e^x(4+x)$

$e^x(4+x)$

plug into $e^{-1}(4-1)$

$e^{-1}(3)$

$3e^{-1}$

Answer:

$\boxed{3e^{-1}}$

Problem 5: (15 points) Evaluate the following limits if they exist. (If the limit is ∞ or $-\infty$, say so. Do not use L'Hôpital's rule.)

(a) (5 points) $\lim_{x\to 1} \sqrt[3]{7+x^5}$

Substitution

$= \sqrt[3]{7+1^5}$

$= \sqrt[3]{7+1}$

$= \sqrt[3]{8}$

$= 2$

Answer: $\lim_{x\to 1} = 2$

(b) (5 points) $\lim_{t\to -1} \dfrac{\sqrt{2t+5}-\sqrt{3}}{t+1}$ on substituting $t=-1$ get $0/0$ indeterminate form hence perform algebraic manipulation to simplify it

$\lim_{t\to -1} \dfrac{(\sqrt{2t+5}-\sqrt{3})(\sqrt{2t+5}+\sqrt{3})}{t+1 \quad (\sqrt{2t+5}+\sqrt{3})}$

$\lim_{t\to -1} \dfrac{2t+5 \ -3}{(t+1)[\sqrt{2t+5}+\sqrt{3}]}$

$\lim_{t\to -1} \dfrac{2t+2}{(t+1)[\sqrt{2t+5}+\sqrt{3}]}$

$\lim_{t\to -1} \dfrac{2\cancel{[t+1]}}{\cancel{[t+1]}\sqrt{2t+5}+\sqrt{3}} = \dfrac{2}{2\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

Answer: $\dfrac{1}{\sqrt{3}}$

(c) (5 points) $\lim_{\theta\to 0} \dfrac{\sin(5\theta)}{\sin(8\theta)}$

$\lim_{\theta\to 0} \dfrac{\sin(5\theta)\cdot\frac{5\theta}{5\theta}}{\sin(8\theta)\cdot\frac{8\theta}{8\theta}}$

$\lim_{\theta\to 0} \dfrac{5\cancel{\theta}\cdot\frac{\sin(5\theta)}{5\theta}}{8\cancel{\theta}\cdot\frac{\sin(8\theta)}{8\theta}}$

$\lim_{\theta\to 0} \dfrac{5\cdot\frac{\sin(5\theta)}{5\theta}}{8\cdot\frac{\sin(8\theta)}{8\theta}} = \dfrac{5\cdot 1}{8\cdot 1} = \dfrac{5}{8}$

Answer: $\dfrac{5}{8}$

Note: in this solution, the student evaluated $\lim_{\theta\to 0} \dfrac{\sin(5\theta)}{5\theta} = 1$ and $\lim_{\theta\to 0} \dfrac{\sin(8\theta)}{5\theta} = 1$ by using the fact that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$, and substituting $x = 5\theta$ and $x = 8\theta$ respectively to find these limits.

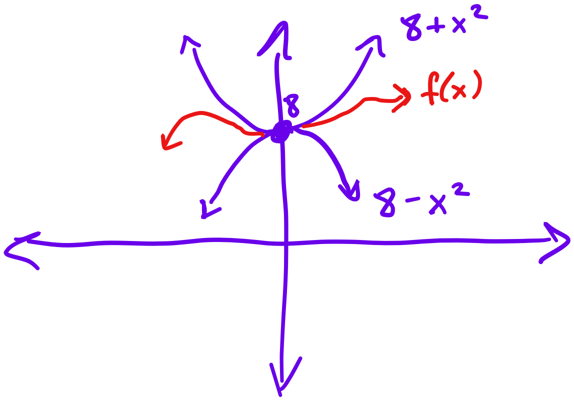
Name:

Problem 6: (10 points) Let $f(x)$ be a function satisfying

$$8 - x^2 \;\leq\; f(x) \;\leq\; 8 + x^2$$

for all values of x in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

(a) (7 points) Find the limit $\lim_{x \to 0} f(x)$ (or state that there is not enough information to find it). Be sure to show your work and justify your reasoning.
(If you are going to use a theorem, name the theorem and verify that any hypotheses are satisfied.)



Answer:

We are given that $8-x^2 \leq f(x) \leq 8+x^2$ on the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$, which contains $x=0$. And we compute:

$$\lim_{x \to 0} 8-x^2 = 8-(0)^2 = 8$$

$$\lim_{x \to 0} 8+x^2 = 8+(0)^2 = 8$$

Since these limits are equal, the <u>Squeeze Theorem</u> tells us that $\lim_{x \to 0} f(x)$ also exists and is equal to $\boxed{8}$.

Name:

Problem 6 (continued)

(b) (3 points) Is f continuous at $x = 0$?

(Answer "yes," "no," or "not enough information," and justify your answer.)

Answer:

Yes. We will use the definition of continuity:

f is continuous at $x=0$ if $\lim_{x\to 0} f(x) = f(0)$.

- In part (a), we showed that $\lim_{x\to 0} f(x)$ exists and is equal to 8.
- And notice that $8-x^2 \le f(x) \le 8+x^2$ for every x in $(-1/2, 1/2)$, including $x=0$. This gives $\underbrace{8-(0)^2}_{8} \le f(0) \le \underbrace{8+(0)^2}_{8}$, which tells us that $f(0)=8$.

Thus $\lim_{x\to 0} f(x) = f(0) = 8$, and f is continuous at $x=0$.