**Announcements:**
- In-person OH Mon 1-2pm at Art of Espresso
- Exam problem suggestions due Mon 9am
- Class survey!

Today §3.3: Product & Quotient rules
+finish §3.2: The Derivative as a Function

Last time

Important examples: $\boxed{\dfrac{d}{dx} x = 1}$ and $\boxed{\dfrac{d}{dx} c = 0}$ for a constant c

Useful rules:

linearity
- Sum rule: $(f+g)' = f' + g'$
- Difference rule: $(f-g)' = f' - g'$
- Constant multiple rule: $(cf)' = c \cdot f'$
- Power rule: $\dfrac{d}{dx} x^n = n x^{n-1}$ (for any constant n)

Warm-up question what happens to $\dfrac{d}{dx}(f(x)g(x))$? Should it be equal to $f'(x)g'(x)$? Can you think of an example where this would fail?

Another important example $\dfrac{d}{dx} e^x = e^x$

Proof $\dfrac{d}{dx} e^x = \lim\limits_{h\to 0} \dfrac{e^{x+h} - e^x}{h} = \lim\limits_{h \to 0} \dfrac{e^x e^h - e^x}{h}$

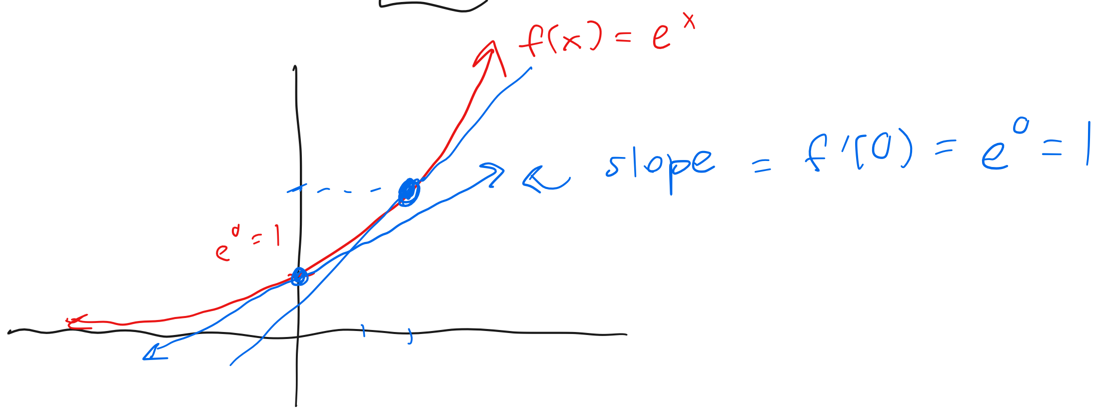
$\dfrac{f(x+h) - f(x)}{h}$ if $f(x) = e^x$

$= \lim\limits_{h\to 0} e^x \dfrac{(e^h - 1)}{h}$

$= e^x \lim\limits_{h\to 0} \dfrac{e^h - 1}{h}$ does not depend on x this limit equals 1 (deep fact)

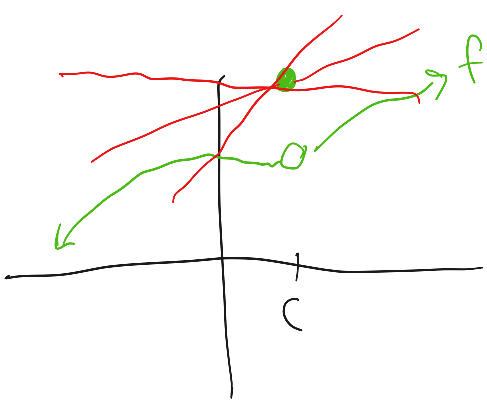
uses the fact that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

$$= \boxed{e^x}$$

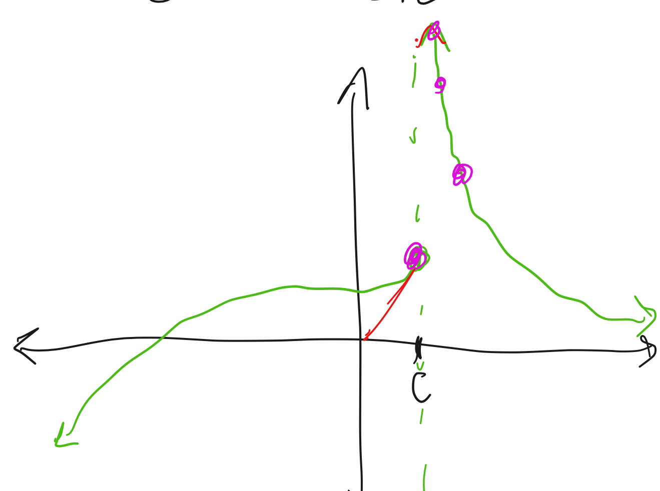



<u>Theorem</u> If $f$ is differentiable at $x = c$, then $f$ is continuous at $x = c$.

<u>Intuition</u>: I'd like to say that if $f$ isn't continuous, it can't be differentiable




removable disc.




infinite discontinuity

<u>Q</u> What is $\frac{d}{dx}[f(x)g(x)]$?

Is it $\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}f(x) \cdot \frac{d}{dx}g(x)$ ?

$f'(x) \cdot g'(x)$

Let's try this with $f(x) = x$ and $g(x) = x$.

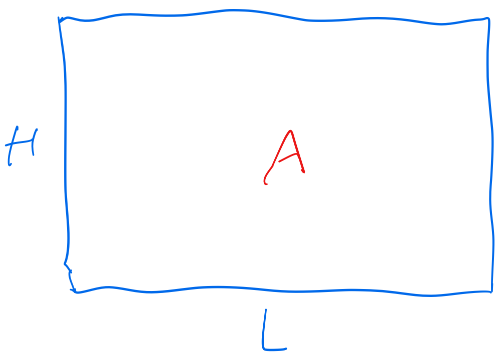
Here, $f(x) \cdot g(x) = x^2$.

$$f'(x) = 1 = g'(x) \implies f'(x) \cdot g'(x) = 1$$

But $\frac{d}{dx}\left[f(x)g(x)\right] = \frac{d}{dx}\left[x^2\right] \overset{\text{power rule}}{=} 2x \neq f'(x)g'(x)$ whoops

Then what <u>is</u> $\frac{d}{dx}\left[f(x)g(x)\right]$ equal to?

Let's look at a physical example.
Imagine a screen of length $L$ and height $H$



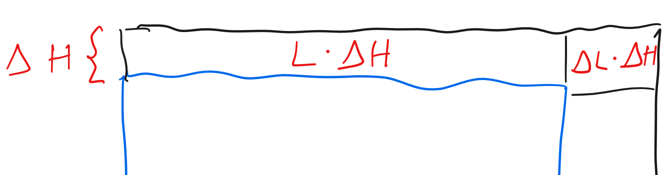

$A$ = area = $L \cdot H$

Imagine that $L$ and $H$ vary as functions of time $t$.
I.e., $L = L(t)$, $H = H(t)$ and $A = A(t) = L(t) \cdot H(t)$.

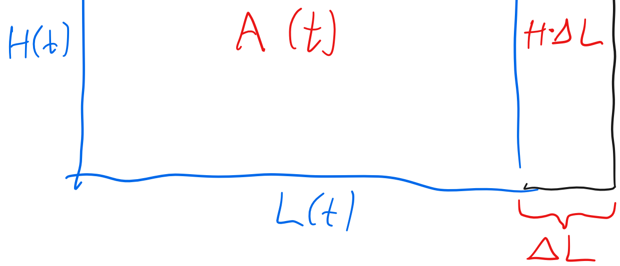
What is $\frac{dA}{dt} = \frac{d}{dt}\left[L(t) \cdot H(t)\right]$?

$$\frac{dA}{dt} = \lim_{\substack{\cancel{h} \to 0 \\ \Delta t}} \frac{A(t + \cancel{h}^{\,\Delta t}) - A(t)}{\cancel{h}\ \Delta t} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t}$$

Let's try to understand $\Delta A$ better




$$\Delta A = L \cdot \Delta H + H \cdot \Delta L + \Delta L \, \Delta H$$




$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \frac{L \Delta H + H \Delta L + \Delta L \Delta H}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left( L \cdot \frac{\Delta H}{\Delta t} + H \frac{\Delta L}{\Delta t} + \frac{\Delta L}{\Delta t} \cdot \Delta H \right)$$

$$= L \lim_{\Delta t \to 0} \frac{\Delta H}{\Delta t} + H \lim_{\Delta t \to 0} \frac{\Delta L}{\Delta t}$$

$$+ \underbrace{\lim_{\Delta t \to 0} \frac{\Delta L}{\Delta t}}_{} \cdot \underbrace{\lim_{\Delta t \to 0} \Delta H}_{}$$

$$\frac{dA}{dt} = \boxed{L \cdot \frac{dH}{dt} + H \frac{dL}{dt}} + \cancel{\frac{dL}{dt} \cdot 0} \to 0$$

Theorem If $f$ and $g$ are differentiable functions, then $fg$ is differentiable, and

$$\frac{d}{dx}(fg) = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx}$$

$$(fg)' = f'g + fg'$$

Check $f(x) = x$, $g(x) = x$. Have $f(x) g(x) = x^2$

$f'(x) = 1$, $g'(x) = 1$ Power rule: $\frac{d}{dx} x^2 = 2x$

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) g'(x)$$

$$= 1 \cdot x + x \cdot 1$$

$$= 2x \quad \checkmark$$

Ex find $\frac{d}{dx}(\underbrace{x}_{f(x)}\underbrace{e^x}_{g(x)}) = \underbrace{1}_{f'(x)} \cdot \underbrace{e^x}_{g(x)} + \underbrace{x}_{f(x)} \cdot \underbrace{e^x}_{g'(x)} = e^x + xe^x$

student questions

$f(x) = x^2, \quad g(x) = e^x, \quad h(x) = e^x$

find $\frac{d}{dx}(f(x)g(x)h(x)) = \frac{d}{dx}(\underbrace{x^2}_{a(x)} \cdot \underbrace{e^x \cdot e^x}_{b(x)})$

Product rule: $\frac{d}{dx}(a(x)b(x)) = a'(x)b(x) + a(x)b'(x)$

$= 2x\, e^x \cdot e^x + x^2 \underbrace{\frac{d}{dx}(e^x \cdot e^x)}_{\text{use the product rule again}}$