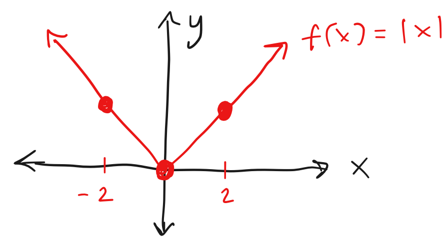
Today §3.2: The Derivative as a Function

- Important examples
- New notation
- Tools for finding derivatives

Warm-up if $f(x) = |x|$, find

- $f'(2)$
- $f'(-2)$
- $f'(0)$



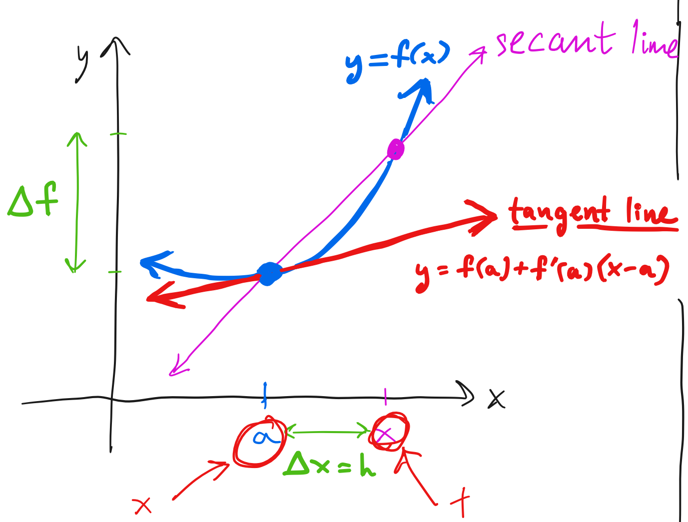
Try it if you have time!

Last time: Derivative of f at a:

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Say that f is differentiable at a if the limit exists.



Notice: we can compute $f'(a)$ at any number a where $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists.

This means that f' is a function

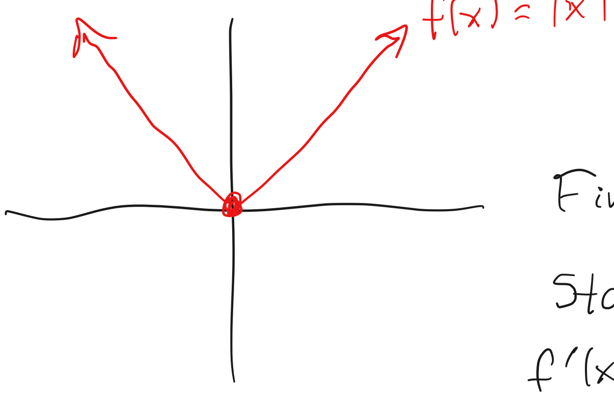
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

at all x where this limit ~~is defined~~ exists.

Vocab: If $f'(x)$ is defined for all x in an interval (a,b), we say that f is <u>differentiable</u> on (a,b).

If $f'(x)$ is defined for all x where $f(x)$ is defined, we simply say that f is <u>differentiable</u>.

<u>Ex</u> Where is $f(x) = |x|$ differentiable, and what is its derivative?



<u>Notice</u> $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Find $f'(x)$.

Start with $x > 0$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{|x+h| - |x|}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{x} + h) - \cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}} = \lim_{h \to 0} 1 = \boxed{1}$$

For $x < 0$, we get $f'(x) = -1$ using a similar approach

What about $f'(0)$?

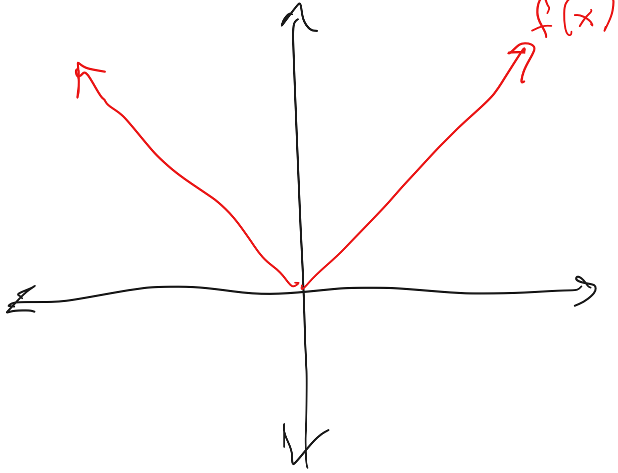
$$f'(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{|\cancel{0} + h| - |\cancel{0}|}{h}$$

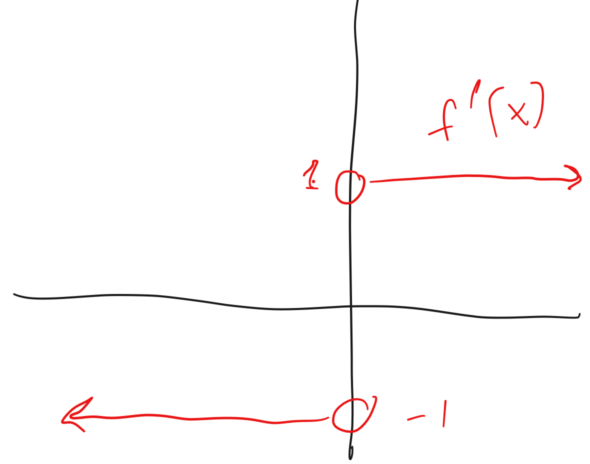
$$= \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{\cancel{h}^{\,1}}{\cancel{h}} = 1$$

Notice $\displaystyle \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$

left-hand and right-hand limits are not equal
limit DNE

So f is <u>not</u> differentiable at 0.





Notice f is continuous at $x=0$, but not differentiable at $x=0$.

$$f'(x) = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \\ \text{undefined} & \text{if } x=0 \end{cases}$$

Q Can a function be differentiable but not continuous?

Notation — Lagrange — Leibniz

$$f'(x) = \frac{df}{dx} = \frac{d}{dx} f(x)$$

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a}$$

"derivative with respect to x of $f(x)$"

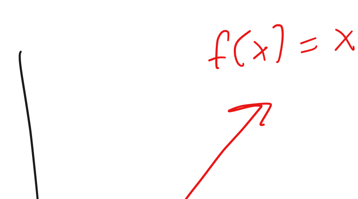
If $y=f(x)$: $\quad y' = \dfrac{dy}{dx}$

Leibniz notation

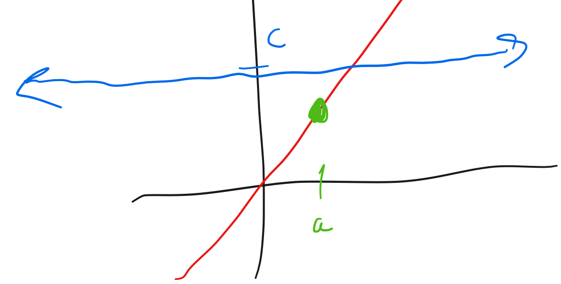
- Helps us remember $f'(x) = \dfrac{df}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta f}{\Delta x}$
- Warning $\dfrac{df}{dx}$ is not a fraction – it's a limit of fractions
- Often used in physical sciences

Important examples

- $\dfrac{d}{dx} x = 1$ $\quad f'(x)$ where $f(x)=x$



- $\dfrac{d}{dx} c = 0$ for any constant c



Useful rules

- Sum rule $(f+g)' = f' + g'$
- Difference rule $(f-g)' = f' - g'$
- Constant multiple rule $(cf)' = c \cdot f'$

Leibniz: $\dfrac{d}{dx}(f+g) = \dfrac{df}{dx} + \dfrac{dg}{dx}$

- Power rule for any constant n,

$$\frac{d}{dx} x^n = n \cdot x^{n-1}$$

Note n has to be constant; doesn't work for x^x

Ex $\dfrac{d}{dx} x^4 = 4 \cdot x^3$

Ex $\dfrac{d}{dx} \dfrac{1}{x^2} = \dfrac{d}{dx}(x^{-2}) = (-2)\, x^{-3} = \dfrac{-2}{x^3}$

Ex $\dfrac{d}{dx}\left(\sqrt[3]{x^5} + 1\right) \overset{\text{sum rule}}{=} \dfrac{d}{dx} \sqrt[3]{x^5} + \cancel{\dfrac{d}{dx} 1}\ 0$

$$= \frac{d}{dx} x^{5/3} + 0$$

$$= \frac{5}{3} x^{5/3 - 1}$$

$$= \frac{5}{3} x^{2/3}$$

This computation was the answer to a student question about how we would find $\frac{d}{dx} x^4$ directly from the definition of derivatives

$$\frac{d}{dx} \underbrace{x^4}_{f(x)} = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x} \frac{t^4 - x^4}{t - x}$$

factor $t^4 - x^4 = (t-x)(t^3 + t^2x + tx^2 + x^3)$

$$= t^4 + (\cancel{+t^3x} + \cancel{t^2x^2} + \cancel{tx^3})$$

$$+ (\cancel{-xt^3} - \cancel{t^2x^2} - \cancel{tx^3} - x^4)$$

$$= t^4 - x^4$$

$$= \lim_{t \to x} \frac{\cancel{(t-x)}(t^3 + t^2x + tx^2 + x^3)}{\cancel{(t-x)}}$$

$$= (x)^3 + (x)^2x + (x)x^2 + x^3$$ ← 4 terms all equal to x^3

$$= 4 \cdot x^3$$

Proof of the Power rule for any positive integer n is the same idea