Today: Finish §2.4: Limits and Continuity
+ §2.5: Indeterminate Forms
+ Start §2.7: Limits at Infinity

_Warm-up problem_ Let \( f(x) = \begin{cases} x^2 + a & \text{if } x > 3 \\ a x - 2 & \text{if } x \leq 3 \end{cases} \)

For what value(s) of \( a \) is \( f \) continuous at \( x = 3 \)?

_Last time:_ A function \( f \) is **continuous at** \( x = c \)

if \( \lim_{x \to c} f(x) = f(c) \).

_(Geometrically: a discontinuity means there's a hole or a jump in the graph of \( f \))_

Removeable discontinuity

Infinite discontinuity

Jump discontinuity: if \( \lim_{x \to c^+} f(x) \) and \( \lim_{x \to c^-} f(x) \) exist, but are not equal.
Def \( f \) is

- **right-continuous** at \( x = c \)
  \[ \lim_{x \to c^+} f(x) = f(c) \]

- **left-continuous** \( \equiv \)
  \[ \lim_{x \to c^-} f(x) = f(c) \]

**Warm-up problem** Let \( f(x) = \begin{cases} x^2 + a & \text{if } x > 3 \\ a \times -2 & \text{if } x \leq 3 \end{cases} \)

For what value(s) of \( a \) is \( f \) continuous at \( x = 3 \)?

Find:

\[ \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x^2 + a = 3^2 + a = 9 + a \]

\[ \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} a \times -2 = 3a - 2 \]

\( \Rightarrow \) \( f \) cont. at \( x = 3 \) if & only if \( 9 + a = 3a - 2 \)

\[ \Leftrightarrow \ 11 = 2a \]

\[ \Leftrightarrow \ \frac{11}{2} = a \]

**Building continuous functions:**

**Theorem** Thm (1, 4, 5) If \( f \) and \( g \) are continuous at \( x = c \), then so are \( f + g \), \( f - g \), \( kf \), \( f/g \) when \( g(c) \neq 0 \).

Also so are \( f^{-1} \) and \( f \circ g(x) = f(g(x)) \).

(but need to be careful with domain)

Concretely, for \( f(g(x)) \): if \( f \) is continuous at \( g(c) \) and \( g \) is continuous at \( c \),
then \( f \circ g \) is continuous at \( c \).
The function \( f(x) \) is continuous at \( x = c \).

**Example**
\[
 f(x) = 3x \cdot e^{\sin(x^2)} + \cos^2 x \quad \text{is continuous at all } x \in \mathbb{R} \text{ since polynomials, } \sin, \cos, e^x \text{ are continuous at all } x \in \mathbb{R}
\]

### 2.5: Indeterminate forms

**Q.** How to evaluate \( \lim_{x \to c} f(x) \)?

Easy when \( f \) is continuous at \( x = c \):
\[
 \lim_{x \to c} f(x) = f(c)
\]

**Examples**
- \[
 \lim_{x \to 2} (x^2 + x) = (2)^2 + (2) = 6
\]
- \[
 \lim_{x \to \pi/2} \cos(x) = \cos(\pi/2) = 0
\]

**Problem**
\[
 \lim_{x \to 2} \frac{x^2 - 4}{x - 2}
\]

If we substitute \( x = 2 \), get \( \frac{(2)^2 - 4}{(2) - 2} = \frac{0}{0} \equiv \text{undefined} \)

But can still find the limit:
\[
 \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \left( \frac{x - 2}{x - 2} \right) \left( \frac{x + 2}{f(x)} \right) g(x)
\]
\[
 = \lim_{x \to 2} f(x) \cdot \lim_{x \to 2} g(x) \quad \text{(Product Law)}
\]
\[
 = 1 \cdot 4 = 4
\]

**Definition:** \( f \) has an indeterminate form at \( x = c \) if \( f(c) \) has the form \( \frac{0}{0}, \infty \cdot 0, \infty - \infty \), or \( \frac{\infty}{\infty} \).
\[
\lim_{x \to 0} \frac{0}{x} \quad \text{not indeterminate}
\]

\[
\lim_{x \to 0} \frac{\cos x}{x} \quad \longrightarrow \quad \frac{1}{0} \quad \text{(diverges to } \infty)\]

\[
\text{Ex} \quad \lim_{x \to 9} \frac{x - 9}{\sqrt{x} - 3} \quad \text{indeterminate}
\]

\[
\lim_{x \to 9} \frac{x - 9}{\sqrt{x} - 3} = \frac{9 - 9}{\sqrt{9} - 3} = \frac{0}{0}
\]

Transform algebraically

\[
\frac{x - 9}{\sqrt{x} - 3} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} \quad \text{(multiply by conjugate)}
\]

\[
= \frac{(x - 9)(\sqrt{x} + 3)}{x - 9} = \sqrt{x} + 3 \quad \text{if } x \neq 9
\]

\[
\lim_{x \to 9} \frac{x - 9}{\sqrt{x} - 3} = \lim_{x \to 9} \sqrt{x} + 3 = \sqrt{9} + 3 = 6
\]

\[
\text{Ex} \quad \lim_{x \to 2^+} \left( \frac{\frac{1}{x} \to \infty}{x - 2} + \frac{1}{\sqrt{x} - \sqrt{2}} \to \infty} \right) = \infty \quad \text{(not indeterminate)}
\]

\[
\text{Answer to student question}
\]

Strategy for indeterminate forms \( f(c) = \frac{0}{0}, \infty - \infty, 0 \cdot \infty, \infty \)

- Manipulate algebraically to "cancel the bad part" (and get a continuous function equal to \( f(x) \) except at \( x = c \))

- Evaluate by substitution

\text{Won't always work}: \text{Eg, we'll need other tools for } \lim_{x \to 0} \frac{\sin x}{x}, \text{which we'll learn in §2.6}

\[
\text{Ex} \quad \lim_{x \to 1} \left( \frac{1}{x - 1} - \frac{2}{x^2 - 1} \right) = \infty - \infty
\]

\text{Ex 5 in book}
\[
\frac{(x+1)}{(x-1)(x+1)} \sim \frac{2}{(x-1)(x+1)} \\
= \frac{x-1}{(x-1)(x+1)}
\]

\[
\lim_{x \to 0} \frac{1}{x+1} = \frac{1}{(1)+1} = \frac{1}{2}
\]