Announcements:
- Midterm 1 Mon. 4-5 pm
- Seat assignments emailed out
- OH/discussion format announced next week (but Monday OH on Zoom)
- Drop deadline (without a "W") today

Today: §3.7: The Chain Rule

Last time: Trig derivatives:
\[
\frac{d}{dx} \sin(x) = \cos(x) \quad \text{use derivatives of } \sin(x), \cos(x) \text{ to find}
\]
\[
\frac{d}{dx} \cos(x) = -\sin(x)
\]
\[
\frac{d}{dx} \tan(x) = \sec^2(x)
\]
\[
\frac{d}{dx} \sec(x) = \sec(x) \tan(x)
\]
\[
\frac{d}{dx} \cot(x) = -\csc^2(x)
\]
\[
\frac{d}{dx} \csc(x) = -\csc(x) \cot(x)
\]

Warm-up: If \( f(x) = \sin(x) \), and \( g(x) = e^x \), write out:
- \( f(\sin(x)) = \sin(e^x) \)
- \( g(\sin(x)) = e^{\sin(x)} \)

Review given functions \( f(x) \) and \( g(x) \), their composition is outside function
\[ f \circ g(x) = f(g(x)) \]
also \( g \circ f(x) = g(f(x)) \)
\[ f(x) = \cos(x), \quad g(x) = x^2 \]

\[ f \circ g \ (x) = f(g(x)) = \cos(x^2) \]

\[ g \circ f \ (x) = g(f(x)) = (\cos(x))^2 = \cos^2(x) \]

Say \( f \) and \( g \) are differentiable functions.

Q: What is \( (f \circ g)'(x) = \frac{d}{dx} f(g(x)) \)

(e.g. \( \frac{d}{dx} \cos(x^2) \))

Use definition:

\[ (f \circ g)'(x) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} \]

As \( h \to 0 \)

notice that \( g(x+h) \to g(x) \).

Why? Because \( g \) is continuous (since it is differentiable)

\[ \begin{align*}
(f \circ g)'(x) &= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \\
&= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} \\
&= f'(g(x)) \cdot g'(x)
\end{align*} \]

**Theorem (Chain Rule)** If \( f \) and \( g \) are differentiable, then the composition \( (f \circ g)(x) = f(g(x)) \) is differentiable, and

\[ \frac{d}{dx} f(g(x)) = (f \circ g)'(x) = f'(g(x)) \cdot g'(x) \]

\[ f'(x) = -\sin(x), \quad g'(x) = 2x \]

\[ (f \circ g)'(x) = f'(g(x)) \cdot g'(x) = -\sin(g(x)) \cdot 2x = -\sin(x^2) \cdot 2x \]
\[ f(x) = \cos(x), \quad g(x) = x^2 \]

\[ (f \circ g)'(x) = f'(g(x)) \cdot g'(x) \]
\[ = -\sin(x^2) \cdot (2x) \]
\[ = -2x \cdot \sin(x^2) \]

\[ (g \circ f)'(x) = \frac{d}{dx} g(f(x)) = \frac{d}{dx} (\cos(x^2)) \]
\[ = 2(\cos(x)) \cdot (-\sin(x)) \]
\[ = -2 \cos(x) \sin(x) \]

**Ex.**
\[ y = \sqrt{x^2 + 5x} \]
Find \( y' = \frac{dy}{dx} \)
\[ = (x^2 + 5x)^{1/2} \]
\[ \frac{dy}{dx} = \frac{1}{2}(x^2 + 5x)^{-1/2} \cdot (2x + 5) \]
\[ = \frac{2x + 5}{2\sqrt{x^2 + 5x}} \frac{1}{f'(g(x))} \frac{1}{g'(x)} \]

**Ex.**
\[ y = \sin(x^2 + 3x) \]
Find \( y' = \frac{dy}{dx} \)
\[ = \cos(x^2 + 3x) \cdot (2x + 3) \]
\[ = (2x + 3) \cos(x^2 + 3x) \]

**Leibniz notation (Chain Rule):** If \( g(x) = u \) and \( f(u) = y \),
so \( f(g(x)) = f(u) = y \), then
\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \]

If \( u = g(x) \), and
\[ y = f(u) \]
then \( y' = f'(u) \cdot g'(x) \)
\[(f \circ g)'(x) = f'(g(x)) \cdot g'(x)\]

Q: What about > 2 functions?

Ex: \(y = e^{ \cos(x^2)} = f(g(h(x)))\)

\[
\frac{dy}{dx} = e^{ \cos(x^2)} \cdot \frac{d}{dx} \cos(x^2)
\]

\[
= e^{ \cos(x^2)} \cdot (-\sin(x^2)) \cdot 2x
\]

\[
= -2x \cdot \sin(x^2) \cdot e^{ \cos(x^2)}
\]

Look at \(f''(g(x)) = f''(u)\)

In Leibniz notation, can rewrite \(f'(u)\) as \(\frac{df}{du} = \frac{dy}{du}\)

\[
h(x) = x^2\]
\[
g(x) = \cos(x)\]
\[
f(x) = e^x\]

Do chain rule again

Answer to a student question about HW bonus problem

Know 1 mile = max height = 5280 feet

\$ 1 for time in air

Want hourly wage \(m \text{ dollars/hr}\)

\[m \cdot h = \$ 1\]

How much time did he spend in air? \(h\) hours

Start with time in seconds and feet

\[
\text{height}(t) = \delta^0 + v_0 t - \frac{1}{2} g t^2
\]

\[
= v_0 t - \frac{1}{2} (32.17) t^2
\]

Max height achieved when velocity is zero