Announcements:

- Midterm 1 Mon. 4-5 pm
- Seat assignments emailed out
- OH/discussion format announced next week (but Monday OH on Zoom)
- Drop deadline (without a "W") today

Today: §3.7: The Chain Rule

Last time Trig derivatives:

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

use derivatives of $\sin(x)$, $\cos(x)$ to find

$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

Warm-up: If $f(x) = \sin(x)$, and $g(x) = e^x$, write out:

- $f(g(x)) = \sin(e^x)$
- $g(f(x)) = e^{\sin(x)}$

Review given functions $f(x)$ and $g(x)$, their **composition** is

$$f\circ g(x) = f(g(x))$$

outside function: f

inside function: $g(x)$

also $g\circ f(x) = g(f(x))$

$$\begin{tikzcd} x \arrow[r, bend left, "g"] \arrow[rr, bend right, "f\circ g"'] & g(x) \arrow[r, bend left, "f"] & f(g(x)) \end{tikzcd}$$

Ex $f(x) = \cos(x)$, $g(x) = x^2$

$$f \circ g(x) = f(g(x)) = \cos(x^2)$$
$$g \circ f(x) = g(f(x)) = (\cos(x))^2 = \cos^2(x)$$

Say f and g are differentiable functions.

Q What is $(f \circ g)'(x) = \frac{d}{dx} f(g(x))$

(e.g. $\frac{d}{dx} \cos(x^2)$)

Know
$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$
$$f'(b) = \lim_{y \to b} \frac{f(y) - f(b)}{y - b}$$

Use definition:

$$(f \circ g)'(x) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h \to 0} \underbrace{\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}}_{\frac{\Delta f \circ g}{\Delta g}} \cdot \underbrace{\frac{g(x+h) - g(x)}{h}}_{\frac{\Delta g}{\Delta x}}$$

As $h \to 0$ notice that $g(x+h) \to g(x)$. Why? Because g is continuous (since it is differentiable)

$$= \boxed{\lim_{h \to 0} \frac{f(\overbrace{g(x+h)}^{y}) - f(\overbrace{g(x)}^{b})}{\underbrace{g(x+h)}_{y} - \underbrace{g(x)}_{b}}} \cdot \boxed{\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}}$$

$$= f'(b) \qquad\qquad = g'(x)$$
$$= f'(g(x))$$

$$= f'(g(x)) \cdot g'(x)$$

Theorem (Chain Rule) If f and g are differentiable, then the composition $(f \circ g)(x) = f(g(x))$ is differentiable, and

$$\frac{d}{dx} f(g(x)) = (f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$f'(x) = -\sin(x)$ $\qquad g'(x) = 2x$

Ex $f(x) = \cos(x)$, $g(x) = x^2$

- $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

$\frac{d}{dx} \underbrace{\cos(x^2)}_{f(g(x))}$

$= -\sin(x^2) \cdot (2x)$

$= -2x \cdot \sin(x^2)$

$\cos^2(x)$ is another way of writing $[\cos(x)]^2$

- $(g \circ f)'(x) = \frac{d}{dx} g(f(x)) = \boxed{\frac{d}{dx} \cos^2(x)}$

$= 2(\cos(x)) \cdot (-\sin(x))$

$= -2\cos(x)\sin(x)$

Ex $y = \sqrt{x^2 + 5x}$. Find $y' = \frac{dy}{dx}$

$= (x^2 + 5x)^{1/2} \leftarrow f(g(x))$ where $g(x) = x^2 + 5x$, $f(x) = x^{1/2}$

$\frac{dy}{dx} = \underbrace{\frac{1}{2}(x^2 + 5x)^{-1/2}}_{f'(g(x))} \cdot \underbrace{(2x + 5)}_{g'(x)}$

$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$

$= \boxed{\frac{2x + 5}{2\sqrt{x^2 + 5x}}}$

Ex $y = \sin(x^2 + 3x)$. Find y' $= f(g(x))$

$g(x) = x^2 + 3x$

$f(x) = \sin(x)$

$\frac{dy}{dx} = \underbrace{\cos(x^2 + 3x)}_{f'(g(x))} \cdot \underbrace{(2x + 3)}_{g'(x)}$

$= \boxed{(2x + 3)\cos(x^2 + 3x)}$

Leibniz notation (Chain Rule): If $g(x) = u$ and $f(u) = y$,

so $f(g(x)) = f(u) = y$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

If $u = g(x)$, and $y = f(u)$

$$(f \circ g)'(x) = f'(\underbrace{g(x)}_{u}) \cdot g'(x)$$

Look at $f'(g(x)) = f'(u)$

in Leibniz notation, can rewrite $f'(u)$ as $\frac{df}{du} = \frac{dy}{du}$

Q What about > 2 functions?

Ex $y = e^{\cos(x^2)} = f(g(h(x)))$

$h(x) = x^2$

$g(x) = \cos(x)$

$f(x) = e^x$

$$\frac{dy}{dx} = e^{\cos(x^2)} \cdot \frac{d}{dx}\cos(x^2)$$

Do chain rule again

$$= e^{\cos(x^2)} \cdot (-\sin(x^2)) \cdot 2x$$

$$= -2x \cdot \sin(x^2) \cdot e^{\cos(x^2)}$$

Answer to a student question about HW bonus problem

Know 1 mile = max height = 5280 feet

\$1 for time in air

Want hourly wage m $\frac{\text{dollars}}{\text{hr}}$

How much time did he spend in air? h hours

$$m \cdot h = \$1$$

Start with time in seconds and feet

$$\text{height}(t) = \cancel{s_0}^{\,0} + \underset{?}{v_0} t - \frac{1}{2} g t^2$$

(t: seconds)

$$= v_0 t - \frac{1}{2}(32.17)t^2$$

max height achieved when velocity is zero