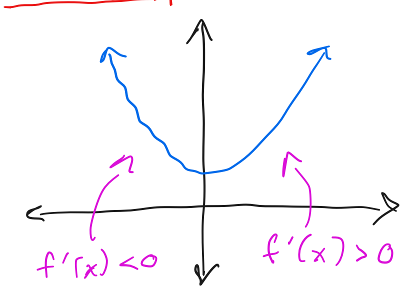
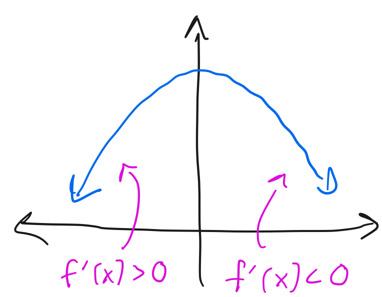
Announcements:
- survey about OH/discussion modality
- Extra OH Friday or Saturday

Today §3.4: Rates of Change
§3.6: Trigonometric ~~Limits~~ Functions ← Not on Midterm 1

Last time:

- Quotient rule: $\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$
- Higher derivatives:
  - second derivative: $f''(x) = \frac{d^2 f}{dx^2} = f^{(2)}(x)$ means $\frac{d}{dx} f'(x)$ "rate of change of the derivative"
  - n^{th} derivative: $\frac{d^n f}{dx^n} = f^{(n)}(x)$
- Graphically: "concave up": $f''(x) > 0$ "concave down" $f''(x) < 0$





Physical example object in motion

$s(t)$ position ← ft (e.g. height in feet after t seconds)

$s'(t) = v(t)$ velocity $\frac{ft}{s}$

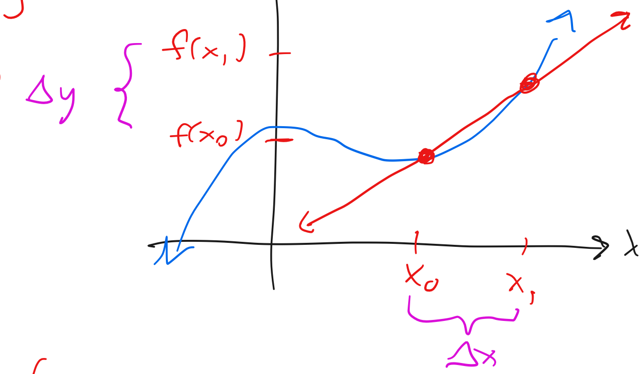
$s''(t) = v'(t) = a(t)$ acceleration $\left(\frac{ft}{s}\right)/s = \frac{ft}{s^2}$

$s'''(t) = v''(t) = a'(t)$ jerk $\frac{ft}{s^3}$

Rates of change (§3.4) Recall from §2.1

Δy $f(x_1) - f(x_0)$ ← change in y

$$\frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad \leftarrow \text{change in } x$$



slope of secant line / average velocity / average rate of change

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(x_0)$$

slope of tangent / instantaneous velocity / instantaneous rate of change

Motion under influence of gravity (Galileo)

Shoot an arrow vertically in the air (near the earth's surface)

Its height is	Its velocity is	Is acceleration is
$s(t) = s_0 + v_0 t - \frac{1}{2} g t^2$ (meters)	$v(t) = \frac{ds}{dt} = V_0 - gt$ m/s	$a(t) = \frac{d^2 s}{dt} = \frac{dv}{dt} = -g$ (m/s^2)
$s_0 = s(0)$ initial height $v_0 = v(0)$ initial velocity $-g$ = acceleration due to gravity $g \approx 9.8\ m/s^2$ or $g \approx 32\ ft/s^2$	$v(t) > 0$ object rising $v(t) < 0$ object falling $v(t) = 0$ at highest point	

<u>Ex</u> (7) Projectile launched upwards from ground with initial velocity 30 m/s

- Find max height of projectile and at which time it reaches that height.

END OF MATERIAL FOR MIDTERM 1

§3.6: Trig functions

What are $\frac{d}{dx}\sin(x)$ and $\frac{d}{dx}\cos(x)$?

Def of derivative:

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}$$

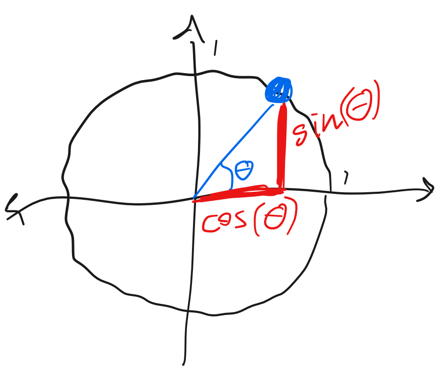
How to evaluate? Two main tools

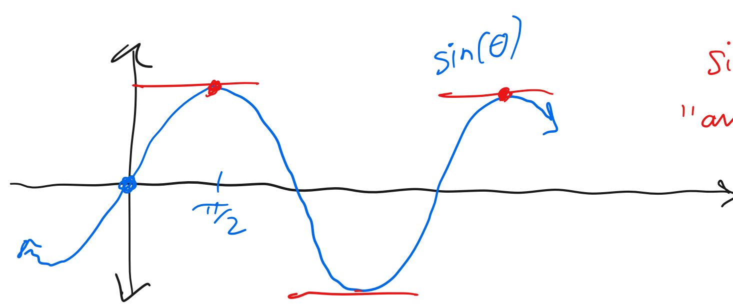
① Trig angle addition formulas (section 1.4 : page 31)

- $\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$
- $\cos(x+h) = \cos(x)\cos(h) - \sin(x)\sin(h)$

② Trig limits

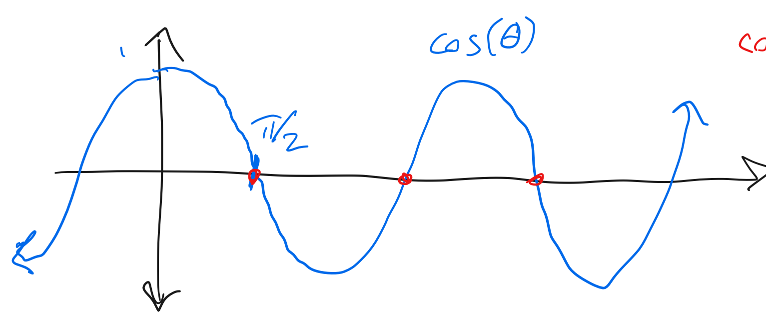
$$\left.\begin{aligned} &\bullet\ \lim_{h\to 0}\frac{\sin(h)}{h} = 1 \\ &\bullet\ \lim_{h\to 0}\frac{1-\cos(h)}{h} = 0 \end{aligned}\right\} \S 2.6$$





$\sin(-\theta) = -\sin(\theta)$

"anti-symmetric"



$\cos(-\theta) = \cos(\theta)$

symmetric

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{[\sin(x)\cos(h) + \sin(h)\cos(x)] - \sin(x)}{h}$$

$$= \lim \left[\frac{\sin(x)\cos(h) - \sin(x)}{h} + \frac{\sin(h)\cos(x)}{h} \right]$$

$$= \lim_{h\to 0}\left[\frac{\sin(x)[\cos(h)-1]}{h}\right] + \lim_{h\to 0}\frac{\sin(h)\cos(x)}{h}$$

$$= \sin(x)\underbrace{\lim_{h\to 0}\frac{\cos(h)-1}{h}}_{0} + \cos(x)\underbrace{\lim_{h\to 0}\frac{\sin(h)}{h}}_{1}$$

$$= \cos(x)$$

$$\Rightarrow \frac{d}{dx}\sin(x) = \cos(x)$$

<u>Exercise</u> Evaluate $\frac{d}{dx}\cos(x) = \lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h} = -\sin(x)$

<u>Theorem</u> 1) $\frac{d}{dx}\sin(x) = \cos(x)$

2) $\frac{d}{dx}\cos(x) = -\sin(x)$

We can find the derivatives of other trig functions from these (using quotient rule).

- $$\frac{d}{dx}\tan(x) = \frac{d}{dx}\frac{\sin(x)}{\cos(x)} = \frac{\cos(x)\cdot\frac{d}{dx}\sin(x) - \sin(x)\cdot\frac{d}{dx}\cos(x)}{\cos^2(x)}$$

$$= \frac{\cos(x)\cos(x) - \sin x(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \leftarrow =1$$

$$= \frac{1}{\cos^2(x)}$$

$$= \boxed{\sec^2(x)}$$

- $$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

- $\frac{d}{dx}\cot(x) = -\csc^2(x)$
- $\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$

double-angle formula for sine: $\sin(2x) = 2\sin(x)\cos(x)$

Notice $\boxed{h(t) = \frac{1}{2}\sin(2x)}$

Ex $h(t) = \sin(t)\cos(t)$. Find $h'(t)$.

$$h'(t) = \frac{d}{dt}\sin(t)\cdot\cos(t) + \sin(t)\cdot\frac{d}{dt}\cos(t)$$

$$= \cos(t)\cdot\cos(t) + \sin(t)\left[-\sin(t)\right]$$

$$= \cos^2(t) - \sin^2(t) \qquad = \boxed{\cos(2t)}$$

(double-angle formula for cos)

Ex $f(x) = \underbrace{x\cdot e^x}_{g}\cdot\underbrace{\sin(x)}_{h}$. Find $f'(x)$.

Product rule

$$f'(x) = g'(x)h(x) + g(x)h'(x)$$

$$= \frac{d}{dx}\left[xe^x\right]\sin(x) + (xe^x)\cdot\cos(x)$$

use prod. rule again

$$= \left(1\cdot e^x + x\cdot e^x\right)\sin(x) + xe^x\cos(x)$$

$$= e^x\left[(1+x)\sin(x) + x\cos(x)\right]$$

Ex Find an equation for the line tangent to

$$f(x) = y = \frac{\sin(x)}{1+\cos(x)} \quad \text{at} \quad x = \pi/3.$$

Two pieces of information define the tangent line:

- slope: $f'(\pi/3)$
- point: $\left(\pi/3, f(\pi/3)\right)$

Tangent line in point-slope form:

$$y - f(\pi/3) = f'(\pi/3)\cdot(x - \pi/3)$$

$$\Rightarrow y = \underbrace{f(\pi/3)}_{1} + \underbrace{f'(\pi/3)}_{2}(x - \pi/3)$$

1 $f(\pi/3) = \dfrac{\sin(\pi/3)}{1+\cos(\pi/3)} = \dfrac{\sqrt{3}/2 \cdot 2}{1 + 1/2 \cdot 2} = \dfrac{\sqrt{3}}{2+1} = \boxed{\dfrac{\sqrt{3}}{3}}$

2 $f'(\pi/3)$. Start by finding $\frac{d}{dx} f(x)$.

$$f'(x) = \frac{[1+\cos(x)]\frac{d}{dx}\sin(x) - \sin(x)\frac{d}{dx}[\overset{\nearrow 0}{\cancel{1}} + \cos(x)]}{[1+\cos(x)]^2}$$

$$= \frac{[1+\cos(x)]\cos(x) - \sin(x)[0 + (-\sin(x))]}{[1+\cos(x)]^2}$$

$$= \frac{\cos(x) + \underbrace{[\cos^2(x) + \sin^2(x)]}_{=1}}{[1+\cos(x)]^2}$$

$$= \frac{\cancel{\cos(x)+1}}{[1+\cos(x)]^{\cancel{2}}}$$

$$= \frac{1}{1+\cos(x)}$$

$$f'(\pi/3) = \frac{1}{1+\cos(\pi/3)} = \frac{1 \cdot 2}{1 + 1/2 \cdot 2} = \frac{2}{2+1} = \boxed{\frac{2}{3}}$$

Tangent line: $y = \dfrac{\sqrt{3}}{3} + \dfrac{2}{3}(x - \pi/3)$