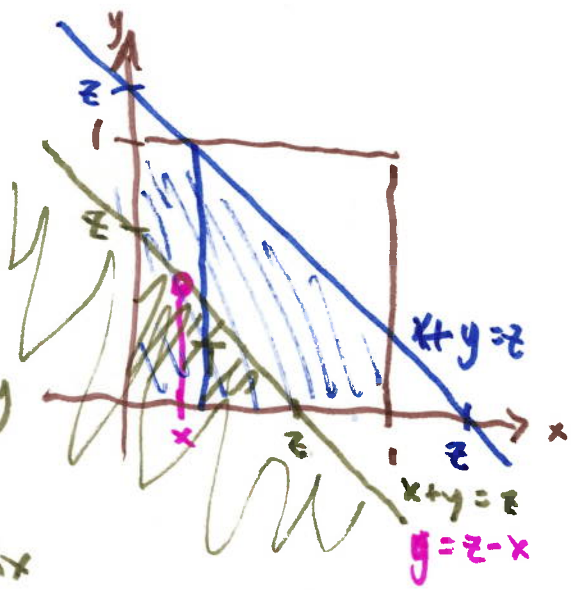
(Example 7.13)

Let X and Y be independent random variables distributed uniformly on $[0, 1]$. What is the probability density function $f_{X+Y}(z)$?



$$f_X(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{else} \end{cases}$$

$$F_{X+Y}(z) = P(X+Y \le z)$$
$$= \iint_{x+y \le z} f_X(x) f_Y(y)\, dx\, dy$$

Case 1 $z \in [0,1]$

$$\int_{x=0}^{z} \int_{y=0}^{z-x} 1 \cdot 1 \, dy \, dx$$
$$= \text{area of } T = \frac{1}{2} \cdot z \cdot z = \frac{1}{2} z^2$$

Case 2 $z \in [1,2]$

$$F_{X+Y}(z) = 1 - \frac{1}{2}(2-z)^2$$

$$\Rightarrow F_{X+Y}(z) = \begin{cases} 0 & \text{if } z < 0 \\ \frac{1}{2} \cdot z^2 & \text{if } z \in [0,1] \\ 1 - \frac{1}{2}(2-z)^2 & \text{if } z \in [1,2] \\ 1 & \text{if } z > 2 \end{cases}$$

$$\Rightarrow f_{X+Y}(z) = \frac{d}{dz} F_{X+Y}(z) = \begin{cases} z & \text{if } z \in [0,1] \\ 2-z & \text{if } z \in [1,2] \\ 0 & \text{else} \end{cases}$$