Name: ______________________________

Problem 1: (1 points) You flip 3 fair coins independently. Let X be the number of heads observed and let Y be the number of tails. Are X and Y independent random variables?
(You do not need to show your work or justify your answers for this problem.)

Choose one:

- ○ Yes
- ● No

Solution: X and Y are not independent, since knowing one determines the other other completely. Concretely, to show that they are not independent, we have (for example) $P(X = 3) = P(HHH) = 1/8$ and $P(Y = 3) = P(TTT) = 1/8$, but $P(X = 3, Y = 3) = 0 \neq P(X = 3) \cdot P(Y = 3)$.

Problem 2: (1 points) If X is an exponential random variable with parameter 2, which of the following is equal to the conditional probability $P(X > 10|X > 9)$?
(You do not need to show your work or justify your answers for this problem.)

Choose one:

- ○ $1 - e^{-2 \cdot 1}$
- ● $P(X > 1)$
- ○ $P(X > 9)$
- ○ $e^{-2 \cdot 10}$

Solution: By the memoryless property of exponential random variables, we have

$$P(X > 9 + 1|X > 9) = P(X > 1).$$

Explicitly, using the CDF of an exponential random variable, this is also equal to $P(X > 1) = 1 - P(X \leq 1) = 1 - (1 - e^{-2 \cdot 1}) = e^{-2 \cdot 1}$.

Problem 3: (2 points) Let X be a normal random variable with mean 1 and variance 4. Which of the following is equal to $P(|X - 1| \geq 2)$?
(You do not need to show your work or justify your answers for this problem.)

Choose one:

- ○ $\Phi(0) + (1 - \Phi(2))$
- ○ $\Phi(1) + \Phi(-1)$
- ● $2 \cdot (1 - \Phi(1))$
- ○ $\Phi(1/2)$

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Solution: You can notice that $P(|X-1| \geq 2)$ is the probability that X is at least one standard deviation from its mean, which is given by $2 \cdot (1 - \Phi(1))$. Or algebraically, using the properties that $\Phi(-x) = 1 - \Phi(x)$ and that $\frac{X-1}{2}$ is a standard normal random variable, we see that

$$\begin{aligned} P(|X-1| \geq 2) &= P(X-1 \geq 2) + P(X-1 \leq -2) \\ &= P\left(\frac{X-1}{2} \geq 1\right) + P\left(\frac{X-1}{2} \leq -1\right) \\ &= \left(1 - P\left(\frac{X-1}{2} \leq 1\right)\right) + \Phi(-1) \\ &= (1 - \Phi(1)) + (1 - \Phi(1)) \\ &= 2 \cdot (1 - \Phi(1)). \end{aligned}$$

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Problem 4: (10 points) Let X and Y be continuous random variables with joint distribution

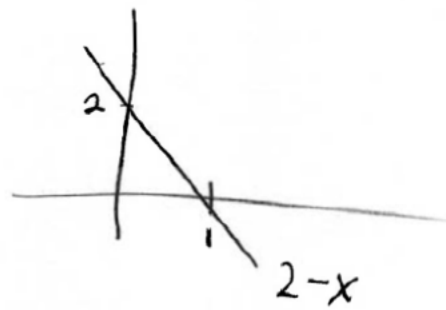
$$f_{X,Y}(x,y) = \frac{6}{7} \cdot y, \quad \text{for } x \in (0,1) \text{ and } y \in (0, x+1)$$

You may express your answers for parts (a) through (c) in terms of explicit but unevaluated integrals.

(a) (3 points) Compute $F_{X+Y}(1)$.

$X+Y=1$

$$F_{X+Y}(1) = \int_{x=0}^{1} \int_{0}^{1-x} \frac{6}{7} \cdot y \, dy \, dx$$



Answer:

$$\int_{x=0}^{1} \int_{0}^{1-x} \frac{6}{7} \cdot y \, dy \, dx$$

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Problem 4 (continued) $x = 1/2$

(b) (3 points) Compute $P(X \leq \frac{1}{2})$.

$$\int_0^{1/2}\int_0^{x+1} \frac{6}{7} y \, dy\, dx$$

$$= \int_0^{1/2} \frac{6}{7}\frac{y^2}{2}\Big|_0^{x+1} dx = \int_0^{1/2} \frac{6y^2}{14}\Big|_0^{x+1} dx = \int_0^{1/2} \frac{6(x+1)^2}{14}\, dx$$

$$= \frac{6}{14}\int_0^{1/2} \cancel{6}(x^2+2x+1)\, dx = \frac{6}{14}\int_0^{1/2} x^2+2x+1\, dx$$

$$= \frac{6}{14}\left[\frac{x^3}{3} + x^2 + x\right]_0^{1/2}$$

$$= \frac{\overset{3}{\cancel{6}}}{14}\left[\frac{1}{24} + \frac{1}{4} + \frac{1}{2}\right]$$

$$\frac{6}{24} \quad \frac{12}{\cancel{24}} \quad \frac{19}{\cancel{24}} \; 12$$

Answer:

$$\boxed{\int_0^{1/2}\int_0^{x+1} \frac{6}{7} y \, dy\, dx \simeq \frac{6}{14}\left(\frac{19}{24}\right)}$$

Problem 4 (continued)

(c) (2 points) Find the marginal PDF of X.

$$f_X(x) = \int_0^{x+1} \frac{6}{7} y \, dy$$

$$= \begin{cases} \frac{3}{7}(x+1)^2 & x \in [0,1] \\ 0 & \text{else} \end{cases}$$

Answer:

$$f_X(x) = \begin{cases} \frac{3}{7}(x+1)^2 & x \in [0,1] \\ 0 & \text{else} \end{cases}$$

(d) (2 points) Are X and Y independent? Why or why not?

No, bc the area of integration isn't a rectangle.

Answer:

No, the area of integration isn't a rectangle.

Problem 5: (10 points) Let X, Y, and Z be independent exponential random variables with parameter 2.

You may express your answers for this problem in terms of explicit but unevaluated integrals; however, your answer for part (b) should not contain any unevaluated "max" terms.

(a) (2 points) Find $E(X^2Y)$.

$X, Y \sim Exp(2)$

Since X, Y indep, $f(x,y) = f_X(x) f_Y(y) = \begin{cases} (2e^{-2x})(2e^{-2y}) & x, y \geq 0 \\ 0 & \text{else} \end{cases}$

Answer:

$$\boxed{\mathbb{E}[X^2Y] = \int_0^\infty \int_0^\infty x^2 y (2e^{-2x})(2e^{-2y})\, dx\, dy}$$

<u>*Alternate solution to 5(a):*</u>

Problem 5: (10 points) Let X, Y, and Z be independent exponential random variables with parameter 2.

You may express your answers for this problem in terms of explicit but unevaluated integrals; however, your answer for part (b) should not contain any unevaluated "max" terms.

(a) (2 points) Find $E(X^2Y)$.

since X, Y independent

$$\mathbb{E}(X^2Y) = E(X^2) \cdot E(Y)$$

$$= (Var(X) + E(X^2)) \cdot E(Y)$$

with parameter 2.

$$= \left(\frac{2}{\lambda^2}\right) \cdot \left(\frac{1}{\lambda}\right) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Answer:

$$\boxed{\frac{1}{4}}$$

(b) (4 points) Find the probability $P(\max(X, Y) \leq Z)$.

Let a, b, c be random real numbers such that $a > b > c$

For X, Y, Z, there are $3! = 6$ options for X, Y, Z

X, Y, Z

$a\ b\ c$

$a\ c\ b$

$b\ a\ c$

$b\ c\ a$

$c\ a\ b$

$c\ b\ a$

If we want $\max(X,Y) \leq Z$, Then Z must be a

$P(\max(X,Y) \leq Z)$

$= \frac{2}{6}$

$= \boxed{\frac{1}{3}}$

Very creative! This is not 100% correct, but can be made into a correct solution once we justifiy that the ordering produced by three exponential random variables is uniform.

Answer:

$\frac{1}{3}$

Alternate solution to 5(b):

(b) (4 points) Find the probability $P(\max(X,Y) \leq Z)$.

$$f_{X,Y,Z}(x,y,z) = \begin{cases} 8e^{-2x-2y-2z}, & x,y,z \geq 0 \\ 0, & \text{else} \end{cases}$$

$$P(\max(X,Y) \leq Z) = P(X \leq Y \leq Z \cap Y < X \leq Z)$$

$$= P(X \leq Y \leq Z) + P(Y < X \leq Z)$$

$$= \iiint_{X \leq Y \leq Z} f_{X,Y,Z}(x,y,z)\,dx\,dy\,dz + \iiint_{Y < X \leq Z} f_{X,Y,Z}(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^\infty \int_0^z \int_0^y 8e^{-2x-2y-2z}\,dx\,dy\,dz + \int_0^\infty \int_0^z \int_0^x 8e^{-2x-2y-2z}\,dy\,dx\,dz$$

Answer:

$$\boxed{\int_0^\infty \int_0^z \int_0^y 8e^{-2x-2y-2z}\,dx\,dy\,dz + \int_0^\infty \int_0^z \int_0^x 8e^{-2x-2y-2z}\,dy\,dx\,dz}$$

5c Solution: Notice that $X - Y = X + (-Y)$, so we can use a formula for the sum of two random variables. Since X and $-Y$ are independent, we can use the convolution formula and we find

$$f_{X-Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_{-Y}(z-x) dx$$

where f_{X-Y} is the density of the difference, X has density

$$f_X(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

and $-Y$ has density

$$f_{-Y}(y) = \begin{cases} 2e^{2y} & y \leq 0 \\ 0 & y > 0 \end{cases}.$$

Now we have to evaluate the integral. The key thing to note here is that X can take any nonnegative value and $-Y$ can take any nonpositive value. So $X+(-Y)$ can take on *any* value in $(-\infty, \infty)$ and our density must reflect that fact. We therefore break the integral into two cases. If $z \leq 0$, then $f_X(x) f_{-Y}(z-x)$ is nonzero for any $x \geq 0$ as the first term is always positive and the second term is always negative. However, if $z > 0$, then we must have $x \geq z$ for $z - x$ to be negative. Thus we have

$$\begin{aligned} f_{X-Y}(z) &= \int_{-\infty}^{\infty} f_X(x) f_{-Y}(z-x) dx \\ &= \begin{cases} \int_0^{\infty} f_X(x) f_{-Y}(z-x) dx & z \leq 0 \\ \int_z^{\infty} f_X(x) f_{-Y}(z-x) & z \geq 0 \end{cases} \\ &= \begin{cases} \int_0^{\infty} 2e^{-2x} 2e^{2(z-x)} dx & z \leq 0 \\ \int_z^{\infty} 2e^{-2x} 2e^{2(z-x)} dx & z \geq 0 \end{cases} \end{aligned}$$

Now we evaluate each integral individually. For $z \leq 0$.

$$\begin{aligned} \int_0^{\infty} 2e^{-2x} 2e^{2(z-x)} dx &= e^{2z} \int_0^{\infty} 4e^{-4x} dx \\ &= e^{2z} \left(-e^{-4x}\right) \Big|_{x=0}^{\infty} \\ &= e^{2z} \end{aligned}$$

Now for $z \geq 0$

$$\begin{aligned} \int_z^{\infty} 2e^{-2x} 2e^{2(z-x)} dx &= e^{2z} \int_z^{\infty} 4e^{-4x} dx \\ &= e^{2z} \left(-e^{-4x}\right) \Big|_{x=z}^{\infty} \\ &= e^{-2z} \end{aligned}$$

So our final density is

$$f_{X-Y}(z) = \begin{cases} e^{2z} & z \leq 0 \\ e^{-2z} & z > 0 \end{cases}$$

Equivalently, we could also write $f_{X-Y}(z) = e^{-2|z|}$