Math 180A Quiz 3 Solutions

Dr. Gwen McKinley, Eva Loeser, Nicholas Sieger

November 5, 2021

1. Let x be a continuous random variable with the following probability density function:

$$f(x) = \frac{1}{\pi(1+x^2)}, \text{ for all } x \in (-\infty, \infty)$$

Which of the following is equal to $P((|X| \geq 1))$

- ☐ $\int_1^\infty \frac{1}{\pi(1+x^2)} dx$
- ☑ $\int_{-\infty}^{-1} \frac{1}{\pi(1+x^2)} dx + \int_1^\infty \frac{1}{\pi(1+x^2)} dx$
- ☐ $\int_{-\infty}^{\infty} |x| \cdot \frac{1}{\pi(1+x^2)} dx$

Solution: If $|X| \geq 1$, then either $X \geq 1$ *or* $X \leq -1$. Hence,

$$\begin{aligned} P((|X| \geq 1)) &= P(X \geq 1 \cup X \leq -1) \\ &= P(X \geq 1) + P(X \leq -1) \\ &= \int_{-\infty}^{-1} \frac{1}{\pi(1+x^2)} dx + \int_1^\infty \frac{1}{\pi(1+x^2)} dx \end{aligned}$$

where we use the fact that the events $X \geq 1$ and $X \leq -1$ are disjoint.

2. Suppose that the average score in a soccer game in the Premier League is 2.69 goals, and that scores follow a Poisson distribution. If there are 380 total games per season, on average how many games in a season will have 0 goals?

- ☑ $380 \cdot e^{-2.69} \frac{(2.69)^0}{0!}$
- ☐ $\left(e^{-2.69} \frac{(2.69)^0}{0!}\right)^3 80$
- ☐ $e^{-380} \frac{(380)^0}{0!}$
- ☐ $e^{-2.69} \frac{(2.69)^0}{0!}$

Solution: Let us compute the expected number of games with zero goals using indicators. Let G be the total number of games in the season with zero goals. Let $Z_1, \ldots, Z_{380}$ be the indicators that each of the 380 games has zero goals. Then,

$$\mathbb{E}[G] = \mathbb{E}[Z_1 + \cdots + Z_{380}] = \mathbb{E}[Z_1] + \cdots + \mathbb{E}[Z_{380}] = 380\mathbb{E}[Z_1]$$

as each indicator is identically distributed. Since we assume that the number of goals in a single game follows a Poisson distribution, we have:

$$\mathbb{E}[Z_1] = e^{-\lambda} \frac{\lambda^0}{0!}$$

where λ is the parameter of the Poisson distribution. Since the expected value of a Poisson random variable with parameter λ is exactly λ, and the expected number of goals is 2.69, it follows that

$$\mathbb{E}[Z_1] = e^{-2.69} \frac{(2.69)^0}{0!}$$

and the expected number of games with zero goals is

$$380 \cdot e^{-2.69} \frac{(2.69)^0}{0!}$$

. Thus the first choice is correct.[1]

We can also see that the second answer is the *probability* that every one of the 380 games has no goals. The third answer is the probability that a game has no goals if the expected number of goals in a game is 380. The fourth answer is the probability that a single game has no goals.

[1]Numerically, this value ≈ 25.79. For the 2020-2021 season, the actual number of $0-0$ games was 30 (https://www.myfootballfacts.com/stats/premier-league-by-season/premier-league-2020-21/premier-league-table-results-2020-21/)

3. In each election in a certain country, the country clerk is responsible for the determining whether candidates from party A or from Party B will appear first on the ballot (the party appearing first tends to have a small advantage in the vote). For each election, he chooses uniformly at random which of the two parties will be listed first. Suppose that out of 41 elections, Party A appears first on the ballot 40 times. [2]

   If the county clerk is not cheating, and did indeed choose uniformly at random each time, what is the probability that Party A would appear first **40 or more times** out of 41 total?

   Solution:

   Since the county clerk choose uniformly at random each time, the probability party A appears first in 1 election $p = \frac{1}{2}$.

   ~~Let X be a binomial random variable~~

   ~~P(party A would appear first~~

   Let X be the number of times that party A appears first.

   $$X \sim \text{Bino}\left(41, \tfrac{1}{2}\right)$$

   $$\begin{aligned} P(X \geq 40) &= P(X=40) + P(X=41) \\ &= \binom{41}{40} \cdot \left(\tfrac{1}{2}\right)^{40} \cdot \left(1-\tfrac{1}{2}\right)^{(41-40)} + \binom{41}{41} \cdot \left(\tfrac{1}{2}\right)^{41} \cdot \left(1-\tfrac{1}{2}\right)^{(41-41)} \\ &= 41 \cdot \left(\tfrac{1}{2}\right)^{41} + \left(\tfrac{1}{2}\right)^{41} \\ &= 42 \cdot \left(\tfrac{1}{2}\right)^{41} \end{aligned}$$

   Answer:

   $$\boxed{42 \cdot \left(\tfrac{1}{2}\right)^{41}}$$

[2] Sometimes truth is stranger than fiction. . . https://www.nytimes.com/1985/07/28/nyregion/politics.html

4. Dr. Gwen decides tonight is the night she will beat the Hotel Ghosts level of Super Mario Sunshine, no matter how long it takes her. Suppose that on each attempt she beats the level with probability $\frac{1}{50}$ (independently of all other attempts). Suppose also that it takes her 10 minutes to turn on and start the game (not including the time to make the first attempt), and that each attempt takes her 5 minutes. Let T be the total amount of time it takes her (in minutes) to beat the level.

(a) Find the probability she beats the level in 1 hour of less.

Let X be # of attempts it takes for her to beat the level

$T = g(X) = 10 + 5X$

$$\mathbb{P}(T \le 60) \underset{\text{>60 min per hour}}{=} \mathbb{P}(10 + 5X \le 60) = \mathbb{P}(5X \le 50) = \mathbb{P}(X \le 10) \Rightarrow$$

Note: $X \sim Geom(\frac{1}{50})$

$$\mathbb{P}(X \le 10) = 1 - \underbrace{\mathbb{P}(X > 10)}_{\text{fails at least 10 times}} = 1 - \overbrace{(1 - \tfrac{1}{50})}^{\text{Prob of failing}}{}^{10} = 1 - (\tfrac{49}{50})^{10}$$

Answer:

$$\boxed{1 - (\tfrac{49}{50})^{10}}$$

(b) Find the standard deviation of T.

(b) (2 points) Find the standard deviation of T.

Since $X \sim Geom(\frac{1}{50})$, $Var(X) = \dfrac{1 - \frac{1}{50}}{(\frac{1}{50})^2} = 49 \times 50$

$T = 10 + 5X$, and hence

$Var(T) = 25\, Var(X)$

$= 25 \times 49 \times 50$

$\sigma = \sqrt{Var(T)}$ $\cancel{= 25 \times 7 \times \sqrt{2}}$

$= 25 \times 7 \times \sqrt{2}$

$= 175\sqrt{2}$

Answer:

$$\boxed{175\sqrt{2}}$$

5. Let X be a (continuous) uniform random variable on the interval $[0, 2]$.

(a) Find the cumulative distribution function $F_X(t)$.

Problem 5: (10 points) Let X be a (continuous) uniform random variable on the interval $\overset{a\ \ b}{[0, 2]}$.

(a) (3 points) Find the cumulative distribution function $F_X(t)$.

CDF

$$F_X(t) = \begin{cases} 0, & t < a \\ \frac{t-a}{b-a}, & t \in [a,b] \\ 1, & t > b \end{cases} = \begin{cases} 0, & t < 0 \\ \frac{t-a}{2}, & t \in [0,2] \\ 1, & t > 2 \end{cases}$$

Answer:

$$F_X(t) = \begin{cases} 0, & t < 0 \\ \frac{t}{2}, & t \in [0,2] \\ 1, & t > 2 \end{cases}$$

(b) Find $\mathbb{E}\left[\cos(X^2)\right]$. (You can leave your answer as an unevaluated integral.)

$$\mathbb{E}(\cos(X^2)) = \int_{-\infty}^{\infty} \cos(x^2)\, f(x)\, dx = \int_{-\infty}^{0} \cancel{0}\, dx + \int_0^2 \frac{1}{2}\cos^2(x)\, dx + \int_2^{\infty} 0\, dx$$

Answer:

$$\mathbb{E}(\cos(X^2)) = \int_0^2 \frac{1}{2}\cos^2(x)\, dx$$

(c) Define a random variable Z as follows. You flip a fair coin; if the flip is heads, you choose Z uniformly at random from the interval $[0,1]$, and if the flip is tails, you choose Z uniformly at random from the interval $[0,2]$. Find the probability density function $f_Z(x)$.

$\frac{1}{2}$ H $\to [0,1]$

$\frac{1}{2}$ T $\to [0,2]$

$f_Z(x) = ?$

if Head

$$f_Z(x) = \begin{cases} 0 & \text{if } x<0 \\ 1 & \text{if } x \in [0,1] \\ 0 & \text{if } x>1 \end{cases}$$

if tail

$$f_Z(x) = \begin{cases} 0 & \text{if } x<0 \\ \frac{1}{2} & \text{if } x \in [0,2] \\ 0 & \text{if } x>2 \end{cases}$$

for $x<0 \to f_Z(x) = 0$

for $x \in [0,1] \to f_Z(x) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$

for $x \in (1,2] \to f_Z(x) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

for $x > 2 \to f_Z(x) = 0$

Answer:

$$f_Z = \begin{cases} 0 & \text{if } x<0 \\ \frac{3}{4} & \text{if } x \in [0,1] \\ \frac{1}{4} & \text{if } x \in [0,2] \\ 0 & \text{if } x>2 \end{cases}$$

Alternate Solution: We can also use the CDF derived in part (a) and differentiate. Let X be uniform on $[0,1]$ and Y be uniform on $[0,2]$. By part (a), we have

$$\mathbb{P}[X \le t] = \begin{cases} 0 & t<0 \\ t & 0 \le t \le 1 \\ 1 & t>1 \end{cases}$$

$$\mathbb{P}[Y \le t] = \begin{cases} 0 & t<0 \\ t/2 & 0 \le t \le 2 \\ 1 & t>2 \end{cases}$$

Thus we can use the Law of Total Probability to find the CDF for Z as follows:

$$\begin{aligned}
\mathbb{P}[Z \le t] &= \mathbb{P}\left[Z \le t|\text{heads}\right]\mathbb{P}[\text{heads}] + \mathbb{P}\left[Z \le t|\text{tails}\right]\mathbb{P}[\text{tails}] \\
&= \mathbb{P}[X \le t]\frac{1}{2} + \mathbb{P}[Y \le t]\frac{1}{2} \\
&= \begin{cases} 0 & t < 0 \\ (t + \frac{t}{2})\frac{1}{2} & 0 \le t \le 1 \\ \frac{t}{2}\frac{1}{2} & 1 \le t \le 2 \\ 1 & t > 2 \end{cases} \\
&= \begin{cases} 0 & t < 0 \\ \frac{3t}{4} & 0 \le t \le 1 \\ \frac{t}{4} & 1 \le t \le 2 \\ 1 & t > 2 \end{cases}
\end{aligned}$$

Now we may find the pdf by differentiating. Note that we must take a derivative separately in each of the four intervals. Hence,

$$\begin{aligned}
f_Z(t) &= \frac{d}{dt}\begin{cases} 0 & t < 0 \\ \frac{3t}{4} & 0 \le t \le 1 \\ \frac{t}{4} & 1 \le t \le 2 \\ 1 & t > 2 \end{cases} \\
&= \begin{cases} \frac{d}{dt}0 & t < 0 \\ \frac{d}{dt}\frac{3t}{4} & 0 \le t \le 1 \\ \frac{d}{dt}\frac{t}{4} & 1 \le t \le 2 \\ \frac{d}{dt}1 & t > 2 \end{cases} \\
&= \begin{cases} 0 & t < 0 \\ \frac{3}{4} & 0 \le t \le 1 \\ \frac{1}{4} & 1 \le t \le 2 \\ 0 & t > 2 \end{cases}
\end{aligned}$$