(Example 7.8)

Let X and Y be independent:

- X is normal with mean μ_1 and variance σ_1^2
- Y is normal with mean μ_2 and variance σ_2^2

Find the probability density function $f_{X+Y}(z)$.

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}\,dx = 1$$

$$f_{X+Y}(z) = \int_{x=-\infty}^{\infty} f_X(x)\, f_Y(z-x)\,dx$$

$$= \int_{x=-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(z-x-\mu_2)^2}{2\sigma_2^2}}\,dx$$

complete the square

$= \text{CALCULUS}$

$$= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} e^{-\frac{(z-(\mu_1+\mu_2))^2}{2(\sigma_1^2+\sigma_2^2)}}$$

PDF of $\mathcal{N}(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

Note $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y) = \mu_1+\mu_2$

$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) = \sigma_1^2+\sigma_2^2$