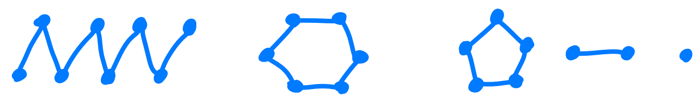
Brooks' Thm Let G be a connected graph with max degree Δ. If G is not a complete graph or an odd cycle, then:

$$\chi(G) \le \Delta.$$

Easy case $\Delta \le 2$. **DONE** (last time)



Remaining $\Delta \geq 3$.

Let G be a connected graph with max degree $\Delta \geq 3$ that is not a complete graph (i.e., $G \neq K_{\Delta+1}$).

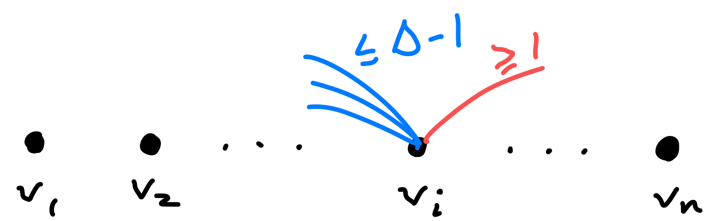
(odd cycle has $\Delta = 2$, so not an option)

Want to show: $\chi(G) \le \Delta$.

Idea do greedy coloring, try to save just a little: remember, greedy coloring always uses $\leq \Delta + 1$ colors, so we only need to reduce by one!

Strategy Do greedy coloring on some vtx order $v_1, v_2, \ldots, v_n$. "Dream ordering" would be:

$|V(G)| = n$



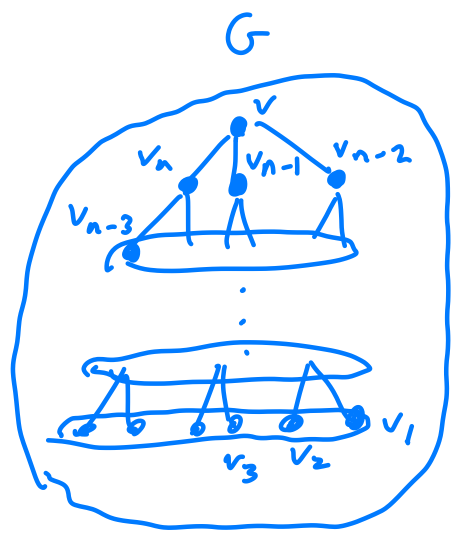
so that each v_i has $\leq \Delta - 1$ neighbors among $v_1, v_2, \ldots, v_{i-1}$ ("backward nbrs").

Then at step i, there is ≥ 1 available color (from among Δ colors) that can be used for v_i.

Surprisingly, this "dream order" is almost possible!

Lemma ✪ For any connected graph G n vtxs, and any $v \in V(G)$, there is a vtx ordering $v_1, v_2, \ldots, v_n = v$ so that v_i has $\leq d_G(v_i) - 1$ nbrs among $v_1, \ldots, v_{i-1}$, for all i <u>except $i = n$</u>.

Pf idea Do BFS (or DFS) order, with v as the root, then reverse the ordering, to get $\underbrace{v_1}_{\text{last vtx added to tree}}, v_2, \ldots, \underbrace{v_{n-1}}_{2^{\text{nd}}\text{ vtx added to tree}}, \underbrace{v_n = v}_{\text{root}}$

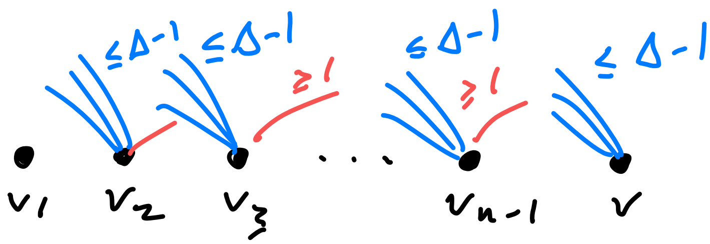


Every vtx v_i other than the root has ≥ 1 "ancestor" in the tree.
$\Rightarrow v_i$ has ≥ 1 nbr <u>later</u> in the ordering, i.e. in $v_{i+1}, \ldots, v_n$.

$\Rightarrow v_i$ has $\leq d_G(v_i) - 1$ nbrs in $v_1, v_2, \ldots, v_{i-1}$.

But what to do about root vtx?

If there is a vtx $v \in V(G)$ with degree $\leq \Delta - 1$, apply lemma ⊛, then do greedy coloring $\Rightarrow$ uses $\leq \Delta$ colors.

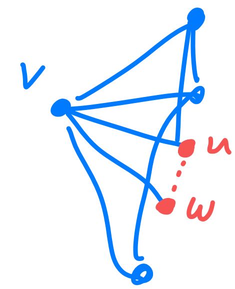


But if <u>every</u> vtx has degree Δ, we're out of luck ☹ Need to "save a color" some other way for last vtx v.

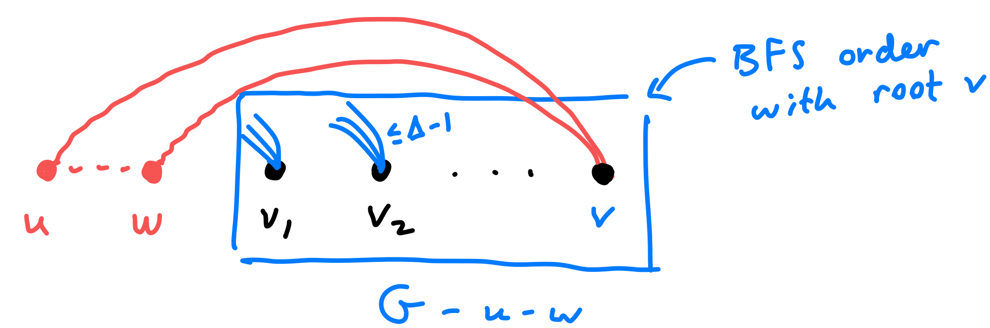
Still haven't used the assumption that $G \neq K_{\Delta+1}$!

Take any vtx v, has degree Δ. Since $G \neq K_{\Delta+1}$, v must have a "missing edge" u, w in its nbhd.

(if every two nbrs of v were adjacent, get $K_{\Delta+1}$.)



<u>Trick</u> apply Lemma ⊛ to $G-u-w$ using v as the root.



Then put u, w at the <u>beginning</u>, and greedily color.

- Since u, w not adjacent, both colored with 1^{st} color.

- For each subsequent vtx before v, $\leq \Delta - 1$ nbrs earlier in ordering, $\Rightarrow \geq 1$ color a ailable.

- And for v, Δ nbrs, but $\leq \Delta - 1$ colors used on nbrs of v (Since u, w have <u>same</u> color) $\Rightarrow \geq 1$ color available.

$\Rightarrow$ colored G with $\leq \Delta$ colors! :)

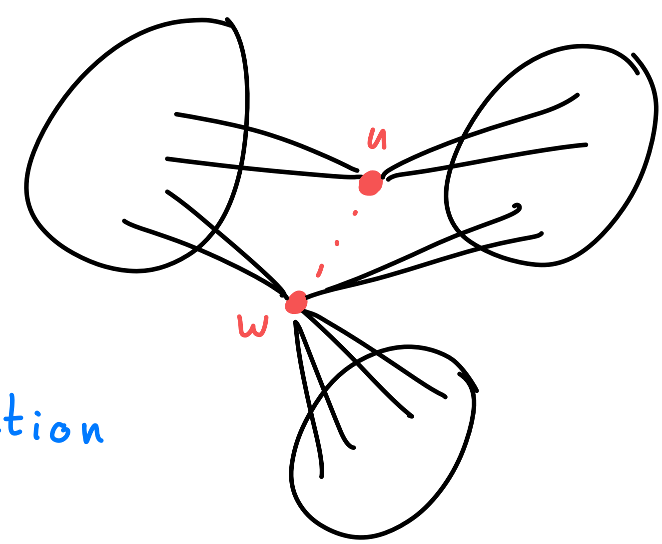
⚠ Not quite done though! There was a subtle flaw in this argument. Can you spot it?

To apply Lemma ⊛ to $G-u-w$ (i.e. to build BFS tree), need $G-u-w$ connected!

What if $G-u-w$ is disconnected?

Then G is made of nearly disconnected "chunks". Maybe we can color the chunks separately, then fuse the colorings @ u and w?

Feels like a divide & conquer algorithm! ⇒ use induction



This is what we'll do!

Note we will probably skip the next two pages in class and go straight to the "remaining cases" page, which is the heart of the induction.

Pf of Brooks' Thm.

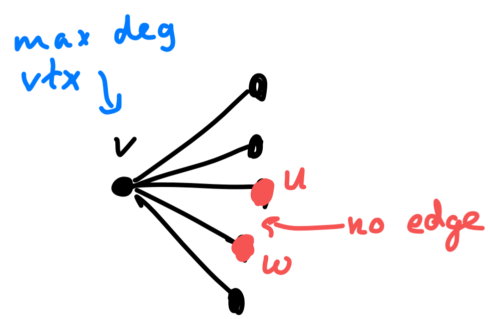
Strategy: Fix $\Delta \geq 3$ and induct on n. $= |V(G)|$

WTS $\forall$ connected G on n vtxs with max deg Δ and $G \neq K_{\Delta+1}$, $\chi(G) \leq \Delta$.

Base step: what's the smallest # of vtxs in a graph with max deg Δ?

A $\Delta+1$ vtxs.

(max deg vtx + nbhd)



Since $G \neq K_{\Delta+1}$,
$\exists$ u, w not adjacent.

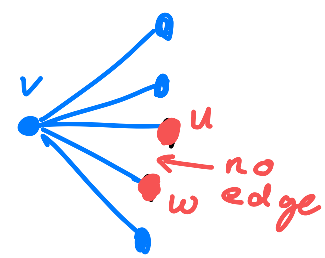
$\Rightarrow$ color u, w same color, this leaves $\Delta - 1$ colors (out of Δ colors total) for the $\Delta - 1$ remaining vtxs.

$\Rightarrow$ can color G with $\leq \Delta$ colors. ✓

Inductive step assume every connected G on n vtxs with max deg Δ and $G \neq K_{\Delta+1}$ has $\chi(G) \leq \Delta$.

Let G be a connected graph on $n+1$ vtxs with max deg Δ and $G \neq K_{\Delta+1}$.
WTS $\chi(G) \leq \Delta$.

Take a vtx v of degree Δ.
Since $G \neq K_{\Delta+1}$, v has nbrs u, w that are not adjacent.

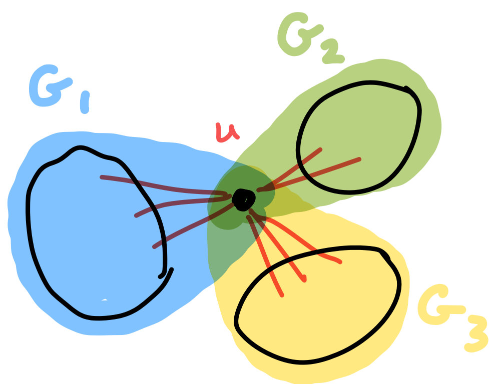


Already showed theorem is true if $G-u-w$ is connected. ← BFS trick

(Also showed true if $\exists$ vtx of degree $< \Delta$, but we don't actually need/use that here.)

Remaining cases:

(1) Removing one of u, v disconnects G. (WLOG, u)



G_i = i^{th} component of $G - u$, together with u.

technicalities: check $G_i \neq K_{\Delta+1}$. And if $\Delta(G_i) < \Delta$, color with greedy algorithm instead of I.H.

By inductive hypothesis, can color each G_i using $\leq \Delta$ colors.

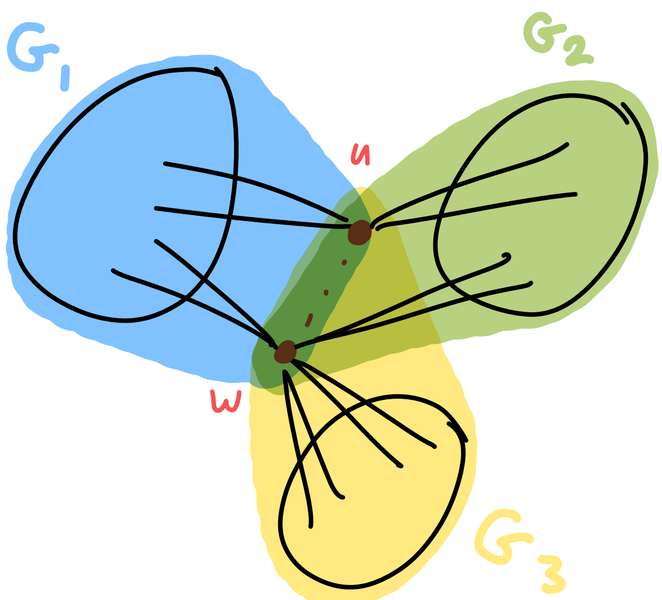
Then in each G_i, permute the colors so that u is colored with color 1.

(colorings now match/agree at u)

$\Rightarrow$ merge colorings to get a coloring of G with $\leq \Delta$ colors.

Remaining case: similar strategy to 1 but a bit trickier since colorings will have to agree on u *and* w.

(2) Removing **both** u, w disconnets G, but removing just one doesn't.



G_i = ith component of $G-u-v$ together with u & v.

We could cololor each G_i by the inductive hypothesis, but it's possible that u, v would get same color as each other in one G_i and different colors in another ⇒ no way to merge colorings ☹

Clever trick notice: $\deg(u), \deg(v) < \Delta$ in *each* G_i.

⇒ if we add edge $\{u,v\}$, then $G_i + \{u,v\}$ still has max deg $\leq \Delta$.

$\Rightarrow$ we can color each $G_i + \{u,v\}$ using $\leq \Delta$ colors (by inductive hyp). And u, v are assigned <u>different</u> colors (since they are adjacent in $G_i + \{u,v\}$).

$\Rightarrow$ can permute colors in each $G_i + \{u,v\}$ so $u =$ color 1, $v =$ color 2. Then we can merge the colorings (and delete edge $\{u,v\}$), to obtain a coloring of G with $\leq \Delta$ colors.

☺