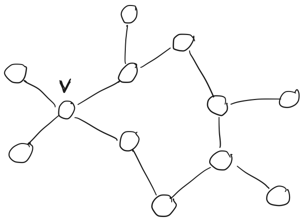
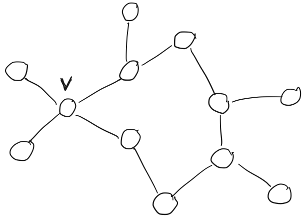
**5-Color Theorem** (7.3.1) *Every planar graph is 5-colorable.*

**Proof idea.** Induction:

- Assume planar graphs on $k$ vertices are 5-colorable.
- To 5-color a planar graph on $k+1$ vertices, first color a $k$-vertex subgraph (induction hypothesis), then strategically "switch" colors to free up a color for the last vertex.

*Useful tool:* (a,b)-**Kempe chain** containing vertex $v$ = maximal connected subgraph with only vtxs of colors **a** and **b** containing $v$.



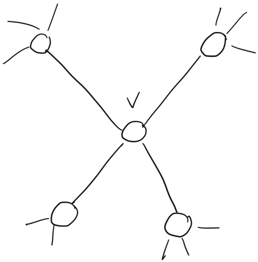
*Proof of 5 Color Theorem.*

**Induction base case(s):** If $G$ is a planar graph with $\leq 5$ vertices, can color with $\leq 5$ colors.

**Inductive step:** assume every planar graph with $k$ vertices is 5-colorable, and let $G$ be a planar graph with $k+1$ vertices.

**Know:** $G$ has a vertex of degree $\leq 5$.

**Easy case:** If $G$ has a vertex $v$ of degree $\leq$ 4, delete $v$. Then by induction, $G - v$ is 5-colorable. Put $v$ back in, color it.



**Hard case:** Every vertex in $G$ has degree $\geq 5$.

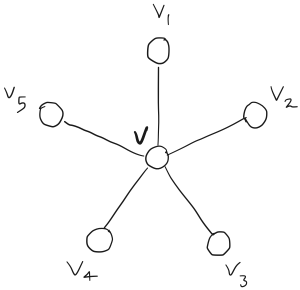
**Still know:** $G$ has a vertex $v$ of degree 5.
And by induction, $G - v$ is 5-colorable.

Color $G - v$ with 5 colors, assume the neighbors $v_1, v_2, v_3, v_4, v_5$ of $v$ all get different colors *(else finish as in easy case)*.

Assume $v_1, v_2, v_3, v_4, v_5$ appear clockwise in this order in our planar embedding of $G$.

Say $v_1$ has color 1 and $v_3$ has color 3.

**Goal:** "switch" $v_1$ to color 3, so color 1 is available for $v$.

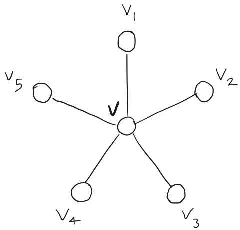


**Setup:** $v_1$ has color 1 and $v_3$ has color 3.

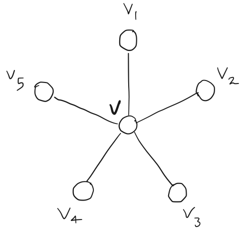
**Goal:** "switch" $v_1$ to color 3, so color 1 is available for $v$.

Consider the (1,3)-Kempe chain containing $v_1$.

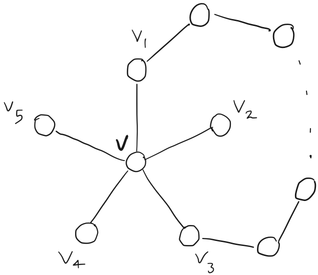
If it **doesn't** contain $v_3$, then switch color 1 and color 3. Now $v_1, v_3$ both have color 3, and we can use color 1 for $v$.




If it **does** contain $v_3$, then there's an alternating path of vertices colored 1 and 3 between $v_1$ and $v_3$.



**Current case:** have an alternating path of vertices colored 1 and 3 between $v_1$ and $v_3$.




**Similar reasoning:** consider $v_2$ and $v_4$, boil down to the case where there is also an alternating path of vertices colored 2 and 4 between $v_2$ and $v_4$.

*Now notice:* these paths must cross. But we are in a **planar** embedding of $G$. Contradiction!!!

Therefore $G$ is 5-colorable ∎

**4-Color Theorem** *Every planar graph is 4-colorable.*

- Proved by Kenneth Appel and Wolfgang Haken in 1976 – first computer-assisted proof of a major result.
- Their strategy: split into cases, use a computer to check them all – more than 1200 hours of computer runtime!
- Some simplifications since then, but still no non-computer-assisted proof!

Cool resources:

- Neat video explaining the flaw in Kempe's original "proof" of the 4-Color Theorem: `https://youtu.be/adZZv4eEPs8`
- Try coloring a randomly-generated map with 4 colors: `https://bit.ly/3PjVHNX`