Math 154 Homework #5

Spring 2022

Due date: **11:59pm** Pacific Time on **Thu, May 12** (via Gradescope)

On the first page of your work, please write a list of everyone with whom you collaborated on this assignment, as well as any outside sources you consulted, apart from the textbook, your notes, and the course staff. If you did not collaborate with anyone, please explicitly write, "No collaborators."

Problem 1.

(a) *(Historical note: this was the original formulation of Hall's Theorem, published in 1935.)*

Let $S_1, S_2, \ldots, S_m$ be sets. These sets have a **system of distinct representatives**, also known as a **transversal**, if we can select $x_1 \in S_1, \ldots, x_m \in S_m$ such that $x_1, x_2, \ldots, x_m$ are all distinct. Prove the following theorem by verifying that the condition below is equivalent to Hall's condition in an appropriately defined bipartite graph:

Theorem. *Sets $S_1, S_2, \ldots, S_m$ have a system of distinct representatives if and only if for every subset $I \subseteq \{1, 2, \ldots, m\}$,*

$$\left| \bigcup_{i \in I} S_i \right| \geq |I|.$$

(b) A standard deck of 52 playing cards is shuffled and then dealt into 13 piles of 4 cards each. Prove that regardless of how the deck is shuffled, there is always a way to select one card from each pile so that *each* of the 13 possible ranks ($A,\ 2,\ 3, \ldots,\ 10,\ J,\ K,\ Q$) occurs once.

(There is an example of a division of the deck into 13 piles on the next page, showing how a card from each pile could be selected.)

Note: you are not required to use part (a) in solving part (b), but you are welcome to do so!

(c) (Bonus – NOT TO BE TURNED IN) Prove that we can in fact go further: we can divide the cards into 4 groups of 13 cards each, where each group has one card from every pile, and one card from every rank.

If you liked this problem, and are curious to see other surprising applications on Hall's Theorem, here is a great list! https://cjquines.com/files/halls.pdf

Division into 13 piles

3♣	4♥	J♠	9♣
J♥	2♣	4♦	5♣
6♦	6♥	6♣	Q♦
7♥	K♣	5♠	6♠
9♠	8♥	10♠	9♦
Q♣	7♦	3♠	9♥
5♦	A♦	J♣	5♥
A♥	7♠	2♠	K♦
K♥	4♠	3♦	4♣
3♥	A♠	10♣	Q♥
A♣	7♣	J♦	2♥
10♥	Q♠	10♦	K♠
8♦	2♦	8♣	8♠

Example for part (b)

3♣	4♥	J♠	9♣
J♥	2♣	4♦	5♣
6♦	6♥	6♣	Q♦
7♥	K♣	5♠	6♠
9♠	8♥	10♠	9♦
Q♣	7♦	3♠	9♥
5♦	A♦	J♣	5♥
A♥	7♠	2♠	K♦
K♥	4♠	3♦	4♣
3♥	A♠	10♣	Q♥
A♣	7♣	J♦	2♥
10♥	Q♠	10♦	K♠
8♦	2♦	8♣	8♠

Example for part (c)

3♣	4♥	J♠	9♣
J♥	2♣	4♦	5♣
6♦	6♥	6♣	Q♦
7♥	K♣	5♠	6♠
9♠	8♥	10♠	9♦
Q♣	7♦	3♠	9♥
5♦	A♦	J♣	5♥
A♥	7♠	2♠	K♦
K♥	4♠	3♦	4♣
3♥	A♠	10♣	Q♥
A♣	7♣	J♦	2♥
10♥	Q♠	10♦	K♠
8♦	2♦	8♣	8♠

Problem 2. Answer parts (a) and (b) of **Question 6.6** at the end of Chapter 6 of the textbook.

Problem 3. Prove that the greedy coloring algorithm always colors a complete bipartite graph with two colors, regardless of the vertex ordering used.