Math 154 Homework #4

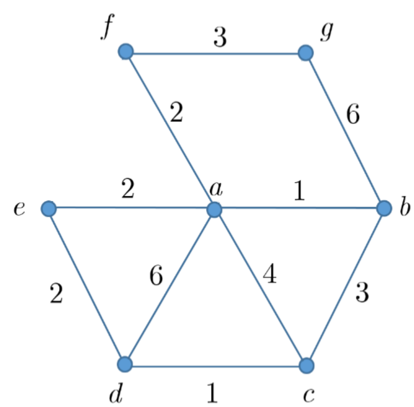
Spring 2022

Due date: **11:59pm** Pacific Time on **Thu, Apr 28** (via Gradescope)

On the first page of your work, please write a list of everyone with whom you collaborated on this assignment, as well as any outside sources you consulted, apart from the textbook, your notes, and the course staff. If you did not collaborate with anyone, please explicitly write, "No collaborators."

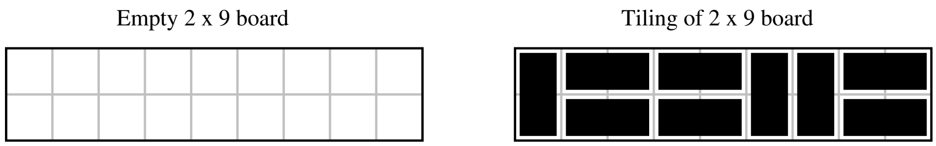
Problem 1. Answer **Question 3.4** at the end of Chapter 3 of the textbook.

Clarification: *For Prim's and Kruskal's Algorithms, specify the order in which the edges are added. For Dijkstra's algorithm, you do not need to show every step, but you should show at least some work to demonstrate that you used Dijkstra's algorithm to solve the problem, rather than solving it by inspection. (For example, you could give the output of the first two iterations of the algorithm, or show a few of the intermediate paths you find while executing the algorithm.) And (reprinted here for your convenience) this is the graph you should reference for this problem:*



Problem 2. Answer **Question 5.15(a)** at the end of Chapter 5 of the textbook.
(There are some small hints on the last page of this document.)

Problem 3. A *tiling* of an $m \times n$ chessboard is a configuration of dominoes which cover all squares on the board exactly once. Each domino covers two adjacent squares, either horizontally or vertically adjacent, and dominoes cannot stick out of the board. Here is a 2×9 example:



This is actually a perfect matching in a graph: the squares of the board are the vertices of a graph, and pairs of adjacent squares are the edges. (An example is shown below.)

(a) For which $m \geq 1$ and $n \geq 1$ does an $m \times n$ chessboard have a tiling? (Justify your answer.)

For parts (b) and (c) of this problem, we will consider $2 \times n$ boards, such as the 2×9 example shown above. Let $T(n)$ be the number of tilings of a $2 \times n$ board, for $n \geq 1$.

(b) Determine the values $T(1)$, $T(2)$, and $T(3)$.

(c) Find a recursive formula for $T(n)$.

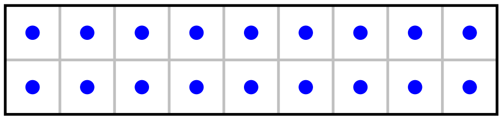
Hint: Consider the ways the first (or last) column of the board can be covered by domino(es), and what that implies about the rest of the board.

Reminder: to define a recursive formula for a function $T(n)$, you should give not only a formula for the function in terms of previous values, but also specify *initial conditions* and the values of n for which the formula holds. For an overview of recurrence relations, see Section 10.2 of these notes: `https://ggc-discrete-math.github.io/induction_recursion.html`

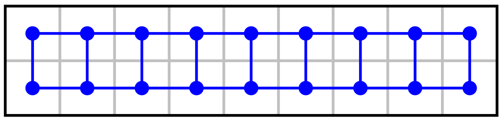
Just for fun (don't turn in): use the recursion to compute $T(4)$, $T(5)$, and $T(6)$ as well. Does this sequence of numbers look familiar?

Example: (illustrating how a domino tiling corresponds to a perfect matching)

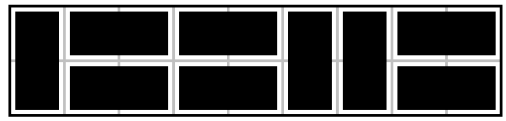
Place blue vertex in each cell of 2 x 9 board



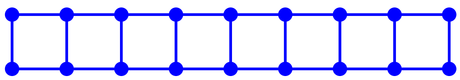
Connect vertices in adjacent cells by blue edges



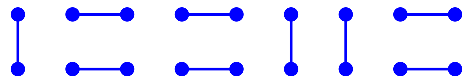
Tiling of 2 x 9 board



Remove original board / keep blue graph



Perfect matching representing above tiling



Note: these "domino tiling" problems display surprisingly rich behavior, and also have applications in physics. Try searching for "The Arctic Circle Theorem" or "Dimer model" if you are interested!

Hints for Problem 2

Hint 1: Try using induction on the number of vertices in the tree.

Hint 2: You may find the result of Problem 1 from the last homework to be helpful!