Math 154 Homework #1

Spring 2022

Due date: **11:59pm** Pacific Time on **Thu, Apr 7** (via Gradescope)

Submission Instructions

1. Log into Gradescope using your `@ucsd.edu` address (either through Canvas or directly), find the course Gradescope page, and select the correct assignment. (Create a Gradescope account linked to your `@ucsd.edu` address if you do not yet have one.)

2. Read the guide for submitting an assignment in Gradescope. (note: this assignment is "variable length" and "not timed")

3. Submit your homework, following the guide in the previous step. Please write the solution for each problem on a page by itself. If submitting as a single PDF, **be sure to "assign pages"** in your submission to the corresponding problems in Gradescope (otherwise the grader may have trouble finding your solution, and you could lose credit for the problem).

On the first page of your work, please write a list of everyone with whom you collaborated on this assignment, as well as any outside sources you consulted, apart from the textbook, your notes, and the course staff. If you did not collaborate with anyone, please explicitly write, "No collaborators."

Problems

Problem 1. Answer parts (a), (b), (c), (d), and (e) of **Question 1.1** at the end of chapter 1 of the textbook. For part (e), only determine whether $K_{1,7}$ is a subgraph of G. You do not need to determine whether K_4 or $K_{2,3}$ are subgraphs of G.

Problem 2. Answer **Question 1.9** at the end of chapter 1 of the textbook.

Problem 3. The Handshaking Lemma, as we have learned it, applies to simple graphs. Assume now that G is a digraph (in which every edge is directed and loops are allowed), but there are no multiedges (i.e., there is at most one directed edge between any two vertices).

By modifying the proof of the Handshaking Lemma to work for G, prove that

$$\sum_{v \in V} d_G^-(v) + \sum_{v \in V} d_G^+(v) = 2|E|.$$

Note: for this graph, the definitions of in- and out-neighborhoods and in- and out-degrees from lecture still apply exactly as given.

Problem 4. Answer **Question 2.11** at the end of chapter 2 of the textbook.

(Hints: there are a couple of possible solutions for this problem; I recommend thinking about it on your own for a while, and trying out some small examples, but if you get stuck, there are hints for one possible solution method on the last page of this document.)

Some additional background: an orientation of a complete graph on n vertices is called a "tournament" because it corresponds to a round-robin tournament (e.g., in chess or boxing or rock-paper-scissors), in which each member of a group of n contestants plays one match against each of the other $n-1$ contestants, and each game results in a win for one player and a loss for the other. For example, an arc $(1, 2)$ could represent a match in which player 1 defeats player 2, while $(2, 1)$ would represent player 2 defeating player 1. The somewhat surprising fact proved in this problem is that, even in games with no element of skill whatsoever, there is always a way to arrange the players in a line so that each player has beaten the next player in the line.

Review resources

Here are review resources on a variety of topics, some of which appear on this homework, and some of which will be used later in the course. You are also encouraged to come and discuss these topics in office hours if you have questions!

I may add resources on other topics later, but these are the ones that many students felt less comfortable with, and/or those that will come up regularly in Math 154 (some others, such as multinomial coefficients and the inclusion-exclusion principle, will probably come up less often).

Counting (permutations, combinations, etc.)

- The counting chapter of Discrete Mathematics (Levin), from LibreTexts, especially Section 1.3 on combinations and permutations.
- Counting, permutations, and combinations from Khan Academy
- Apppendices A.2 and A.3 in our textbook

Proof techniques

Induction and strong induction

- These notes by Matt DeVos (includes helpful graph theory examples!)
- Apppendix A.6 in our textbook

Pigeonhole principle

- This video (gives an overview of the principle, and several nice examples)
- These notes by Veselin Jungic
- Apppendix A.7 in our textbook

Bonus resource

While searching for review resources, I came across this page of nice advice on solving problems in graph theory, written by Matt DeVos at Simon Fraser University. It summarizes a number of great strategies! (Though it may be more useful after you've done a couple of homework assignments and have a feel for the kind of problems we'll be seeing.)

Hints for Problem 4

One possible solution approach for this problem uses induction on the number of vertices, which we will denote by n. Here are some of the steps:

Base step: choose an appropriate value of n, and prove that the result holds for a tournament with this number of vertices.

Inductive step: assume that the result holds for every tournament with n vertices, up to some n (at least the value chosen in the base step). Consider a tournament T with $n+1$ vertices, and remove any vertex v. Apply the induction hypothesis to obtain a directed path $v_1, v_2, \ldots, v_n$ containing every vertex in $T - v$. To show that there is a directed path containing all the vertices in T, try breaking your analysis into the following cases:

Case 1: (v, v_1) is an arc in T.

Case 2: (v_n, v) is an arc in T.

Case 3: neither of the previous conditions hold.