Menger's Theorems

- Vertices (4.5.2): Let u, v be distinct non-adjacent vtxs in a connected graph G. Then

$$\underbrace{\kappa(u,v)}_{\text{min \# of vtxs to separate } u \text{ from } v} = \text{maximum \# of internally vtx-disjoint } uv\text{-paths in } G$$

- Edges (4.5.3): Let u, v be distinct vertices in a connected graph G. Then

$$\underbrace{\lambda(u,v)}_{\text{min \# of edges to separate } u \text{ from } v} = \text{maximum \# of edge-disjoint } uv\text{-paths in } G$$

We'll prove edge form using max-flow min-cut

Notice one direction is simple:

$$\begin{matrix}\text{\# edges needed to} \\ \text{separate } u,v\end{matrix} \;\geq\; \underbrace{\begin{matrix}\text{\# edge-disjoint} \\ uv\text{-paths}\end{matrix}}_{\ell}$$

(If we delete fewer than ℓ edges, some uv-path is untouched, so u, v are still connected)

Goal prove the *other* direction.

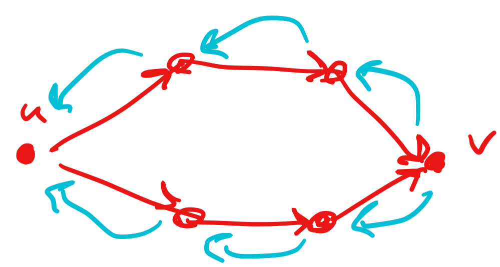
Proof of Menger's Thm, edge form:

Strategy: Given a connected graph G and vtxs u, v, build a directed graph $\vec{G}$ with source u and sink v:

- Replace each edge $\{x,y\}$ in G with two arcs: (x,y) and (y,x).
- Give all arcs capacity 1

flow ≈ disjoint paths

Claim ① $\exists$ (integer) flow f in $\vec{G}$ of value $k \Leftrightarrow \exists$ collection of k edge-disjoint uv-paths in G.



cut ≈ edges separating u from v

Claim ② $\exists$ cut of capacity $\leq k$ in $\vec{G} \Leftrightarrow u$ can be separated from v in G removing some set F of k edges.

If we prove both:

max # edge disjoint uv-paths in G = max flow value in $\vec{G}$

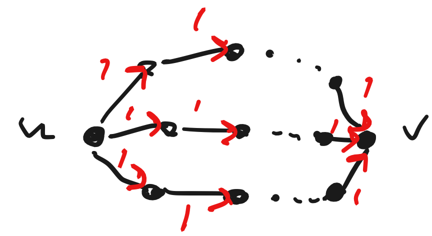
max flow min cut $\rightarrow$ = min cut capacity in $\vec{G}$

= min # edges needed to separate u from v in G

= $\lambda(u,v)$

$\Rightarrow$ DONE.

First prove ①: "$\Leftarrow$" If $\exists$ k edge-disjoint uv-paths in G, route 1 unit of flow along each path to obtain a flow of value k in $\vec{G}$.



Notice since paths are edge-disjoint, this flow doesn't violate capacity constraints.

" $\Rightarrow$ "

Now, given an integer flow f of value k in $\vec{G}$.
(all arcs have flow 0 or 1)

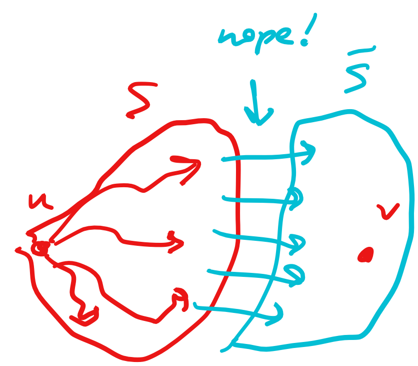
Want k edge-disjoint uv-paths in G.
Start by finding 1 path:

Notice $\exists$ a directed path P from u to v in $\vec{G}$ using arcs of flow 1 in f.

(Since flow is positive. Then gives undirected path in G)

ASK IF THEY WANT TO SEE PF.

Pf $\nexists$ not. Let S be the set of vtxs reachable from u by a directed path using arcs of flow 1.



Since $v \notin S$, $(S, \bar{S})$ is a cut, and by construction, flow $= 0$ on all arcs from $S \to \bar{S}$.

$\Rightarrow v(f) =$ net flow from $S \to \bar{S} = 0$

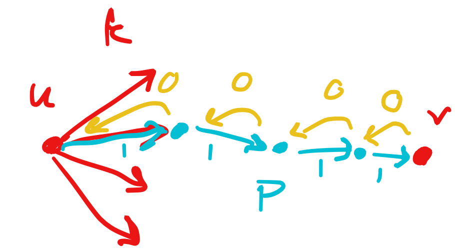
$\lightning$ since $v(f) = k > 0$.

So there is an st-path P in G using arcs of flow 1.

Assume no arcs with $f(x,y)=1$ and $f(y,x)=1$. If so, can set $f(x,y)=0=f(y,x)$ to obtain a flow of the same value ("flow in = flow out" preserved).

Now for all arcs (x,y) in P, delete (x,y) and (y,x) from $\vec{G}$.

Get a flow of value $k-1$ in $\vec{G}_{new}$. Now find a path in $\vec{G}_{new}$



Kirchoff's laws satisfied in $\vec{G}_{new}$

Continue iterating to obtain k uv-paths in $\vec{G}$, which correspond to edge-disjoint uv-paths in G by construction.

This proves Claim 1.

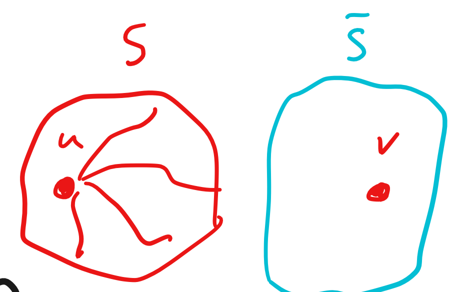
Now, Claim ②: $\exists$ cut of capacity $\le k$ in $\vec{G}$ $\Longleftrightarrow$

u can be separated from v in G removing some set F of k edges.

(Basically like Problem 3 from HW)

"$\Leftarrow$" u is separated from v in $G-F$:

define S as the set of vtxs reachable by paths from u in $G-F$. (Then $v \notin S$.)



Then all edges btwn S and $\bar{S}$ in G must belong to F (none in $G-F$).

Now consider the cut $(S, \bar{S})$ in $\vec{G}$:

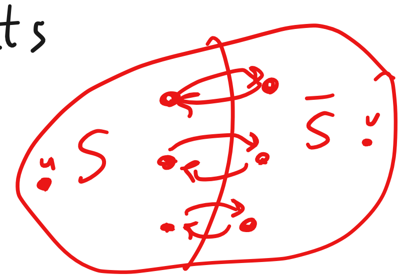
$$c(S,\bar{S}) = \begin{pmatrix}\#\text{edges btwn} \\ S, \bar{S} \text{ in } G\end{pmatrix} \le |F| = k$$

$\Rightarrow$ $(S, \bar{S})$ is a cut of capacity $\le k$.

Now ② "$\Rightarrow$" Have cut, want "separating set"

Consider a cut $(S, \bar{S})$ in $\vec{G}$ of capacity $\leq k$. Since every arc has capacity 1, $c(S, \bar{S})$ is just the # of arcs from S to $\bar{S}$ in $\vec{G}$, which is just the # of edges between S and $\bar{S}$ in G.

Deleting these edges disconnects S from $\bar{S}$ in G, separating u from v.



$\Rightarrow$ u and v can be separated by removing $\leq k$ edges in G.

This proves Claim ②.

As argued before, this completes the proof of Menger's Theorem:

max # edge disjoint paths = max flow

max flow min cut → = min cut

= min # edges needed to separate u from v