# SOME CALCULATIONS FOR THE OUTER HOOKWALK

## Introduction

Given a partition $\mu$ we shall represent it as customary by a Ferrers diagram. We shall use the French convention here and, given that the parts of $\mu$ are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0$, we let the corresponding Ferrer's diagram have $\mu_i$ lattice squares in the $i^{th}$ row (counting from the bottom up). We shall also adopt the Macdonald convention of calling the *arm*, *leg*, *co-arm* and *co-leg* of a lattice square $s$ the parameters $a(s), l(s), a'(s)$ and $l'(s)$ giving the number of cells of $\mu$ that are respectively *strictly* EAST, NORTH, WEST and SOUTH of $s$ in $\mu$. We recall that Macdonald in [11] defines the symmetric function basis $\{P_\mu(x;q,t)\}_\mu$ as the unique family of polynomials satisfying the following conditions

$$a) \quad P_\lambda = S_\lambda + \sum_{\mu<\lambda} S_\mu \, \xi_{\mu\lambda}(q,t)$$

$$b) \quad \langle P_\lambda \, , \, P_\mu \rangle_{q,t} \; = \; 0 \quad for \quad \lambda \neq \mu$$

Where $\langle \, , \, \rangle_{q,t}$ denotes the scalar product of symmetric polynomials defined by setting for the power basis $\{p_\rho\}$

$$\langle p_{\rho^{(1)}} \, , \, p_{\rho^{(2)}} \rangle_{q,t} \; = \; \begin{cases} z_\rho \, \prod_i \frac{1-q^{\rho_i}}{1-t^{\rho_i}} & \text{if } \rho^{(1)} = \rho^{(2)} = \rho \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

where $z_\rho$ is the integer that makes $n!/z_\rho$ the number of permutations with cycle structure $\rho$. Macdonald shows that the basis $\{Q_\mu(x;q,t)\}_\mu$ , dual to $\{P_\mu(x;q,t)\}_\mu$ with respect to this scalar product, is given by the formula

$$Q_\lambda(x;q,t) \; = \; d_\lambda(q,t) \, P_\lambda(x;q,t) \; ,$$

where

$$d_\lambda(q,t) \; = \; \frac{h_\lambda(q,t)}{h'_\lambda(q,t)}$$

and

$$h_\lambda(q,t) \; = \; \prod_{s\in\lambda} (1 - q^{a_\lambda(s)} t^{l_\lambda(s)+1}) \quad , \quad h'_\lambda(q,t) \; = \; \prod_{s\in\lambda} (1 - q^{a_\lambda(s)+1} t^{l_\lambda(s)}) \qquad I.1$$

Macdonald sets

$$J_\mu(x;q,t) \; = \; h_\mu(q,t) P_\mu(x;q,t) \; = \; h'_\mu(q,t) Q_\mu(x;q,t) \; , \qquad I.2$$

and then defines his $q,t$-analogs of the Kostka coefficients by means of an expansion that in $\lambda$-ring notation may be written as

$$J_\mu(x;q,t) \; = \; \sum_\lambda S_\lambda[X(1-t)] \, K_{\lambda\mu}(q,t) \; , \qquad I.3$$

This note is concerned with the modified basis $\{\tilde{H}_\mu(x;q,t)\}_\mu$ defined by setting

$$\tilde{H}_\mu(x;q,t) \; = \; \sum_\lambda S_\lambda(x) \, \tilde{K}_{\lambda\mu}(q,t) \; , \qquad I.4$$

where we have set

$$\tilde{K}_{\lambda\mu}(q,t) \; = \; K_{\lambda\mu}(q,1/t) t^{n(\mu)} \qquad I.5$$

with

$$n(\mu) \;=\; \sum_{s\in\mu} l_\mu(s)$$

To this date it is know that the $K_{\lambda\mu}(q,t)$ (and the $\tilde{K}_{\lambda\mu}(q,t)$ as well) are polynomials in q and t, but it is still a conjectured open problem to show that the coefficients are positive integers. In [2] Garsia and Haiman have conjectured that $\tilde{H}_\mu(x;q,t)$ is in fact (for a given $\mu \vdash n$) the bivariate Frobenius characteristic of a certain $S_n$-module $\mathbf{H}_\mu$ yielding a bigraded version of the left regular representation of $S_n$. In particular this would imply that the expression

$$F_\mu(q,t) \;=\; \sum_{\lambda} f_\lambda\, \tilde{K}_{\lambda\mu}(q,t)$$

should be the Hilbert series of $\mathbf{H}_\mu$. Here, $f_\lambda$ denotes the number of standard tableaux of shape $\lambda$. Since Macdonald proved that

$$K_{\lambda\mu}(1,1) \;=\; f_\lambda \;, \qquad I.6$$

we see that we must necessarily have

$$F_\mu(1,1) \;=\; \sum_{\lambda} f_\lambda^2 \;=\; n! \qquad I.7$$

According to our conjectures in [2] the polynomial

$$\partial_{p_1} \tilde{H}_\mu(x;q,t)$$

should give the Frobenius characteristic of the action of $S_{n-1}$ on $\mathbf{H}_\mu$. Using the fact that the operator $\partial_{p_1}$ is in a sense (*) dual to multiplication by the elementary symmetric function $e_1$, we can transform one of the Pieri rules given by Macdonald in [12] into the expansion of $\partial_{p_1}\tilde{H}_\mu(x;q,t)$ in terms of the polynomials $\tilde{H}_\nu(x;q,t)$ whose index $\nu$ immediately precedes $\mu$ in the Young partial order. More precisely we obtain

$$\partial_{p_1}\tilde{H}_\mu(x;q,t) \;=\; \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\, \tilde{H}_\nu(x;q,t) \qquad I.8$$

with

$$c_{\mu\nu}(q,t) \;=\; \prod_{s\in\mathcal{R}_{\mu/\nu}} \frac{t^{l_\mu(s)} - q^{a_\mu(s)+1}}{t^{l_\nu(s)} - q^{a_\nu(s)+1}} \prod_{s\in\mathcal{C}_{\mu/\nu}} \frac{q^{a_\mu(s)} - t^{l_\mu(s)+1}}{q^{a_\nu(s)} - t^{l_\nu(s)+1}} \;, \qquad I.9$$

where $\mathcal{R}_{\mu/\nu}$ (resp. $\mathcal{C}_{\mu/\nu}$) denotes the set of lattice squares of $\nu$ that are in the same row (resp. same column) as the square we must remove from $\mu$ to obtain $\nu$. This given, an application of $\partial_{p_1}^{n-1}$ to both sides of I.8 yields the recursion

$$F_\mu(q,t) \;=\; \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\, F_\nu(q,t) \;, \qquad I.10$$

(*) This will be made precise in section 1.

which together with the initial condition $F_{(1)}(q,t) = 1$ permits the computation of extensive tables of $F_\mu(q,t)$. Of course all the data so obtained not only confirms the polynomiality and positive integrality of the coefficients of $F_\mu(q,t)$ but exhibits some truly remarkable symmetries under various transformations of the variables $\mu$, $q$ and $t$.

Since $\{H_\mu(x;q,t)\}_\mu$ is a basis for the symmetric functions, it follows that the polynomial $e_1\tilde{H}_\nu$ has an expansion in terms of the polynomials $\tilde{H}_\mu(x;q,t)$ for which $\mu/\nu$ is a single cell. In other words, there exists coefficients $d_{\mu\nu}(q,t)$ such that

$$e_1(x)\tilde{H}_\nu(x;q,t) = \sum_{\mu\leftarrow\nu} d_{\mu\nu}(q,t)\tilde{H}_\mu(x;q,t) \qquad I.11$$

The coefficents $d_{\mu\nu}$ can be calculated explicitly and since multiplication by $e_1$ is dual to the operator $\partial_{p_1}$ it can be shown that

$$\begin{aligned} d_{\mu\nu}(q,t) &= c_{\mu\nu}(q,t)(1-t)(1-q)\,\frac{\tilde{h}_\nu(q,t)\ \tilde{h}'_\nu(q,t)}{\tilde{h}_\mu(q,t)\ \tilde{h}'_\mu(q,t)} \\ &= \prod_{s\in\mathcal{R}_{\mu/\nu}} \frac{q^{a_\nu(s)} - t^{l_\nu(s)+1}}{q^{a_\mu(s)} - t^{l_\mu(s)+1}} \prod_{s\in\mathcal{C}_{\mu/\nu}} \frac{t^{l_\nu(s)} - q^{a_\nu(s)+1}}{t^{l_\mu(s)} - q^{a_\mu(s)+1}} \end{aligned} \qquad I.12$$

where we define the quantities

$$\tilde{h}_\mu(q,t) = \prod_{s\in\mu}(q^{a_\mu(s)} - t^{l_\mu(s)+1}) \quad , \quad \tilde{h}'_\mu(q,t) = \prod_{s\in\mu}(t^{l_\mu(s)} - q^{a_\lambda(s)+1}) \ . \qquad I.13$$

By applying the operator $\partial^n_{p_1}$ to both sides of I.11 we obtain the relationship of the $d_{\mu\nu}$ to the $F_\mu$.

$$n\ F_\nu(q,t) = \sum_{\mu\leftarrow\nu} F_\mu(q,t)\ d_{\mu\nu}(q,t) \qquad I.14$$

The idea that a "hook walk" of sorts is involved here stems from noticing what takes place if we successively make the substitutions $t\to 1/t$ then $t\to q$. To this end, setting

$$G_\mu(q) = \big(\ F_\mu(q,1/t)\ t^{n(\mu)}\ \big)\,|_{t\to q}$$

routine manipulations yield that the recursion in I.10 becomes

$$G_\mu(q) = \sum_{\nu\to\mu} \gamma_{\mu\nu}(q)\ G_\nu(q) \qquad I.15$$

and the recursion in I.14 becomes

$$n\,G(q) = \sum_{\mu\leftarrow\nu} G_\mu(q)/\gamma_{\mu\nu}(q)\ , \qquad I.16$$

with

$$\gamma_{\mu\nu}(q) = \prod_{s\in\mathcal{R}_{\mu/\nu}} \frac{1-q^{h_\mu(s)}}{1-q^{h_\nu(s)}} \prod_{s\in\mathcal{C}_{\mu/\nu}} \frac{1-q^{h_\mu(s)}}{1-q^{h_\nu(s)}}$$

where

$$h_\mu(s) = l_\mu(s) + a_\mu(s) + 1 \quad \text{and} \quad h_\nu(s) = l_\nu(s) + a_\nu(s) + 1$$

However, now these coefficients $\gamma_{\mu\nu}(q)$ may be given a very revealing form. Indeed, since when $s$ is not in $\mathcal{R}_{\mu/\nu}$ or $\mathcal{C}_{\mu/\nu}$ we do have

$$h_\mu(s) = l_\mu(s) + a_\mu(s) + 1 = l_\nu(s) + a_\nu(s) + 1 = h_\nu(s)$$

we may write

$$\prod_{s \in \mathcal{R}_{\mu/\nu} + \mathcal{C}_{\mu/\nu}} \frac{1 - q^{h_\mu(s)}}{1 - q^{h_\nu(s)}} \;=\; \frac{1}{1-q} \frac{\prod_{s\in\mu}(1 - q^{h_\mu(s)})}{\prod_{s\in\nu}(1 - q^{h_\nu(s)})}$$

where the divisor $1 - q$ compensates for the fact that $\mu$ differs from $\nu$ by a corner square (of hook length $= 1$). Using the notation

$$[m]_q \;=\; 1 + q + \cdots + q^{m-1} \;=\; \frac{1 - q^m}{1 - q}$$

we can finally rewrite the recursion in I.15 in the form

$$\frac{G_\mu(q)}{\prod_{s\in\mu}[h_\mu(s)]_q} \;=\; \sum_{\nu\to\mu} \frac{G_\nu(q)}{\prod_{s\in\nu}[h_\nu(s)]_q} \;.$$

This means that the expression $G_\mu(q)/\prod_{s\in\mu}[h_\mu(s)]_q$ satisfies the same recursion as the number of of standard tableau $f_\mu$. Since the initial condition is $G_{(1)} = 1$, we deduce that for all partitions $\mu$ we must have

$$G_\mu(q) \;=\; f_\mu \prod_{s\in\mu}[h_\mu(s)]_q \qquad I.17$$

The derivation of I.17 suggests that the coefficient $c_{\mu\nu}(q)$ is some sort of $q,t$-analog of the ratio $h_\mu/h_\nu$ and that the coefficients $d_{\mu\nu}(q)$ are a $q,t$-analog of the ratio $h_\nu/h_\mu$, where $h_\mu$ and $h_\nu$ denote the hook-products for $\mu$ and $\nu$ respectively. This given, the recursion in I.10 may be viewed as a $q,t$-analog of the identity

$$n! \;=\; \sum_{\nu\to\mu} \frac{h_\mu}{h_\nu}\,(n-1)! \;.$$

Dividing both sides of this identity by $n!$ we get

$$1 \;=\; \frac{1}{n}\sum_{\nu\to\mu}\frac{h_\mu}{h_\nu} \;, \qquad I.18$$

which is precisely what Greene, Nijenhuis and Wilf prove by means of their random hook-walk. We also note that the recursion in I.14 should then be considered as a $q,t$-analog of the identity

$$n\,(n-1)! \;=\; \sum_{\mu\leftarrow\nu} n!\,\frac{h_\nu}{h_\mu}.$$

This is equivalent to the identity

$$1 \;=\; \sum_{\mu\leftarrow\nu} \frac{f_\mu}{n f_\nu} \qquad I.19$$

that Green, Nijenhuis, and Wilf prove in their second paper by using the random hook-walk machinery on a complementary shape to $\mu$.

In [] Garsia and Haiman show that an appropriate $q,t$-extension of the Green, Nihenhuis and Wilf argument can be used to prove the identity

$$1 \;=\; \frac{1}{B_\mu(q,t)} \sum_{\nu\rightarrow\mu} c_{\mu\nu}(q) \;. \qquad I.20$$

where

$$B_\mu(q,t) \;=\; \sum_{s\in\mu} t^{l'_\mu(s)} q^{a'_\mu(s)} \qquad I.21$$

which is an $q-t$-analog to the identity I.18. Here we shall show that applying Garsia and Haiman's $q,t$-hook walk to a complementary shape to $\mu$ in the same way that Green, Nijenhuis and Wilf proved I.19 yields the identity

$$1 \;=\; \sum_{\mu\leftarrow\nu} d_{\mu\nu}(q,t)\, T_{\mu/\nu}, \qquad I.22$$

where $T_{\mu/\nu} = t^{r_\nu} q^{s_\nu}$ with $r_\nu, s_\nu$ the coleg and coarm of the corner cell $\mu/\nu$. This identity is again a $q,t$-analog of the formula I.19.

The contents of this note are divided into two sections. In the first section we give the description of the $q,t$-hook walk and the identities proven by Garsia and Haiman that we shall use. In the second section we show how the $q,t$-hook walk applied to a complementary shape in the same way that Green, Nijenhuis and Wilf applied their hook walk to prove I.19 yields I.22 and how similar calculations can be done to prove the additional identity

$$1 \;=\; \sum_{\mu\leftarrow\nu} d_{\mu\nu}(q,t) \qquad I.23.$$

## 1. The $q,t$-hook walk

We shall start with a brief review of the Greene-Nijenhuis-Wilf proof of the identity

$$1 \;=\; \frac{1}{n} \sum_{\nu\rightarrow\mu} \frac{h_\mu}{h_\nu} \qquad 1.1$$

To simplify our language we need to make some notational conventions. To begin with we shall hereafter identify a partition $\mu$ with its Ferrers diagram. We should also recall that the *hook* of a cell $s$ of $\mu$ consists of $s$ together with its *arm*, whose length we have denoted by $a_\mu(s)$ and its *leg* whose length we have denoted by $l_\mu(s)$. Since we use the French convention of depicting Ferrers diagrams, the arm of $s$ consists of the cells of $\mu$ which are strictly east of $s$ and the leg consists of the cells of $\mu$ which are strictly north of $s$. Likewise the coarm and coleg consist of the cells respectively strictly west and strictly south. We shall often use the words *arm, coarm, leg* and *coleg* to refer to their respective lengths. We set $h_\mu(s) = 1 + a_\mu(s) + l_\mu(s)$ and refer to it as the *hook length* of $s$ in $\mu$. We shall also set, $h_\mu = \prod_{s\in\mu} h_\mu(s)$. When $\nu$ immediately precedes $\mu$ (which we have expressed by writing $\nu\rightarrow\mu$) it will be convenient to denote by $\mu/\nu$ the corner cell we must remove from $\mu$ to obtain $\nu$. A cell $s$ with coarm $a'$ and coleg $l'$ will be represented by the pair $(a'+1, l'+1)$. If $s = (x, y)$ and $s' = (x', y')$ we shall write $s \ll s'$ if and only if $x < x'$ and $y < y'$ and $s \ll= s'$ if and only if $x \le x'$ and $y \le y'$ . The collection of cells that are weakly NORTH-EAST of $s$ will be denoted by $NE(s)$ and referred to as the *shadow* of $s$. That is

$$NE(s) \;=\; \{s' \,:\, s \ll= s'\,\}$$

We shall also express the inequality $s \ll s'$ by saying that *$s$ is covered by $s'$*. Here the symbols $R_{\mu/\nu}$ and $C_{\mu/\nu}$ will have the same meaning as in the introduction, but in addition, for a given cell $s$ we shall denote by $R_{\mu/\nu}(s)$ and $C_{\mu/\nu}(s)$ the cells of $R_{\mu/\nu}$ and $C_{\mu/\nu}$ that are **strictly** NORTH-EAST of $s$. Note that both $R_{\mu/\nu}(s)$ and $C_{\mu/\nu}(s)$ are empty when $\mu/\nu$ is not in the shadow of $s$. When $\mu/\nu$ is in the shadow of $s$ we shall denote by $r[s]$ the element of $R_{\mu/\nu} \cup \{\mu/\nu\}$ that is directly NORTH of $s$. Likewise $c[s]$ will denote the element of $C_{\mu/\nu} \cup \{\mu/\nu\}$ that is directly EAST of $s$.

Given $\mu \vdash n$, the basic ingredient in [9] is a random walk $Z_1, Z_2, \ldots, Z_m, \ldots$ over the cells of $\mu$ which is constructed according to the following mechanism.

(1) The initial point $Z_1 = (x_1, y_1)$ is obtained by selecting one of the cells of $\mu$ at random and with probability $1/n$.

(2) After $k$ steps, given that $Z_k = s$,
   (a) the walk stops if $s$ is a corner cell of $\mu$
   (b) if $s$ is not a corner cell, then $Z_{k+1}$ is obtained by selecting at random and with equal probability $\frac{1}{a_\mu(s)+l_\mu(s)} = \frac{1}{h_\mu(s)-1}$ one of the cells of the arm or the leg of $s$ in $\mu$.

Greene-Nijenhuis-Wilf establish 1.1 by showing that for any $\nu\rightarrow\mu$ the quantity $\frac{1}{n}\frac{h_\mu}{h_\nu}$ gives the probability that the random walk ends at the corner cell $\mu/\nu$. Denoting by $Z_{end}$ the ending position of the random walk, we may express this by writing

$$P[Z_{end} = \mu/\nu] \;=\; \frac{1}{n}\frac{h_\mu}{h_\nu}\;. \qquad 1.2$$

Clearly, if the random walk starts at the cell $s$ then it can only end on a corner cell that is in the shadow of $s$. In fact, the G-N-W proof yields that for $s \ll \mu/\nu$

$$P[Z_{end} = \mu/\nu\,|Z_1 = s] \;=\; \frac{1}{h_\mu(r[s])-1}\frac{1}{h_\mu(c[s])-1}\prod_{r\in R_{\mu/\nu}(s)}\frac{h_\mu(r)}{h_\nu(r)}\prod_{c\in C_{\mu/\nu}(s)}\frac{h_\mu(c)}{h_\nu(c)}\;. \qquad 1.3$$

for $r[s] = \mu/\nu$

$$P[Z_{end} = \mu/\nu\,|Z_1 = s] \;=\; \frac{1}{h_\mu(c[s])-1}\prod_{c\in C_{\mu/\nu}(s)}\frac{h_\mu(c)}{h_\nu(c)} \qquad 1.3r$$

and for $c[s] = \mu/\nu$

$$P[Z_{end} = \mu/\nu \,|Z_1 = s] \;=\; \frac{1}{h_\mu(r[s]) - 1} \prod_{r \in R_{\mu/\nu}(s)} \frac{h_\mu(r)}{h_\nu(r)} \qquad 1.3c$$

This given, 1.2 follows from the identity

$$P[Z_{end} = \mu/\nu] \;=\; \sum_{s \ll = \mu/\nu} P[Z_1 = s]\, P[Z_{end} = \mu/\nu \,|Z_1 = s] \;. \qquad 1.4$$

Garsia and Haiman show that there exists a $q,t-$analogue of the hook walk that can be used to prove the identity

$$B_\mu(q,t) \;=\; \sum_{\nu \to \mu} c_{\mu\nu}(q,t). \qquad 1.5$$

In order to use the probabilistic jargon in our argument, it is necessary to view the parameters $q$ and $t$ as positive numbers. In fact, it will be convenient to let $0 < q < 1$ and $t > 1$. However, the trained combinatorial eye should have no difficulty seeing that this condition is totally artificial. In fact, it can be completely done without by viewing each random walk as a *lattice path* and its probability as the *weight* of the path. In this setting $q$ and $t$ may be left as they should be, namely two independent indeterminates. From this point of view our proof may viewed as a modification of the G-N-W proof obtained by simply *changing weights*.

Following the G-N-W scheme our random walk may be described as follows.

(1) The initial point $Z_1$ is obtained by selecting the cell $(x,y)$ of $\mu$ with probability $\frac{q^{x-1}t^{y-1}}{B_\mu(q,t)}$.

(2) After $k$ steps, given that $Z_k = s = (x,y)$,

 (a) the walk stops if $s$ is a corner cell of $\mu$

 (b) if $s$ is not a corner cell then $Z_{k+1}$ is obtained by selecting

  (i) the cell $(x, y+j)$ of the leg of $s$ with probability $t^{j-1}\frac{q^{a_\mu(s)}(t-1)}{t^{l_\mu(s)} - q^{a_\mu(s)}}$

  (ii) the cell $(x+i, y)$ of the arm of $s$ with probability $q^{i-1}\frac{t^{l_\mu(s)}(1-q)}{t^{l_\mu(s)} - q^{a_\mu(s)}}$.

Note that the probability of $Z_{k+1}$ landing anywhere in the leg of $s$ is given by the sum

$$\sum_{j=1}^{l_\mu(s)} t^{j-1}\frac{q^{a_\mu(s)}(t-1)}{t^{l_\mu(s)} - q^{a_\mu(s)}} \;=\; q^{a_\mu(s)}\,\frac{t^{l_\mu(s)} - 1}{t^{l_\mu(s)} - q^{a_\mu(s)}}$$

and the the probability of $Z_{k+1}$ landing anywhere in the arm of $s$ is given by

$$\sum_{i=1}^{a_\mu(s)} q^{i-1}\frac{t^{l_\mu(s)}(1-q)}{t^{l_\mu(s)} - q^{a_\mu(s)}} \;=\; t^{l_\mu(s)}\,\frac{1 - q^{a_\mu(s)}}{t^{l_\mu(s)} - q^{a_\mu(s)}}$$

and we see that we do have, as necessary

$$q^{a_\mu(s)}\,\frac{t^{l_\mu(s)} - 1}{t^{l_\mu(s)} - q^{a_\mu(s)}} \;+\; t^{l_\mu(s)}\,\frac{1 - q^{a_\mu(s)}}{t^{l_\mu(s)} - q^{a_\mu(s)}} \;=\; 1$$

It will be convenient to set for any cell $s \in \mu$

$$A(s) \;=\; \frac{t^{l_\mu(s)}(1-q)}{t^{l_\mu(s)} - q^{a_\mu(s)}} \qquad \text{and} \qquad B(s) \;=\; \frac{q^{a_\mu(s)}(t-1)}{t^{l_\mu(s)} - q^{a_\mu(s)}} \;\; . \tag{1.6}$$

Garsia and Haiman then establish the following theorem: **Theorem**
*For any $s \ll \mu/\nu$*

$$P\big[\, Z_{end} = \mu/\nu \mid Z_1 = s \,\big] \;=\; A(r[s])\, B(c[s]) \prod_{r \in R_{\mu/\nu}(s)} (q + A(r)) \prod_{c \in C_{\mu/\nu}(s)} (t + B(c)) \;\; . \tag{1.8a}$$

*Or, equivalently*

$$\begin{aligned} P\big[\, Z_{end} = \mu/\nu \mid Z_1 = s \,\big] \;&=\; A(r[s])\, B(c[s]) \prod_{r \in R_{\mu/\nu}(s)} \frac{\tilde{h}'_\mu(r)}{\tilde{h}'_\nu(r)} \prod_{c \in C_{\mu/\nu}(s)} \frac{\tilde{h}_\mu(c)}{\tilde{h}_\nu(c)} \\ &=\; A(r[s]) B(c[s]) \prod_{r \in R_{\mu/\nu}(s)} \frac{t^{l_\mu(r)} - q^{a_\mu(r)+1}}{t^{l_\nu(r)} - q^{a_\nu(r)+1}} \prod_{c \in C_{\mu/\nu}(s)} \frac{q^{a_\mu(c)} - t^{l_\mu(c)+1}}{q^{a_\nu(c)} - t^{l_\nu(c)+1}} \end{aligned} \tag{1.8b}$$

Next they use this theorem to prove the identity that for any $\nu \rightarrow \mu$

$$P[Z_{end} = \mu/\nu] \;=\; \tfrac{1}{B_\mu(q,t)} \;\; c_{\mu\nu}(q,t) \;\; . \tag{1.9}$$

The following theorem follows as a corollary **Theorem**

$$c_{\mu\nu}(q,t) \;=\; \sum_{s \ll = \mu/\nu} q^{a'(s)} t^{l'(s)} \, P\big[Z_{end} = \mu/\nu \mid Z_1 = s\,\big] \;\; . \tag{1.10}$$

*In particular we derive that*

$$\sum_{\nu \rightarrow \mu} c_{\mu\nu}(q,t) \;=\; B_\mu(q,t) \;\; .$$

The last assertion follows immediately from 1.10 and and the fact that for any $s \in \mu$ we must have

$$\sum_{\nu \rightarrow \mu} P\big[Z_{end} = \mu/\nu\,\big] \;=\; 1 \;\; .$$

## 3. Further $q,t$-analogues

In their second paper [10], Green, Nijenhuis and Wilf show that their hook walk mechanism can be used to give a probabilist proof of the so called *upper recursion* for the number of standard tableaux. This an identity due to A. Young [16], which is obtained by summing $f_\mu$ over partitions which immediately follow a fixed partition $\nu$. More precisely, for a given $\nu \vdash n-1$ we have

$$n\, f_\nu \;=\; \sum_{\mu \leftarrow \nu} f_\mu \;\; . \tag{3.1}$$

This identity, was used by Rutherford [15] to give a proof of the Young's formula

$$n! \;=\; \sum_{\mu\vdash n} f_\mu^2 \qquad 3.2$$

which follows from an inductive argument based on the following steps

$$\begin{aligned}(n+1)\,n! \;&=\; \sum_{\mu\vdash n} f_\mu\,(n+1)\,f_\mu \;= \\ &=\; \sum_{\mu\vdash n} f_\mu \sum_{\lambda\leftarrow\mu} f_\lambda \\ &=\; \sum_{\lambda\vdash n+1} f_\lambda \sum_{\mu\to\lambda} f_\mu \;=\; \sum_{\lambda\vdash n+1} f_\lambda^2\end{aligned} \qquad 3.9$$

The Theory of Macdonald polynomials produces several $q,t$-analogues of 3.1 and 3.2. In [] Garsia and Haiman suggest that the $q,t$-hook walk mechanism should have an extension that yields proofs of these further identities.

**Proposition 3.1**

*For every $\nu\vdash n-1$ we have*

$$\begin{aligned}&a)\quad 1 \;=\; \sum_{\mu\leftarrow\nu} d_{\mu\nu}(q,t) \\ &b)\quad 1 \;=\; \sum_{\mu\leftarrow\nu} d_{\mu\nu}(q,t)\,T_{\mu/\nu} \\ &c)\quad n\,F_\nu(q,t) \;=\; \sum_{\mu\leftarrow\nu} F_\mu(q,t)\,d_{\mu\nu}(q,t)\end{aligned} \qquad 3.3$$

*where the coefficients $d_{\mu\nu}(q,t)$ are as given in 1.16 and $T_{\mu/\nu}=t^{r_\nu}q^{s_\nu}$ with $r_\nu, s_\nu$ the coleg and coarm of the corner cell $\mu/\nu$.*

It develops that 3.3 a), b) & c) are but three different variants of the upper recursion. To see this note that, dividing both sides of 3.1 by $nf_\nu$ , the resulting identity may be rewritten in the form

$$1 \;=\; \sum_{\mu\leftarrow\nu} h_\nu/h_\mu \;\; . \qquad 3.6$$

On the other hand, from the definition 1.16, we can deduce (as we did for $c_{\mu\nu}$) that making the replacement $t\to 1/q$ and then letting $q=1$ reduces $d_{\mu\nu}$ to the ratio $h_\nu/h_\mu$. Thus we see that the same replacements reduce 3.3 a) & b) to 3.6 and 3.1 c) to

$$n\,(n-1)! \;=\; \sum_{\mu\to\nu} n!\,\frac{h_\nu}{h_\mu}$$

which is yet another way of writing 3.6.

Now perform the hook walk on a complementary shape of $\nu$ and write the expression in terms of quantities on the shape $\nu$.

Let $(\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_n,\beta_n)$ denote the coordinates of the corners of $\nu$ numbering the corners so that $(\alpha_1,\beta_1)$ is the north-west most corner and $(\alpha_n,\beta_n)$ is the south-east most corner. It will be convenient in the simplification of the following calculations to use the convention that $\alpha_0=\beta_{n+1}=0$. Since the square $\mu/\nu$ lies between two of these corners we let the number k be the index of the corner just north and east of $\mu/\nu$.

Let $Z_1=s=(p,r)$ where $p>\alpha_n$ and $r>\beta_1$. The coordinates $(0,0)$ and $(p,r)$ define a square $X$ with those points at alternate corners that contains $\nu$. Define $\bar\mu$ to be the complementary shape to $\nu$ (i.e. $\bar\mu=X/\nu$). The single square $\mu/\nu$ lies in the shape $\bar\mu$ so we define the shape $\bar\nu$ to be $\bar\mu/(\mu/\nu)$.

Now consider the quantities $R_{\bar\mu/\bar\nu},C_{\bar\mu/\bar\nu}$ (the north and east definitions are incorrect for these quantities unless the whole shape is rotated 180°). Then we have that $r[s]=(p,\beta_{k+1}+1)$ and $c[s]=(\alpha_k+1,r)$. We also have that $l_{\bar\mu}(r[s])=\beta_{k+1}$, $a_{\bar\mu}(r[s])=p-\alpha_k-1$, $l_{\bar\mu}(c[s])=r-\beta_{k+1}-1$, $a_{\bar\mu}(c[s])=\alpha_k$.

Hence we have that:

$$A(r[s])=\frac{t^{l_{\bar\mu}(r[s])}(1-q)}{t^{l_{\bar\mu}(r[s])}-q^{a_{\bar\mu}(r[s])}}=\frac{t^{\beta_{k+1}}(1-q)}{t^{\beta_{k+1}}-q^{p-\alpha_k-1}} \tag{5}$$

$$B(c[s])=\frac{q^{a_{\bar\mu}(c[s])}(t-1)}{t^{l_{\bar\mu}(c[s])}-q^{a_{\bar\mu}(c[s])}}=\frac{q^{\alpha_k}(t-1)}{t^{r-\beta_{k+1}-1}-q^{\alpha_k}} \tag{6}$$

**Proposition:** The products that appear in (4) are:

$$\prod_{r\in R_{\mu/\nu}(s)}\frac{t^{l_{\bar\mu}(r)}-q^{a_{\bar\mu}(r)+1}}{t^{l_{\bar\nu}(r)}-q^{a_{\bar\nu}(r)+1}}=\frac{t^{\beta_{k+1}}-q^{p-\alpha_k-1}}{t^{\beta_{k+1}}(1-q)}\prod_{i=k+1}^{n}\frac{q^{\alpha_k}t^{\beta_{k+1}}-q^{\alpha_i}t^{\beta_i}}{q^{\alpha_k}t^{\beta_{k+1}}-q^{\alpha_i}t^{\beta_{i+1}}} \tag{7}$$

and

$$\prod_{c\in C_{\mu/\nu}(s)}\frac{q^{a_{\bar\mu}(c)}-t^{l_{\bar\mu}(c)+1}}{q^{a_{\bar\nu}(c)}-t^{l_{\bar\nu}(c)+1}}=\frac{t^{r-\beta_{k+1}-1}-q^{\alpha_k}}{q^{\alpha_k}(t-1)}\prod_{i=1}^{k}\frac{q^{\alpha_k}t^{\beta_{k+1}}-q^{\alpha_i}t^{\beta_i}}{q^{\alpha_k}t^{\beta_{k+1}}-q^{\alpha_{i-1}}t^{\beta_i}} \tag{8}$$

**Proof:** By expanding out the product and noticing that $l_\mu$ and $l_\nu$ are constant on the cells between $(\alpha_i,r[s])$ and $(\alpha_{i+1},r[s])$ and that the product collapse for cells in this region. We have then that,

$$\prod_{r\in R_{\mu/\nu}(s)}\frac{t^{l_{\bar\mu}(r)}-q^{a_{\bar\mu}(r)+1}}{t^{l_{\bar\nu}(r)}-q^{a_{\bar\nu}(r)+1}}=\begin{array}{l}
\frac{t^{\beta_{k+1}}-q^{p-\alpha_k-1}}{t^{\beta_{k+1}}-q^{p-\alpha_k-2}}\frac{t^{\beta_{k+1}}-q^{p-\alpha_k-2}}{t^{\beta_{k+1}}-q^{p-\alpha_k-3}}\cdots\frac{t^{\beta_{k+1}}-q^{\alpha_n-\alpha_k+1}}{t^{\beta_{k+1}}-q^{\alpha_n-\alpha_k}}\\
\frac{t^{\beta_{k+1}-\beta_n}-q^{\alpha_n-\alpha_k}}{t^{\beta_{k+1}-\beta_n}-q^{\alpha_n-\alpha_k-1}}\frac{t^{\beta_{k+1}-\beta_n}-q^{\alpha_n-\alpha_k-1}}{t^{\beta_{k+1}-\beta_n}-q^{\alpha_n-\alpha_k-2}}\cdots\frac{t^{\beta_{k+1}-\beta_n}-q^{\alpha_{n-1}-\alpha_k+1}}{t^{\beta_{k+1}-\beta_n}-q^{\alpha_{n-1}-\alpha_k}}\\
\frac{t^{\beta_{k+1}-\beta_{n-1}}-q^{\alpha_{n-1}-\alpha_k}}{t^{\beta_{k+1}-\beta_{n-1}}-q^{\alpha_{n-1}-\alpha_k-1}}\frac{t^{\beta_{k+1}-\beta_{n-1}}-q^{\alpha_{n-1}-\alpha_k-1}}{t^{\beta_{k+1}-\beta_{n-1}}-q^{\alpha_{n-1}-\alpha_k-2}}\cdots\frac{t^{\beta_{k+1}-\beta_{n-1}}-q^{\alpha_{n-2}-\alpha_k+1}}{t^{\beta_{k+1}-\beta_{n-1}}-q^{\alpha_{n-2}-\alpha_k}}\\
\vdots\\
\frac{t^{\beta_{k+1}-\beta_{k+1}}-q^{\alpha_{k+1}-\alpha_k}}{t^{\beta_{k+1}-\beta_{k+1}}-q^{\alpha_{k+1}-\alpha_k-1}}\frac{t^{\beta_{k+1}-\beta_{k+1}}-q^{\alpha_{k+1}-\alpha_k-1}}{t^{\beta_{k+1}-\beta_{k+1}}-q^{\alpha_{k+1}-\alpha_k-2}}\cdots\frac{t^{\beta_{k+1}-\beta_{k+1}}-q^{\alpha_k-\alpha_k+2}}{t^{\beta_{k+1}-\beta_{k+1}}-q^{\alpha_k-\alpha_k+1}}
\end{array}=$$

$$\frac{t^{\beta_{k+1}} - q^{p-\alpha_k-1}}{t^{\beta_{k+1}} - q^{\alpha_n-\alpha_k}} \frac{t^{\beta_{k+1}-\beta_n} - q^{\alpha_n-\alpha_k}}{t^{\beta_{k+1}-\beta_n} - q^{\alpha_{n-1}-\alpha_k}} \frac{t^{\beta_{k+1}-\beta_{n-1}} - q^{\alpha_{n-1}-\alpha_k}}{t^{\beta_{k+1}-\beta_{n-1}} - q^{\alpha_{n-2}-\alpha_k}} \cdots \frac{1 - q^{\alpha_{k+1}-\alpha_k}}{1-q} =$$

$$\frac{q^{\alpha_k}t^{\beta_{k+1}} - q^{p-1}}{q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_n}} \frac{q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_n}t^{\beta_n}}{q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_{n-1}}t^{\beta_n}} \frac{q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_{n-1}}t^{\beta_{n-1}}}{q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_{n-2}}t^{\beta_{n-1}}} \cdots \frac{q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_{k+1}}t^{\beta_{k+1}}}{q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_k+1}t^{\beta_{k+1}}} =$$

$$\frac{q^{\alpha_k}t^{\beta_{k+1}} - q^{p-1}}{q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_k+1}t^{\beta_{k+1}}} \prod_{i=k+1}^{n} \frac{q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_i}t^{\beta_i}}{q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_i}t^{\beta_{i+1}}}$$

Similarly,

$$\prod_{c \in C_{\mu/\nu}(s)} \frac{q^{a_{\tilde{\mu}}(c)} - t^{l_{\tilde{\mu}}(c)+1}}{q^{a_{\tilde{\nu}}(c)} - t^{l_{\tilde{\nu}}(c)+1}} = \begin{array}{l} \frac{q^{\alpha_k} - t^{r-\beta_{k+1}-1}}{q^{\alpha_k} - t^{r-\beta_{k+1}-2}} \frac{q^{\alpha_k} - t^{r-\beta_{k+1}-2}}{q^{\alpha_k} - t^{r-\beta_{k+1}-3}} \cdots \frac{q^{\alpha_k} - t^{\beta_1-\beta_{k+1}+1}}{q^{\alpha_k} - t^{\beta_1-\beta_{k+1}}} \\ \frac{q^{\alpha_k-\alpha_1} - t^{\beta_1-\beta_{k+1}}}{q^{\alpha_k-\alpha_1} - t^{\beta_1-\beta_{k+1}-1}} \frac{q^{\alpha_k-\alpha_1} - t^{\beta_1-\beta_{k+1}-1}}{q^{\alpha_k-\alpha_1} - t^{\beta_1-\beta_{k+1}-2}} \cdots \frac{q^{\alpha_k-\alpha_1} - t^{\beta_2-\beta_{k+1}+1}}{q^{\alpha_k-\alpha_1} - t^{\beta_2-\beta_{k+1}}} \\ \frac{q^{\alpha_k-\alpha_2} - t^{\beta_2-\beta_{k+1}}}{q^{\alpha_k-\alpha_2} - t^{\beta_2-\beta_{k+1}-1}} \frac{q^{\alpha_k-\alpha_2} - t^{\beta_2-\beta_{k+1}-1}}{q^{\alpha_k-\alpha_2} - t^{\beta_2-\beta_{k+1}-2}} \cdots \frac{q^{\alpha_k-\alpha_2} - t^{\beta_3-\beta_{k+1}+1}}{q^{\alpha_k-\alpha_2} - t^{\beta_3-\beta_{k+1}}} \\ \vdots \\ \frac{q^{\alpha_k-\alpha_k} - t^{\beta_k-\beta_{k+1}}}{q^{\alpha_k-\alpha_k} - t^{\beta_k-\beta_{k+1}-1}} \frac{q^{\alpha_k-\alpha_k} - t^{\beta_k-\beta_{k+1}-1}}{q^{\alpha_k-\alpha_k} - t^{\beta_k-\beta_{k+1}-2}} \cdots \frac{q^{\alpha_k-\alpha_k} - t^2}{q^{\alpha_k-\alpha_k} - t} \end{array} =$$

$$\frac{q^{\alpha_k} - t^{r-\beta_{k+1}-1}}{q^{\alpha_k} - t^{\beta_1-\beta_{k+1}}} \frac{q^{\alpha_k-\alpha_1} - t^{\beta_1-\beta_{k+1}}}{q^{\alpha_k-\alpha_1} - t^{\beta_2-\beta_{k+1}}} \frac{q^{\alpha_k-\alpha_2} - t^{\beta_2-\beta_{k+1}}}{q^{\alpha_k-\alpha_2} - t^{\beta_3-\beta_{k+1}}} \cdots \frac{1 - t^{\beta_k-\beta_{k+1}}}{1-t} =$$

$$\frac{q^{\alpha_k}t^{\beta_{k+1}} - t^{r-1}}{q^{\alpha_k}t^{\beta_{k+1}} - t^{\beta_1}} \frac{q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_1}t^{\beta_1}}{q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_1}t^{\beta_2}} \cdots \frac{q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_{k-1}}t^{\beta_{k-1}}}{q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_{k-1}}t^{\beta_k}} \frac{q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_k}t^{\beta_k}}{q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_k}t^{\beta_{k+1}+1}} =$$

$$\frac{t^{r-\beta_{k+1}-1} - q^{\alpha_k}}{q^{\alpha_k}(t-1)} \prod_{i=1}^{k} \frac{q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_i}t^{\beta_i}}{q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_{i-1}}t^{\beta_i}}$$

Hence we have that the result from (4)-(8) that:

$$P(Z_{end} = \mu/\nu | Z_1 = s) = \frac{\prod_{i=1}^{n} q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_i}t^{\beta_i}}{\prod_{i=0}^{n}{}^{(k)} q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_i}t^{\beta_{i+1}}}$$

By rewriting (formula for dmn) and expressing it in terms of the coordinate of the corners of $\nu$ produces the formula:

$$d_{\mu\nu} = \frac{1}{q^{\alpha_k}t^{\beta_{k+1}}} \frac{\prod_{i=1}^{n} q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_i}t^{\beta_i}}{\prod_{i=0}^{n}{}^{(k)} q^{\alpha_k}t^{\beta_{k+1}} - q^{\alpha_i}t^{\beta_{i+1}}}$$

Thus we can conclude that $P(Z_{end} = \mu/\nu | Z_1 = s) = T_{\mu/\nu} d_{\mu\nu}$.

The notation that we chose for the weighted hook path was suggestive of a probablity because if we choose $t > 1$ and $0 < q < 1$ we have that the quantities do in fact represent probablities so that for any of these values of $q$ and $t$ the sum over all possible values of $\mu$ of $P(Z_{end} = \mu/\nu | Z_1 = s)$ must be 1. From this we can conclude that for any value of $q$ and $t$:

$$1 = \sum_{\mu \leftarrow \nu} P(Z_{end} = \mu/\nu | Z_1 = s) = \sum_{\mu \leftarrow \nu} T_{\mu/\nu} d_{\mu\nu}$$

Next, if in the expansion of $d_{\mu\nu}$ we make the replacements $x_i = q^{\alpha_i} t^{\beta_{i+1}}$ and $u_i = q^{\alpha_i} t^{\beta_i}$ then $d_{\mu\nu}$ takes the form:

$$d_{\mu\nu} = \frac{1}{x_k} \frac{\prod_{i=1}^{n} u_i - x_k}{\prod_{i=0}^{n}{}^{(k)} x_i - x_k}$$

Now consider the following rational function of $z$ that has a helpful partial fraction expansion:

$$\frac{\prod_{i=1}^{n} u_i - z}{\prod_{i=0}^{n} x_i - z} = \sum_{k=0}^{n} \frac{A_k}{x_k - z}$$

where

$$A_k = \frac{\prod_{i=1}^{n} u_i - x_k}{\prod_{i=0}^{n}{}^{(k)} x_i - x_k}$$

Notice that $\frac{A_k}{x_k}$ is precisely $d_{\mu\nu}$. Setting $z = 0$ in () allows us to derive that

$$\sum_{\mu \leftarrow \nu} d_{\mu\nu} = \sum_{k=0}^{n} \frac{A_k}{x_k} = \frac{\prod_{i=1}^{n} u_i}{\prod_{i=0}^{n} x_i} = \frac{\prod_{i=1}^{n} q^{\alpha_i} t^{\beta_i}}{\prod_{i=0}^{n} q^{\alpha_i} t^{\beta_{i+1}}} = 1$$