Combinatorial Aspects of the Baker-Akhiezer functions for S_2

by

A. M. Garsia and N. Wallach

Bien que S_2 n'est pas assez symetrique
On t'offre sa fonction de BA qui est bien diabolique.
Bon 60-ème anniversaire Alain.

Astract

We show here that a certain sequence of polynomials arising in the study of S_2 m-quasi invariants satisfies a 3-term recursion. This leads to the discovery that these polynomials are closely related to the Bessel polynomials studied by Luc Favreau in [3]. This connection reveals a variety of combinatorial properties of the sequence of Baker-Akhiezer functions for S_2. In particular we obtain in this manner their generating function and show that it is equivalent to several further identities satisfied by these functions.

I. Introduction

Recent work [1], [2], [4], [5] on m-Quasi-Invariants has brought to focus certain remarkable sequences of multivariate polynomials associated to each Coxeter group W. There are strong indications that these sequences of polynomials have a rich combinatorial underpinning. This fact is somewhat obscured by the complexity and generality with which the subject is treated in present literature. In this paper we report our first findings in an attempt to develop a more transparent and accessible development of the subject by a close study of special cases. It develops that the sequence of Baker-Akhiezer functions for S_2, which is one of the simplest special cases, has a beautiful combinatorial description from which several of its basic properties may be derived with the greatest of ease. The reader is referred to the survey paper of Etingof and Strickland [2] for the general definition of the Baker-Akhiezer functions arising in the study of m-quasi-invariants and their significance for the theory. In the case of the symmetric group S_2 the Baker-Akhiezer function $\Psi_m^{S_2}(x;y) = \Psi_m^{S_2}(x_1,x_2;y_1,y_2)$ should be a formal power series in $x_1,x_2;y_2,y_2$ satisfying the following conditions:

$$
\begin{aligned}
&(1)\ \ \Psi_m^{S_2}(X_2;Y_2) = P_m^{S_2}(X_2;Y_2)e^{x_1y_1+x_2y_2} \quad \text{with } P_m^{S_n}(X_2;Y_2) \text{ a polynomial in } x_1,x_2;y_1,y_2.\\
&(2)\ \ P_m^{S_2}(X_2;Y_2) \ = \ (x_1-x_2)^m(y_1-y_2)^m + \cdots (terms\ of\ \ x-degree\ \ <\ \ m),\\
&(3)\ \ \Psi_m^{S_2}(X_2;Y_2) = \Psi_m^{S_2}(Y_2;X_2)\ ,\\
&(i)\ \ (x_1-x_2)^{2m+1} \,\Big|\, \Big(\Psi_m^{S_2}(x_1,x_2;Y_2) - \Psi_m^{S_2}(x_2,x_1;Y_2)\Big)\ ^{(\dagger)},\\
&(ii)\ \ L(m)\Psi_m^{S_2}(x;y) = (y_1^2+y_2^2)\,\Psi_m^{S_2}(x;y) \quad \text{with} \quad L(m) = \partial_{x_1}^2 + \partial_{x_2}^2 - 2m\,\tfrac{1}{x_1-x_2}(\partial_{x_1}-\partial_{x_2})\,.
\end{aligned}
\qquad \text{I.1}
$$

We show here that in the presence of (1) and (2) conditions (i) and (ii) are equivalent and either of them uniquely determines $\Psi_m^{S_2}(X_2;Y_2)$. More precisely we show that (1), (2) and (i) imply that the polynomial $P_m^{S_2}(x;y)$ has the simple form

$$
P_m^{S_2}(x;y) \ = \ 2^m \sum_{k=1}^{m} (-1)^{m-k} I_{2m-k}^{k} (x_1-x_2)^k (y_1-y_2)^k / 2^k \qquad \text{I.2}
$$

with I_{2m-k}^k giving the number of involutions of S_{2m-k} with k fixed points. Moreover we show that I.2 implies

(†) The symbol " $B|A$" means "B divides A".

that the sequence $\{\Psi_m^{S_2}(X_2;Y_2)\}_{m\geq 0}$ satisfies the identity

$$e^{x_1y_1+x_2y_2} + \tfrac{(x_1-x_2)(y_1-y_2)}{2^m}\sum_{m\geq 1}\Psi_{m-1}^{S_2}(X_2;Y_2)\frac{u^m}{m!} \;=\; e^{\frac{(x_1+x_2)(y_1+y_2)}{2}+\frac{(x_1-x_2)(y_1-y_2)}{2}\sqrt{1+2u}} \qquad \text{I.3}$$

This of course immediately implies (3). We also show that in the presence of (1) and (2) we have the sequence of implications

$$I.3 \implies (ii) \implies (i)$$

completing the proof of existence and uniqueness of a formal power series satisfying all the properties in I.1.

These results are derived from a parallel set of results for the dihedral group D_2. This approach leads to several identities which should be of independent interest. More precisely, we start by giving an explicit construction of the sequence of polynomials $\{P_m(x,y)\}_{m\geq 0}$, in two single variables x and y, which satisfy the following conditions

$$(A) \qquad P_m(x,y) \;=\; x^m y^m + \cdots (terms\ \ of\ \ x-degree\ \ <\ \ m)$$

and such that if we set

$$\Psi_m(x,y) \;=\; P_m(x,y)e^{xy} \qquad \text{I.4}$$

then

$$(B) \qquad x^{2m+1}\,\big|\,\big(\Psi_m(x,y)-\Psi_m(-x,y)\big)\,,$$

and

$$(C) \qquad \big(\partial_x^2 - \frac{2m}{x}\big)\Psi_m(x,y) \;=\; y^2\Psi_m(x,y)\,,$$

It is easy to see that (A) and (B) force the initial conditions

$$P_0(x,y) \;=\; 1\,, \qquad P_1(x,y) \;=\; x\,y-1 \qquad \text{I.5}$$

This given, we show that (A) and (B) uniquely determine that $P_m(x,y)$ can be written in the form

$$P_m(x,y) \;=\; (-1)^m(2m-1)!!W_m(x\,y) \qquad \text{I.6}$$

with

$$(2m-1)!! \;=\; (2m-1)\cdot(2m-3)\cdots 3\cdot 1$$

and $\{W_m(z)\}_{m\geq 0}$ a sequence of polynomials satisfying the 3-term recursion

$$W_{m+1}(z) \;=\; W_m(z) + \frac{z^2}{(2m+1)(2m-1)}\,W_{m-1}(z) \qquad \text{I.7}$$

and initial conditions

$$W_0(z) \;=\; 1\,, \qquad W_1(z) \;=\; 1-z$$

The recursion in I.6 reveals that the polynomials $W_n(z)$ are closely related to the Bessel polynomials $B_n(z)$ studied by Luc Favreau in [3]. This connection and I.6 yields the explicit expressions

$$a)\quad W_m(z) \;=\; \frac{1}{(2m-1)!!}\sum_{k=0}^{m}(-1)^k I_{2m-k}^{k}\, z^k\,, \qquad b)\quad P_m(x,y) \;=\; \sum_{k=0}^{m}(-1)^{m-k} I_{2m-k}^{k}\,(xy)^k \qquad \text{I.8}$$

We then derive from I.8 b) that the sequence $\{\Psi_m(x,y)\}_{m\geq 0}$ satisfies the identity

$$e^{xy} + xy\sum_{m\geq 1}\Psi_{m-1}(x,y)\frac{u^m}{m!} \;=\; \exp\Big(xy\sqrt{1+2u}\,\Big) \qquad \text{I.9}$$

This done we derive the sequence of implications

$$\text{I.9} \Longrightarrow (\text{C}) \Longrightarrow (B) \qquad \text{I.10}$$

completing the proof that the polynomial in I.8 is the unique solution of (A) and (B) as well as (A) and (C).

We should mention that (A) (B) and (C) identify $\{\Psi_m(x,y)\}_{m\geq 0}$ to be the sequence of Baker-Akhiezer functions of the dihedral group D_2 (see [2]). Now it is stated in [2] that this sequence may also be constructed from the single initial condition

$$\Psi_0(x,y) \;=\; 1$$

and the recursion

$$\Psi_m(x,y) \;=\; x\partial_x\,\Psi_{m-1}(x,y) \;-\; (2m-1)\Psi_{m-1}(x,y)\,. \qquad \text{I.11}$$

We show that also this recursion follows from I.9.

This paper is divided into four sections. In the first section we derive I.6 and I.7. In the second section we present the combinatorial setting that yields I.8. This is obtained by showning that the polynomials $\{I_m(z)\}_{m\geq 1}$ satisfying

$$I_m(z) \;=\; (2m-1)I_{m-1}(z) \,+\, z^2 I_{m-2}(z) \qquad (\text{with } I_0(z)=1 \text{ and } I_1(z)=1+z)$$

are given by the formula

$$I_m(z) \;=\; \sum_{k=1}^{m} I_{2m-k}^{k}\, z^k\,. \qquad \text{I.12}$$

In section 3 we use a bijection of L. Favreau to show that I.12 is equivalent to the generating function identity

$$1+z\sum_{m\geq 0} I_{m-1}(z)\frac{u^m}{m!} \;=\; e^{z(1-\sqrt{1-u})}\,. \qquad \text{I.13}$$

This yields I.9. We terminate the section by proving the implications in I.10. Moreover, we show there that I.9 is the only exponential formula which is consistent with I.9. In section 4 we show that I.13 can also be obtained by a simple manipulatorial argument. We also prove I.11 and again show that I.9 is the only exponential formula which is consistent with the recursion in I.11. We terminate the section with the proof of I.2 and I.3. We should mention that polynomials closely related to I.12 have also emerged from a different context in the works of B. Leclerc [6] and B. Leclerc and J.-Y. Thibon [7]. In [6] these polynomials are obtained as univariate specializations of staircase Schur functions. This suggests the possibility that the Baker-Akhiezer functions of S_n might be related to appropriate multivariate specializations of staircase Schur functions. Since all that is known so far about the Baker-Akhiezer functions is anything but explicit, such a development would be quite remarkable and worth further investigation.

1. The recursion

We shall start by constructing all polynomials $P_m(x,y)$ of degree m in x such that the formal power series

$$\Psi_m(x,y) \;=\; P_m(x,y)\sum_{s\geq 0}\frac{x^s y^s}{s!}$$

satisfies (B). To this end it will be convenient to write $P_m(x,y)$ in the form

$$P_m(x,y) \;=\; \sum_{r=0}^{m} a_r(y)\, y^r x^r\,. \qquad 1.1$$

A priori, doing this, may result in the coefficients $a_r(y)$ having powers of y in the denominator. However this will not happen as we shall see.

To begin note that we can write

$$\Psi_m(x,y)-\Psi_m(-x,y) \;=\; 2\sum_{r+s=odd}\frac{a_r(y)}{s!}\, y^{r+s}x^{r+s} \qquad 1.2$$

with the convention that $a_r(y)=0$ for $r>m$. Thus (B) may be expressed as the system of equations

$$\sum_{r+s=2k+1}\frac{a_r(y)}{s!}\, x^{r+s} \;=\; 0 \qquad (\text{for } k=0,1,\ldots,m-1) \qquad 1.3$$

It will be illuminating to view this system in a special case. For instance for $m=6$ we get

$$\begin{array}{lllllllll}
\frac{a_0}{1!} & +\frac{a_1}{0!} & & & & & & & =0\\[4pt]
\frac{a_0}{3!} & +\frac{a_1}{2!} & +\frac{a_2}{1!} & +\frac{a_3}{0!} & & & & & =0\\[4pt]
\frac{a_0}{5!} & +\frac{a_1}{4!} & +\frac{a_2}{3!} & +\frac{a_3}{2!} & +\frac{a_4}{1!} & +\frac{a_5}{0!} & & & =0\\[4pt]
\frac{a_0}{7!} & +\frac{a_1}{6!} & +\frac{a_2}{5!} & +\frac{a_3}{4!} & +\frac{a_4}{3!} & +\frac{a_5}{2!} & +\frac{a_6}{1!} & & =0\\[4pt]
\frac{a_0}{9!} & +\frac{a_1}{8!} & +\frac{a_2}{7!} & +\frac{a_3}{6!} & +\frac{a_4}{5!} & +\frac{a_5}{4!} & +\frac{a_6}{3!} & & =0\\[4pt]
\frac{a_0}{11!} & +\frac{a_1}{10!} & +\frac{a_2}{9!} & +\frac{a_3}{8!} & +\frac{a_4}{7!} & +\frac{a_5}{6!} & +\frac{a_6}{5!} & & =0
\end{array} \qquad 1.4$$

Thus we may express $a_1,a_2,\ldots,a_6$ in terms of a_0 by solving the system

$$\begin{array}{llllllll}
\frac{a_1}{0!} & & & & & & & =-\frac{a_0}{1!}\\[4pt]
\frac{a_1}{2!} & +\frac{a_2}{1!} & +\frac{a_3}{0!} & & & & & =-\frac{a_0}{3!}\\[4pt]
\frac{a_1}{4!} & +\frac{a_2}{3!} & +\frac{a_3}{2!} & +\frac{a_4}{1!} & +\frac{a_5}{0!} & & & =-\frac{a_0}{5!}\\[4pt]
\frac{a_1}{6!} & +\frac{a_2}{5!} & +\frac{a_3}{4!} & +\frac{a_4}{3!} & +\frac{a_5}{2!} & +\frac{a_6}{1!} & & =-\frac{a_0}{7!}\\[4pt]
\frac{a_1}{8!} & +\frac{a_2}{7!} & +\frac{a_3}{6!} & +\frac{a_4}{5!} & +\frac{a_5}{4!} & +\frac{a_6}{3!} & & =-\frac{a_0}{9!}\\[4pt]
\frac{a_1}{10!} & +\frac{a_2}{9!} & +\frac{a_3}{8!} & +\frac{a_4}{7!} & +\frac{a_5}{6!} & +\frac{a_6}{5!} & & =-\frac{a_0}{11!}
\end{array} \qquad 1.5$$

Thus the existence and uniqueness of the solution, given a_0, depends on the non-vanishing of the determinant

$$D_5 = \det \begin{Vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} \\ \frac{1}{9!} & \frac{1}{8!} & \frac{1}{7!} & \frac{1}{6!} & \frac{1}{5!} \end{Vmatrix} \tag{1.6}$$

In the general case the corresponding determinant is

$$D_{m-1} \;=\; \det \left\| \frac{1}{(2i-j)!} \right\|_{i,j=1}^{m-1} . \tag{1.7}$$

Now it is well known (see [8]) that for a given $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0) \vdash n$ the expression

$$f_\lambda \;=\; n! \det \left\| \frac{1}{(\lambda_i + j - i)!} \right\|_{i,j=1}^{k}$$

gives the number of standard tableaux of shape λ. Thus from the Frame-Robinson-Thrall formula we derive that

$$\det \left\| \frac{1}{(\lambda_i + j - i)!} \right\|_{i,j=1}^{k} \;=\; \frac{1}{h_\lambda} \tag{1.8}$$

where h_λ is the product of the hooks of λ. In particular, for $\lambda = (m-1, m-2, \ldots, 2, 1)$ 1.8 reduces to

$$\det \left\| \frac{1}{(m-i+j-i)!} \right\|_{i,j=1}^{m-1} \;=\; \frac{1}{(2m-3)!! \cdots 3!! \cdot 1!!} . \tag{1.9}$$

Since

$$\det \left\| \frac{1}{(m-i+j-i)!} \right\|_{i,j=1}^{m-1} \;=\; \det \left\| \frac{1}{(2i-j)!} \right\|_{i,j=1}^{m-1}$$

from 1.7 we get that

$$D_{m-1} \;=\; \frac{1}{(2m-3)!! \cdots 3!! \cdot 1!!} . \tag{1.10}$$

In particular

$$D_5 \;=\; \frac{1}{9!! \cdot 7!! \cdot 5!! \cdot 3!! \cdot 1!!} .$$

This proves that the system in 1.5 and more generally the system in 1.3 has a unique solution for any given choice of $a_0(y)$.

Thus we can state

Theorem 1.1

If the constants $c_1^{(m)}, c_2^{(m)}, \ldots, c_m^{(m)}$ are obtained by solving 1.3 for $a_0 = 1$, then every polynomial $P_m(x,y)$, of degree $\leq m$ in x such that $\Psi_m(x,y) = P_m(x,y)e^{xy}$ satisfies (B) may be written in the form

$$P_m(x,y) \;=\; u(y)p_m(x,y)$$

with

$$p_m(x,y) \;=\; 1 + \sum_{r=1}^{m} c_r^{(m)}(xy)^r \qquad 1.11$$

In particular the symmetry condition $P_m(x,y) = P_m(y,x)$ forces $u(y)$ to be a constant as well.

To identify the polynomial in 1.11 we need two auxiliary results

Proposition 1.1

For all $m \geq 1$ we have

$$p_m(x,y)\big|_{x^m} \;=\; \tfrac{(-1)^m}{(2m-1)!!}\, y^m \qquad 1.12$$

Proof

Let us view again the special case $m = 6$. Here, an application of Cramer's rule to the system in 1.5 (with $a_0 = 1$) yields that

$$c_6^{(6)} \;=\; \frac{1}{D_5}\det\left\|\begin{matrix} \frac{1}{0!} & 0 & 0 & 0 & 0 & -\frac{1}{1!} \\ \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & -\frac{1}{3!} \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & -\frac{1}{5!} \\ \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & -\frac{1}{7!} \\ \frac{1}{8!} & \frac{1}{7!} & \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & -\frac{1}{9!} \\ \frac{1}{10!} & \frac{1}{9!} & \frac{1}{8!} & \frac{1}{7!} & \frac{1}{6!} & -\frac{1}{11!} \end{matrix}\right\|$$

or better

$$c_6^{(6)} \;=\; \frac{(-1)^5}{D_5}\det\left\|\begin{matrix} -\frac{1}{1!} & \frac{1}{0!} & 0 & 0 & 0 & 0 \\ -\frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 & 0 \\ -\frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} \\ -\frac{1}{7!} & \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} \\ -\frac{1}{9!} & \frac{1}{8!} & \frac{1}{7!} & \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} \\ -\frac{1}{11!} & \frac{1}{10!} & \frac{1}{9!} & \frac{1}{8!} & \frac{1}{7!} & \frac{1}{6!} \end{matrix}\right\| \;=\; \frac{(-1)^6}{D_5}\,D_6 \;=\; \frac{(-1)^6\, 9!!\cdot 7!!\cdot 5!!\cdot 3!!}{11!!\cdot 9!!\cdot 7!!\cdot 5!!\cdot 3!!} = \frac{(-1)^6}{11!!}$$

It is easily seen that in the general case we will get

$$c_m^{(m)} \;=\; \frac{(-1)^m}{D_{m-1}}\, D_m \;=\; \frac{(-1)^m}{(2m-1)!!}\,.$$

This proves 1.12.

Remark 1.1

We should note that Theorem 1.1 together with 1.12 imply that any polynomial $P_m(x,y)$ which satisfies (B) must be of degree at least m in x or identically vanish. Let us keep in mind this fact since it will play a crucial role later.

Proposition 1.2

The polynomials of degree at most $m+k$

$$q(x,y) \;=\; \sum_{r=1}^{m+k} d_r (xy)^r$$

such that $\Psi_m(x,y) = q(x,y)e^{xy}$ satisfies (B) span a $k+1$-dimensional vector space.

Proof

Let us view the argument in a special case. For instance for $m = 6$ and $k = 1$ the constants $d_0, d_1, d_2, \ldots, d_7$ must satisfy the homogeneous system

$$\begin{array}{ccccccccccccc}
\frac{d_1}{0!} & & & & & & & & & & & = & -\frac{d_0}{1!} \\
\frac{d_1}{2!} & + & \frac{d_2}{1!} & + & \frac{d_3}{0!} & & & & & & & = & -\frac{d_0}{3!} \\
\frac{d_1}{4!} & + & \frac{d_2}{3!} & + & \frac{d_3}{2!} & + & \frac{d_4}{1!} & + & \frac{d_5}{0!} & & & = & -\frac{d_0}{5!} \\
\frac{d_1}{6!} & + & \frac{d_2}{5!} & + & \frac{d_3}{4!} & + & \frac{d_4}{3!} & + & \frac{d_5}{2!} & + & \frac{d_6}{1!} & = & -\frac{d_0}{7!} - \frac{d_7}{0!} \\
\frac{d_1}{8!} & + & \frac{d_2}{7!} & + & \frac{d_3}{6!} & + & \frac{d_4}{5!} & + & \frac{d_5}{4!} & + & \frac{d_6}{3!} & = & -\frac{d_0}{9!} - \frac{d_7}{2!} \\
\frac{d_1}{10!} & + & \frac{d_2}{9!} & + & \frac{d_3}{8!} & + & \frac{d_4}{7!} & + & \frac{d_5}{6!} & + & \frac{d_6}{5!} & = & -\frac{d_0}{11!} - \frac{d_7}{4!}
\end{array} \qquad 1.13$$

and the non-vanishing of the determinant in 1.6 yields that $d_1, d_2, \ldots, d_6$ are uniquely determined by d_0 and d_7. Since the latter may be arbitrarily prescribed we see that in this case the polynomials $q(x,y)$ span a 2-dimensional space. It should be clear that an analogous argument can be carried out in the general case.

The special case $k = 1$ of this result completely characterizes the sequence $\{p_m(x,y)\}_{m\geq 0}$. More precisely we have

Theorem 1.2

The polynomials $p_m(x,y)$ given by 1.11 satisfy the recurrence

$$p_{m+1}(x,y) \;=\; p_m(x,y) \;+\; \frac{(xy)^2}{(2m+1)(2m-1)}\, p_{m-1}(x,y) \qquad 1.14$$

together with the initial conditions

$$p_0(x,y)=1\,, \qquad p_1(x,y)=1-xy\,. \qquad 1.15$$

Proof

Our definition of $p_{m+1}(x,y), p_m(x,y)$ and $p_{m-1}(x,y)$ assures that the differences

$$p_{m+1}(x,y)e^{xy} - p_{m+1}(-x,y)e^{-xy}$$
$$p_m(x,y)e^{xy} - p_m(-x,y)e^{-xy}$$
$$(xy)^2 p_{m-1}(x,y)e^{xy} - (-xy)^2 p_{m-1}(-x,y)e^{-xy}$$

are all divisible by x^{2m+1}. Thus these three polynomials belong to a vector space of polynomials of degree at most $m+1$ which, by Proposition 1.1, is 2 dimensional. Since $p_m(x,y)$ and $(xy)^2 p_{m-1}(x,y)$ are linearly independent, there must be coefficients $a(y)$ and $b(y)$ yielding

$$p_{m+1}(x,y) \;=\; a(y)\, p_m(x,y) \;+\; b(y)\,(xy)^2 p_{m-1}(x,y)\,. \qquad 1.16$$

Since by construction $p_{m+1}(0,y)=p_m(0,y)=1$, setting $x=0$ in 1.16 gives $a(y)=1$. On the other hand, equating coefficients of x^{m+1}, Proposition 1.1 gives

$$\frac{(-1)^{m+1}}{(2m+1)!!}\, y^{m+1} \;=\; b(y)\frac{(-1)^{m+1}}{(2m-3)!!}\, y^{m+1}$$

or better

$$b(y) \;=\; (2m+1)(2m-1)\,.$$

Thus 1.14 must hold true precisely as asserted. The initial conditions in 1.15 are forced by the condition $p_m(0)=1$ and 1.12 for $m=1$.

It will be convenient here and after to set

$$W_m(z) \;=\; 1+\sum_{r=1}^{m} c_r^{(m)} z^m \qquad 1.17$$

so that we may write

$$p_m(x,y) \;=\; W(xy)\,. \qquad 1.18$$

In this notation, Theorem 1.2 simply states that the sequence $\{W_m(z)\}_{m\geq 0}$ statisfies the recursion

$$W_m(z) = W_{m-1}(z) + \frac{z^2}{(2m-1)(2m-3)} W_{m-2}(z) \qquad 1.19$$

with initial conditions

$$W_0(z) = 1\,, \quad W_1(z) = 1 - z\,. \tag{1.20}$$

Combining 1.12 and 1.18 we get that the polynomial $P_m(x,y)$ satisfying (A) and (B) may now be expressed as

$$P_m(x,y) \;=\; (-1)^m(2m-1)!!W_m(xy) \tag{1.21}$$

2. Counting involutions.

Solving the recursion in 1.19 with the initial conditions in 1.20 produces a sequence of polynomials with rational coefficients of alternating signs. However, the underlying combinatorial mechanism yielding these polynomials quickly emerges by working with the sequence

$$I_m(z) \;=\; (2m-1)!!W_m(-z) \tag{2.1}$$

In fact, multiplying both sides of 1.19 by $(2m-1)!!$ and replacing z by $-z$ we get

$$(2m-1)!!W_m(-z) = (2m-1)\times(2m-3)!!W_{m-1}(-z) + z^2(2m-5)!!W_{m-2}(z)$$

and 2.1 gives

$$I_m(z) \;=\; (2m-1)I_{m-1}(z) + z^2 I_{m-2}(z)\,, \tag{2.2}$$

and the initial conditions

$$I_0(z) = 1\,, \quad I_1(z) = 1 + z\,. \tag{2.3}$$

Starting from 2.3 and iterating according to 2.2 we obtain

$$\begin{gathered}
I_0 = 1\\
I_1 = 1 + z\\
I_2 = 3 + 3\,z + z^2\\
I_3 = 15 + 15\,z + 6\,z^2 + z^3\\
I_4 = 105 + 105\,z + 45\,z^2 + 10\,z^3 + z^4\\
I_5 = 945 + 945\,z + 420\,z^2 + 105\,z^3 + 15\,z^4 + z^5
\end{gathered}$$

It develops that a work of L. Favreau [3] yields a combinatorial interpretations for these coefficients. To be precise, it is stated in [3] (see pp. 72 and 76) that the sequence of polynomials $B_m(z)$ which satisfies the recursion

$$B_m(z) \;=\; (2m-1)zB_{m-1}(z) + B_{m-2}(z)\,, \tag{2.4}$$

with initial conditions $B_o(z) = 1$ and $B_1(z) = 1 + z$ may be expressed in the form

$$B_m(z) \;=\; \sum_{k=0}^{m} I^k_{2m-k} z^{m-k} \tag{2.5}$$

where I^k_{2m-k} gives the number of involutions in S_{2m-k} with k fixed points. Note that if we replace z by $1/z$ in 2.4 and multiply both sides by z^m we get

$$z^m B_m(1/z) \;=\; (2m-1)z^{m-1}B_{m-1}(1/z) + z^2 z^{m-2}B_{m-2}(1/z)\,,$$

Comparing with 2.2 we see that we must have

$$I_m(z) \;=\; z^m\, B_m(1/z) \qquad 2.6$$

thus from 2.5 we derive that

$$I_m(z) \;=\; \sum_{k=0}^{m} I^k_{2m-k} z^k \qquad 2.7$$

and 2.1 gives

$$W_m(z) \;=\; \frac{1}{(2m-1)!!}\sum_{k=0}^{m}(-1)^k I^k_{2m-k}\, z^k\,. \qquad 2.8$$

The relation in 2.5 is proved in [3] by a combinatorial argument. We can show 2.7 here by appropriately counting involutions. To be precise the expansion in 2.7 is an immediate corollary of the following identity

Proposition 2.1

For all $1 \leq k \leq 2m-k$ we have

$$I^k_{2m-k} \;=\; (2m-1)I^k_{2m-k-2} \;+\; I^{k-2}_{2m-k-2} \qquad 2.9$$

Proof

Note that we may construct an involution in S_n by first choosing the fixed points and then pairing-off the remaining letters into two cycles. For $n = 2m-k$, k fixed points can be chosen in $\binom{2m-k}{k}$ distinct ways. This done, the remaing $2m-2k$ letters can paired off in $(2(m-k)-1)!!$ distinct ways. This gives

$$I^k_{2m-k} \;=\; \binom{2m-k}{k}(2(m-k)-1)!!$$

Thus 2.9 simply states that

$$\binom{2m-k}{k}(2(m-k)-1)!! \;=\; (2m-1)\binom{2m-k-2}{k}(2(m-k)-3)!! \;+\; \binom{2m-k-2}{k-2}(2(m-k)-1)!!$$

But this is

$$\frac{(2m-k)!}{k!(2m-2k)!}(2(m-k)-1)!! \;=\; (2m-1)\frac{(2m-k-2)!}{k!(2m-2k-2)!}\,(2(m-k)-3)!! \;+\; \frac{(2m-k-2)!}{(k-2)!(2m-2k)!}\,(2(m-k)-1)!!$$

cancelling common factors and multiplying by $k(k-1)(2m-2k)$ gives

$$(2m-k)(2m-k-1) \;=\; (2m-1)(2m-2k) \;+\; k(k-1)$$

which is easily seen to be true for $k=1$, $k=0$ and $k=m$. This proves 2.9.

From 2.9 we derive that the polynomial in the right hand side of 2.7 satisfies the same recursion as $I_m(z)$. Since we may take $I^0_0 = 1$ and $I^0_1 = I^1_1 = 1$ these polynomials satisfy also the same initial conditions, and the equalities in 2.7 and 2.8 necessarily follow.

We can thus state the basic result of this section

Theorem 2.1

The polynomial $P_m(x,y)$ uniquely characterized by (A) and (B) has the explicit expansion

$$P_m(x,y) \;=\; \sum_{k=0}^{m}(-1)^{m-k} I_{2m-k}^{k}\,(xy)^k \qquad 2.10$$

Proof

Formula 2.10 is an immediate consequence of 2.8 and 1.21.

Remark 2.1

The combinatorial proof of the identity in 2.9 given in [3] is quite interesting and is worth including here. To begin, set $N=2m-k$ and denote by $\mathcal{I}_N^k$, $\mathcal{I}_{N-2}^k$ and $\mathcal{I}_{N-2}^{k-2}$ respectively the collections of involutions on N, $N-2$ letters with k and $k-2$ fixed points. This given, to establish 2.8 we need only exhibit a bijection sending $\mathcal{I}_N^k$ onto the union of $2m-1$ copies of $\mathcal{I}_{N-2}^k$ together with a copy of $\mathcal{I}_{N-2}^{k-2}$. We can easily see that this bijection is simply obtained by "*removing*" N and $N-1$ from the graphic representation of elements of $\mathcal{I}_N^k$. Indeed, if an element $e\in\mathcal{I}_N^k$ has both N and $N-1$ as fixed points the resulting element e' falls in $\mathcal{I}_{N-2}^{k-2}$. If neither N and $N-1$ are fixed points, and $(N,N-1)$ is a cycle of e then this removal produces an element of $\mathcal{I}_{N-2}^k$. If N and $N-1$ occur in different cycles (N,i) and $(N-1,j)$, we simply remove $N,N-1$ and join i,j into a cycle. To remember how this cycle was created by the operation, we consider the resulting involution of S_{N-2} as the element e' of $\mathcal{I}_{N-2}^k$ with the cycle (i,j) rendered with i in red and j in blue. This operation requires $2m-2k-2$ copies $\mathcal{I}_{N-2}^k$, since there are 2 different ways of coloring one of the $m-k-1$ 2-cycles of an element of $\mathcal{I}_{N-2}^k$. Finally two possibilities remain. We could have $N-1$ as fixed point and a cycle (N,i), or N as fixed point and a cycle $(N-1,i)$. Here we remove N and $N-1$ and make i into a fixed point colored red in the first case and blue in the second case. In this instance we need $2k$ copies of $\mathcal{I}_{N-2}^k$ depending on which fixed point is given which color. In summary the removal of $N,N-1$ in the manner indicated above produces: One copy of $\mathcal{I}_{N-2}^{k-1}$ and

$$1 \;+\; 2m-k-2 \;+\; 2k \;=\; 2m-1$$

copies of $\mathcal{I}_{N-2}^k$.The reversibility of the process yields then the equivalence

$$\mathcal{I}_N^k \;\approx\; \mathcal{I}_{N-2}^{k-2} \cup \bigcup_{i=1}^{2m-1} \mathcal{I}_{N-2}^k$$

establishing 2.8

3. The generating function

Luc Favreau in [3] goes on to construct by a beautiful combinatorial argument the exponential generating function of the polynomials $B_m(z)$. To be precise, in the present notation, it is proved in [3] (section 4.4) that

$$1+\sum_{m\geq 1} B_{m-1}(x)\frac{t^m}{m!} \;=\; e^{\frac{1-\sqrt{1-2xt}}{x}}\,. \qquad 3.1$$

Replacing x by $1/z$ and t by uz we get

$$1+\sum_{m\geq 1} B_{m-1}(1/z)\frac{(uz)^m}{m!} \;=\; e^{z(1-\sqrt{1-2u}\,)}\,.$$

and 2.6 gives

$$1+z\sum_{m\geq 1} I_{m-1}(z)\,\frac{u^m}{m!} \;=\; e^{z(1-\sqrt{1-2u}\,)}\,. \tag{3.2}$$

Luc Favreau's argument may be used to give a direct and slightly simpler proof of this identity. We reproduce it here for sake of completeness. To do this we need to review the combinatorics of "*exponential structures* ".

For each $k\geq 1$ we are given a finite deck of cards $\mathcal{D}_k$. In each card $\gamma\in\mathcal{D}_k$ there is a figure which contains k circles respectively indexed by the letters $1,2,\ldots,k$. Given a partition $\pi=(E_1,E_2,\ldots,E_r)$ of the set $\{1,2,\ldots,n\}$ with parts of cardinalities $k_1,k_2,\ldots,k_r$, a composite deck of r cards is obtained by picking a card γ_i from the deck $\mathcal{D}_{k_i}$ and filling its circles with the elements of E_i. To be precise, if $E_i=\{j_1,j_2,\ldots,j_{k_i}\}$ then in the circle of γ_i indexed by s we place j_s. Let us denote the resulting card by $\gamma_i(E_i)$. We thus obtain a composite deck

$$\rho(\pi,\gamma) \;=\; \big(\gamma_1(E_1),\gamma_2(E_2),\ldots,\gamma_r(E_r)\big)\,. \tag{3.3}$$

Denoting by $\Pi_{n,r}$ the collection of set partitions $\pi=(E_1,E_2,\ldots,E_r)$ of $\{1,2,\ldots,n\}$ the family of composite decks

$$\mathcal{E}(\{\mathcal{D}_k\}_{k\geq 1}) \;=\; \bigcup_{n\geq 1}\bigcup_{r=1}^{n}\Big\{\big(\gamma_1(E_1),\gamma_2(E_2),\ldots,\gamma_r(E_r)\big)\,:\,(E_1,E_2,\ldots,E_r)\in\Pi_{n,r}\ \&\ \gamma_i\in\mathcal{D}_{|E_i|}\Big\}$$

is called "*the exponental structure* " generated by the collection $\mathcal{D}=\{\mathcal{D}_k\}_{k\geq 1}$. It is customary to give the composite deck $\rho(\pi,\gamma)$ in 3.3 a weight $w(\rho(\pi,\gamma))$ which is a monomial whose factors account for some characteristic features of the deck. For our purposes here it is sufficient to set

$$w(\rho(\pi,\gamma)) \;=\; \frac{z^r u^n}{n!}\,.$$

This given, one of the earliest results of enumeration theory states that

$$1+\sum_{\rho\in\mathcal{E}(\mathcal{D})} w(\rho) \;=\; exp\Big(z\sum_{k\geq 1}|\mathcal{D}_k|\frac{u^k}{k!}\Big) \tag{3.4}$$

Denoting by $p_n(z,\mathcal{D})$ the polynomial

$$p_n(z,\mathcal{D}) \;=\; \sum_{r=1}^{n}\ \sum_{\substack{(E_1,E_2,\ldots,E_r)\in\Pi_{n,r}\\ \gamma_i\in\mathcal{D}_{|E_i|}}} z^r \tag{3.5}$$

the identity in 3.4 may be rewritten as

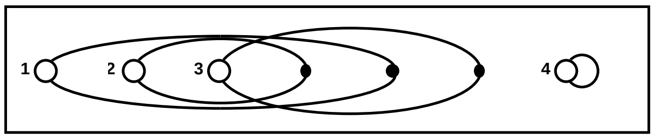
$$1+\sum_{n\geq 1} p_n(z,\mathcal{D})\frac{u^n}{n!} \;=\; exp\Big(z\sum_{k\geq 1}|\mathcal{D}_k|\frac{u^k}{k!}\Big)\,. \tag{3.6}$$

We call this the "*Exponential Formula* " yielded by the family of decks $\mathcal{D}=\{\mathcal{D}_k\}_{k\geq 1}$

Note now that we have the power series expansion

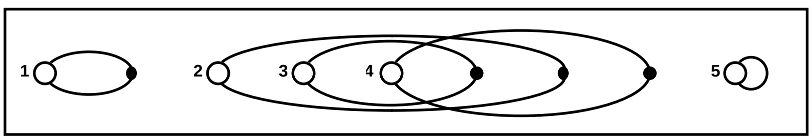
$$1-\sqrt{1-2u} \;=\; u+\sum_{k\geq 2}(2k-3)!!\,\frac{u^k}{k!}$$

Thus to prove 3.2 we need to work with a family $\mathcal{D}=\{\mathcal{D}_k\}_{k\geq 1}$ where $\mathcal{D}_k$ for $k\geq 2$ has cardinalty $(2k-3)!!$. The present context strongly suggests that we should take $\mathcal{D}_k$ (for $k\geq 2$) to be a family of cards each depicting an involution consisting of $k-1$ *two-cycles*. Luc Favreau's ingenious choice is simply to insert an additional fixed point. The cards used in [3] are best understood through examples. For instance the figure below gives a typical card in $\mathcal{D}_4$



3.7

Each of the first 3 ovals here represents a 2-cycle, the last represents a fixed point. Note that the first oval can be constructed in 5 different ways, this done for the second we have only 3 choices and the last is forced. So we see that our deck $\mathcal{D}_4$ contains $5\times 3=15$ cards. As required each card has four circles indexed $1,2,3,4$. The figure below exhibits a card of $\mathcal{D}_5$

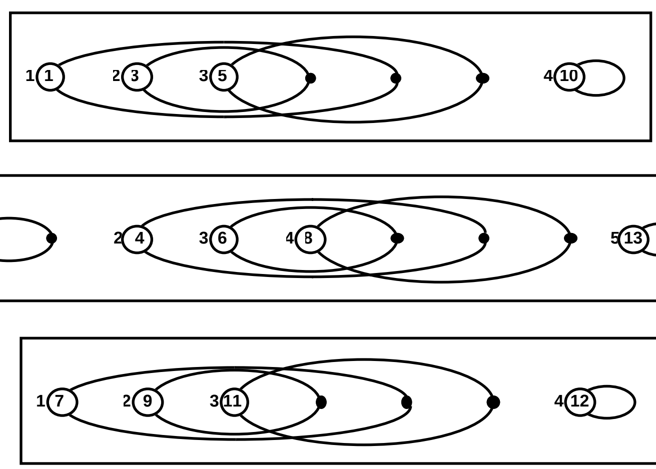


3.8

Now suppose we are given a 3-part partition $\Pi=(E_1,E_2,E_3)$ of the set of $\{1.2.\dots,13\}$ with

$$E_1=\{1,3,5,10\} \quad ,E_2=\{2,4,6,8,13\} \quad E_3=\{7,9,1,12\}\big) \;.$$

As customary, the parts of a partitions are ordered by increasing minimal elements. To construct a composite deck from this partition we must select $\gamma_1,\gamma_3\in\mathcal{D}_4$ and $\gamma_2\in\mathcal{D}_5$. Suppose we take the card in 3.7 for both γ_1 and γ_3 and the card in 3.8 for γ_2. Then according to the recipe above the resulting composite deck is as illustrated below.



3.9

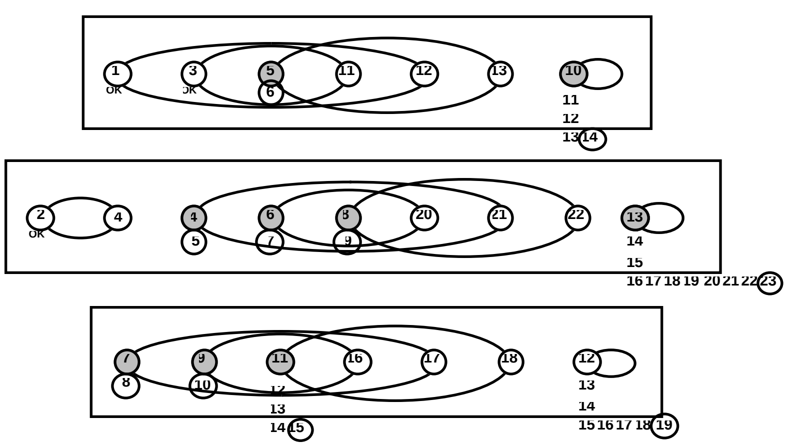
The trick now is to see that this is none other than an involution in disguise. To be precise, in order to obtain 3.2 from 3.6. we must show that for this particular collection of decks $\mathcal{D}$ we have

$$p_m(z;\mathcal{D}) \;=\; z\, I_{m-1}(z) \tag{3.10}$$

Since the composite deck in 3.9 has weight $z^3\, u^{13}/10!$, it must turn out to be one of those contributing to $I_{12}(z)$. More specifically, the fact that z factors out in 3.10, forces this deck to one of those counted by $I^2_{2\times 12-2}$. In summary, 3.9 must represent an involution with two fixed points in S_{22}. This requires an algorithm which transforms the labeling in 3.9 into an appropriate labelling, by $1, 2, \ldots, 23$, of all the nodes in 3.9 (circles and dots included). The algorithm presented in [3] produces such a relabeling with the highest label falling on a fixed point. The removal of this fixed point produces the desired target involution.

In the general case Luc Favreau's relabeling process defines a map Φ which sends a composite deck $\rho = \rho(\pi, \gamma)$ with $k+1$ cards and labels $1, 2, \ldots, m$ onto an involution $\sigma = \Phi(\rho)$ with k fixed points in $S_{2(m-1)-k}$. The invertibility of Φ assures that we have the bijection between composite decks and involutions that is needed to establish 3.2.

It will suffice to give a brief description of Luc Favreau's relabeling process in the particular case of the composite deck of 3.9. This process is quite simple. In this case we are to ultimately place the labels $1, 2, \ldots, 23$ in the 23 nodes of the composite deck. At the i^{th} step of the process the final labels $1, 2, \ldots i-1$ have been placed. At the same time there will be a number of "*discarded* " labels and a number of "*candidate* " labels which include the label i. The card γ that contains the label i is located and i is converted into a final label if to the left of i in γ there are no unlabelled nodes, otherwise the left-most unlabelled node of γ is given the label i and each candidate label $j > i$ is discarded and replaced by a new candidate label $j+1$. At the start of the process all the labels are candidate. In this example the starting configuration is 3.9 and $1, 2, \ldots, 13$ are all candidate labels. Now 1 has no unlabeled node to the left, so it becomes final. The same for 2 and 3. However 4 has an unlabelled node to the left so that node is given the final label 4 and the candidate labels $4, 5, 6, 7, 8, 9, 10, 11, 12, 13$ are discarded and respectively replaced by the new candidate labels $5, 6, 7, 8, 9, 10, 11, 12, 13, 14$. Now the candidate label 5 has no unlabelled nodes to its left and it becomes final. The same holds true for the candidate labels $6, 7, 8, 9, 10$. Next, 11 has a unlabelled node to its left. That node gets the final label 11 and the candidate labels $11, 12, 13, 14$ are discarded and replaced by the candidate labels $12, 13, 14, 15$. It should be now quite clear how this labeling process is continued. The display below gives the target involution. All the final labels are in clear circles. The labels in the shaded circles are the discarded original labels. We have also listed in succession all the discarded candidate labels. The target involution is obtained by removing the fixed point labeled 23.



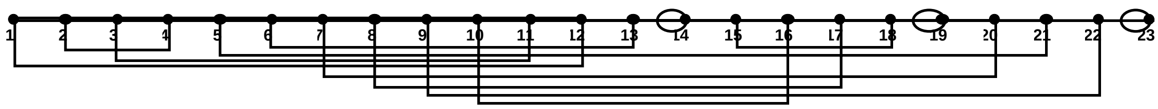
3.11

We should note that at any step of the process the final labels as well as the candidate labels increase from left to right in each card. This fact uniquely determines the reverse process which yields the inverse of the map Φ.

To be precise, given an involution $\sigma \in S_N$ $(N = 2(m-1)-k)$ with k fixed points we are to construct the composite deck $\rho = \rho(\pi, \gamma)$ with $k+1$ cards and labels $1, 2, \ldots, m$ such that .

$$\Phi(\rho) = \sigma$$

Of course the reverse process must start by adding a fixed point with label $N+1$. This done, the process must relable all the nodes of σ and at the same time construct the $k+1$ cards of ρ by assigning each two-cycle of σ to an appropriate fixed point. To carry this process it is helpful to draw σ with its nodes labelled $1, 2, \ldots, N+1$ drawn from left to right on a straight line. In this manner each two cycle (i, j), with $i < j$, will appear with i to the left of j as in the figure below where for clarity we have represented a two-cycle (i, j) as a path from i to j. We will call i the "*head* " and j the "*tail* " of (i, j)



3.12

The reader may recognize this as the involution in 3.11 with $N+1 = 23$ and $k = 2$. It should be clear at the start that all the two-cycles whose tail is to the right of the k^{th} fixed point must be assigned to the last fixed point. Thus in this case the first step of the relabeling process is to assign the cycle $(9, 22)$ to the last fixed point, remove the label 22, discard 23 and replace it by 22. In the next step, we assign $(5, 21)$ to the last fixed point, remove the label 21, discard 22 and replace it by 21. Next, we assign $(7, 20)$ to the last fixed point, remove the label 20, discard 21 and replace it by 20.

In this process, when we operate on the label r all the cycles (i, j) with $j > r$ have already been assigned to a fixed point and their tails have no longer a label. There are two cases to be considered according as a) r is a head of a cycle c or b) r is a tail of c. In the first case c has already been assigned and its tail has no label. We leave r alone and proceed to operate on $r-1$. In case b) we know that in the construction of Φ the label r was inherited from the head of a cycle c' to the right of it in *the same card*. At this stage c' is easily identified. It is precisely the cycle whose head has label $r+1$. If c' is a fixed point then we assign c to c'. If c' is a two-cycle then at this time c' has already been assigned to a fixed point and we are forced to assign c to the same fixed point. In either case we move r from the tail of c to the head of c' and decrease by one all the labels greater than r. After we finish processing the label 1 only the heads of the $m-k-1$ two-cycles have a label and since there are $k+1$ fixed points the end product is a composite deck $\rho(\pi, \gamma)$ with $k+1$ cards with labels $1, 2, \ldots, m$. The parts of the partition π are then the $k+1$ subsets of $\{1, 2, \ldots, m\}$ obtained by reading the labels from each of the cards. The $k+1$ components of γ will then be given by the geometry of the cards induced by the rectilinear diagram of the original involution σ.

This completes the proof of 3.2.

We can now derive the main result of this section

Theorem 3.1

The sequence

$$\Psi_m(x,y) = P_m(x,y)\, e^{xy} \tag{3.13}$$

is generated by the exponential formula

$$e^{xy} + xy \sum_{m\geq 1} \Psi_{m-1}(x,y)\frac{u^m}{m!} \;=\; \exp\Big(xy\sqrt{1+2u}\,\Big) \tag{3.14}$$

Proof

Note that 2.1 and 1.21 give

$$P_m(x,y) \;=\; (-1)^m I_m(-xy)\,. \tag{3.15}$$

Making the replacement $z\rightarrow -xy$ in 3.2 we obtain

$$1 - xy\sum_{r\geq 1} I_{m-1}(-xy)\frac{u^m}{m!} \;=\; e^{xy\,(\sqrt{1-2u}-1)}$$

and the change of sign $u\rightarrow -u$ gives

$$1 + xy\sum_{r\geq 1} (-1)^{m-1} I_{m-1}(-xy)\frac{u^m}{m!} \;=\; e^{xy\,(\sqrt{1+2u}-1)}\,.$$

Using 3.15 we now get

$$1 + xy\sum_{r\geq 1} P_{m-1}(x,y)\frac{u^m}{m!} \;=\; e^{xy\,(\sqrt{1+2u}-1)}\,.$$

Multiplying both sides by $e^{x\,y}$ and using 3.13 yields 3.14 as desired.

Our next task is to show that 3.14 implies (C). What is remarkable at this point is that 3.14 is in fact, equivalent to (C). To begin with we have

Theorem 3.2

The sequence $\Psi_m(x,y)$ *with initial conditions*

$$\Psi_0(x,y) = e^{xy}\,, \qquad \Psi_1(x,y) = (xy-1)e^{xy}$$

generated by the exponential formula

$$e^{xy} + xy\sum_{r\geq 1} \Psi_{m-1}(x,y)\frac{u^m}{m!} \;=\; e^{xy\,f(u)}\,, \tag{3.16}$$

satisfies the differential equation

$$\Big(\partial_x^2 - \;\frac{2m}{x}\partial_x\Big)\Psi_m(x,y) = y^2\,\Psi_m(x,y)\,. \tag{3.17}$$

if and only if

$$f(u) = \sqrt{1+2u}\,. \tag{3.18}$$

Proof

It is convenient to simplify our notation and set

$$\Psi_m(x,y) \;=\; \Psi_m(xy)$$

with $\{\Psi_m(z)\}_{m\geq 0}$ a sequence of formal power series in z satisfying the initial conditions

$$\Psi_0(z) = e^z\,, \qquad \Psi_1(z) = (z-1)e^z\,. \tag{3.19}$$

With this notation we see that 3.17 will hold true if and only if for all $m \geq 1$ we have

$$\Psi''_{m-1}(z) - \frac{2(m-1)}{z}\Psi'_{m-1}(z) \;=\; \Psi_{m-1}(z)\,. \tag{3.20}$$

Thus we need to show that if

$$e^z + z\sum_{m\geq 1}\Psi_{m-1}(z)\frac{u^m}{m!} \;=\; e^{z\,f(u)}\,, \tag{3.20}$$

then 3.19 and 3.20 are equivalent to 3.18.

To this end note that applying the operator $u\partial_u$ to both sides of 3.20 we get

$$\sum_{m\geq 1} m\,\Psi_{m-1}(z)\frac{u^m}{m!} \;=\; uf'(u)\,e^{z\,f(u)}\,.$$

and this differentiated by z gives

$$\sum_{m\geq 1} m\,\Psi'_{m-1}(z)\frac{u^m}{m!} \;=\; uf(u)f'(u)\,e^{z\,f(u)}\,. \tag{3.21}$$

On the other hand differentiating twice by z both sides of 3.20 we get

$$e^z + 2\sum_{m\geq 1}\Psi'_{m-1}(z)\frac{u^m}{m!} \;+\; z\sum_{r\geq 1}\Psi''_{m-1}(z)\frac{u^m}{m!} \;=\; \big(f(u)\big)^2 e^{z\,f(u)}\,,$$

and subtracting from this twice 3.21 gives

$$e^z \;+\; z\sum_{m\geq 1}\Big(\Psi''_{m-1}(z) - \frac{2(m-1)}{z}\,\Psi'_{m-1}(z)\Big)\frac{u^m}{m!} \;=\; \big(f^2 - 2uff'\big)e^{z\,f}\,.$$

Subtracting 3.20 we are finally brought to the identity

$$z\sum_{m\geq 1}\Big(\Psi''_{m-1}(z) - \frac{2(m-1)}{z}\,\Psi'_{m-1}(z) - \Psi_{m-1}(z)\Big)\frac{u^m}{m!} \;=\; \big(f^2 - 2uff' - 1\big)e^{z\,f}\,,$$

which shows that 3.20 is equivalent to

$$f^2 - 2uff' \;=\; 1\,.$$

Now this may be rewritten in the form

$$\frac{\big(f^2-1\big)'}{f^2-1} \;=\; \frac{1}{u}$$

So for some constant c we must have

$$f^2-1 \;=\; c\,u$$

or better

$$f(u) = \pm\sqrt{1+c\,u} \qquad 3.22$$

The argument is completed by showing that the initial conditions in 3.19 force "+" and $c=2$ in 3.22 giving

$$f(u) \;=\; \sqrt{1+2\,u}$$

as desired.

To complete the implications in I.10 we need to show

$$(C) \implies (i) \qquad 3.23$$

But to do that we need the following auxiliary result

Proposition 3.1

Let $P(x,y)$ and $Q(x,y)$ be homogeneous polynomials of degrees r and $r-2$ such that

$$(\partial_x^2 - \tfrac{2m}{x}\partial_x)P(x,y) = Q(x,y) \qquad 3.24$$

then

$$x^{2m+1}\,\big|\,\big(Q(x,y)-Q(-x,y)\big) \qquad 3.25$$

implies

$$x^{2m+1}\,\big|\,\big(P(x,y)-P(-x,y)\big) \qquad 3.26$$

Proof

Let us write

$$Q(x,y) = \sum_{s=0}^{r-2} q_s x^s y^{r-2-s} \quad \text{and} \quad P(x,y) = \sum_{s=0}^{r} p_s x^s y^{r-s} \qquad 3.27$$

then we have have 3.25 if and only if

$$q_{2s+1} = 0 \quad \text{for } 0 \le s \le m-1 \qquad 3.28$$

and analogously we have 3.26 if and only if

$$p_{2s+1} = 0 \quad \text{for } 0 \le s \le m-1 \qquad 3.29$$

Thus we need to show that, in the presence of 3.24, 3.28 implies 3.29. To this end note first that since $\partial_x^2 P(x,y)$ and $Q(x,y)$ are polynomials it follows from 3.24 that $\partial_x P(x,y)$ must be divisible by x. This already gives

$$p_1 = 0\,.$$

This given, using the expansions in 3.27, equation 3.2 translates into the identity

$$\sum_{s=2}^{r} s(s-1-2m))p_s x^{s-2} y^{r-s} \;=\; \sum_{s=0}^{r-2} q_s x^s y^{r-2-s}\,. \tag{3.30}$$

Equating coefficients of powers of x 3.30 gives:

(a) for an even power x^{2i}

$$p_{2i} \;=\; \frac{q_{2i-2}}{2i(2i-1-2m)}$$

(b) and for an odd power x^{2i+1} with $i \neq m$

$$p_{2i+1} \;=\; \frac{q_{2i-1}}{(2i+1)2(im)}$$

Thus 3.28 implies 3.29 as desired, completing our proof.

As a corollary we derive that

Theorem 3.3

Let $\Psi(x,y)$ be a formal power series

$$\Psi(x,y) \;=\; \sum_{r\geq 0} \Psi^{(r)}(x,y) \tag{3.31}$$

with $\Psi^{(r)}(x,y)$ a homogeneous polynomial of degree r and suppose that

$$\big(\partial_x^2 - \tfrac{2m}{x}\partial_x\big)\Psi(x,y) \;=\; y^2\Psi(x,y) \tag{3.32}$$

then

$$x^{2m+1} \,\big|\, \big(\Psi(x,y) - \Psi(-x,y)\big) \tag{3.33}$$

Proof

Note that 3.31 and 3.32 give

$$\sum_{r\geq 0} \big(x\partial_x^2 - 2m\,\partial_x\big)\Psi^{(r)}(x,y) \;=\; \sum_{r\geq 0} x\,y^2\Psi^{(r)}(x,y)\,.$$

So we must have

$$\big(x\partial_x^2 - 2m\,\partial_x\big)\Psi^{(r)}(x,y) \;=\; 0 \qquad \text{for} \quad 1\leq r\leq 3 \tag{3.34}$$

and

$$\big(x\partial_x^2 - 2m\,\partial_x\big)\Psi^{(r)}(x,y) \;=\; x\,y^2\Psi^{(r-4)}(x,y) \qquad \text{for } r\geq 4 \tag{3.35}$$

In either case $\partial_x\Psi^{(r)}(x,y)$ must be divisible by x. Thus writing

$$\Psi^{(r)}(x,y) \;=\; \sum_{s=0}^{r} c_s^{(r)} x^s y^{r-s}$$

we must have $c_1^{(r)} = 0 \quad \forall\; r \geq 1$.

Using this fact we can easily derive that

$$\Psi^{(1)}(x,y) = c_0^{(1)} y \quad , \quad \Psi^{(2)}(x,y) = c_0^{(2)} y^2 \quad , \quad \Psi^{(3)}(x,y) = \begin{cases} c_0^{(3)} y^3 \;+\; c_3^{(3)} x^3 & \text{if } m = 1, \\ c_0^{(3)} y^3 & \text{if } m \neq 1, \end{cases}$$

and we see that

$$x^{2m+1} \,\big|\, \big(\Psi^{(r)}(x,y) - \Psi^{(r)}(-x,y)\big) \qquad 3.36$$

for $r = 0,1,2,3$. But then 3.35 and Proposition 3.1 give that 3.36 must hold as well for all $r \geq 4$ proving the theorem.

Remark 3.1

It is interesting to see how condition (B) comes out of I.9. To this end note that I.9 gives

$$\begin{aligned} e^{xy} + e^{-xy} + x\,y \sum_{m \geq 1} \big(\Psi_{m-1}(x,y) - \Psi_{m-1}(-x,y)\big) \frac{u^m}{m!} \quad &= \\ = \;\; & exp\Big(xy\sqrt{1+2u}\Big) + exp\Big(-xy\sqrt{1+2u}\Big) \\ = \;\; & \sum_{k \geq 0} \frac{1}{k!}\Big[\Big(xy\sqrt{1+2u}\Big)^k + \Big(-xy\sqrt{1+2u}\Big)^k\Big] \;\; = \;\; 2\sum_{k \geq 0} \frac{(xy)^{2k}}{2k!}\big(1+2u\big)^k \\ = \;\; & 2\sum_{k \geq 0} \frac{(xy)^{2k}}{2k!} \sum_{r=0}^{k} \binom{k}{r} (2u)^r \\ = \;\; & 2\sum_{r \geq 0} (2u)^r \sum_{k \geq r} \binom{k}{r} \frac{(xy)^{2k}}{2k!} \,. \end{aligned}$$

Thus equating coefficients of u^{m+1} yields

$$\big(\Psi_m(x,y) - \Psi_m(-x,y)\big)\frac{xy}{(m+1)!} \;\; = \;\; 2^{m+2} \sum_{k \geq m+1} \binom{k}{m+1} \frac{(xy)^{2k}}{2k!}$$

or better yet

$$\Psi_m(x,y) - \Psi_m(-x,y) \;\; = \;\; 2^{m+2} \sum_{k \geq m+1} \frac{1}{(k-m-1)!} \; \frac{(xy)^{2k-1}}{2^k(2k-1)!!} \;\; ,$$

which is a rather nifty version of (B).

4. Further amenities

In this section, to recover from the combinatorial extravanganzas of last two sections, we shall only derive manipulatorial consequences of our definitions. The sobering thought is that there is a rather simple purely manipulatorial proof of the exponential formula in I.9. In fact, it is easily seen that to by-pass the combinatorics of the last two sections we need only show that the polynomials $I_m(z)$ defined by the generating function in 3.2 satisfy the recurrence in 2.2 together with the initial conditions in 2.3.

To this end note that we can write

$$\begin{aligned} 1 + z\sum_{m\geq 1} I_{m-1}(z)\frac{u^m}{m!} &= e^{z(1-\sqrt{1-2u})} = e^z \sum_{k\geq 0} \frac{(-z)^k (1-2u)^{k/2}}{k!} \\ &= e^z \sum_{k\geq 0} \frac{(-z)^k}{k!} \sum_{r\geq 0} \frac{k/2(k/2-1)\cdots(k/2-r+1)}{r!}(-2u)^r \\ &= e^z \sum_{k\geq 0} \frac{(-z)^k}{k!} \sum_{r\geq 0} \frac{k(k-2)\cdots(k-2r+2)}{r!}(-u)^r \\ &= e^z \sum_{r\geq 0} \frac{(-u)^r}{r!} \sum_{k\geq 1} \frac{k(k-2)\cdots(k-2r+2)}{k!}(-z)^k \end{aligned}$$

Thus equating coefficients of u^{m+1} gives

$$z\, I_m(z) = (-1)^{m+1} e^z \sum_{k\geq 1} \frac{k(k-2)\cdots(k-2m-2+2)}{k!}(-z)^k$$

or better

$$I_m(z) = e^z \sum_{k\geq 1} (-1)^{m+1-k} \frac{k(k-2)\cdots(k-2m)}{k!} z^{k-1}\,. \qquad 4.1$$

Before we proceed any further we should note that, in spite of appearences, the right hand side of 4.1 can only yield a polynomial of degree m in z. The reason for this is simple. If the sequence $p_m(z)$ is given by the identity

$$1 + \sum_{m\geq 1} p_m(z)\frac{u^m}{m!} = e^{zf(u)}\,,$$

with $f(u)$ a formal power series

$$f(u) = u + f_1 u^2 + \cdots + f_r u^r + \cdots$$

then for $m \geq 1$ we have

$$p_m(z) = m! \sum_{k\geq 1} z^k \frac{(u + f_1u^2 + \cdots)^k}{k!}\Big|_{u^m} = m! \sum_{k=1}^{m} z^k \frac{(u + f_1u^2 + \cdots + f_m u^m)^k}{k!}\Big|_{u^m}$$

Keeping this in mind, to complete our argument we need only show that the right-hand side of 4.1 satisfies the recursion in 2.2 and the initial conditions in 2.3. Now note that 4.1 gives

$$
\begin{aligned}
I_m(z) &= e^z \sum_{k\geq 1}(-1)^{m+1-k}\frac{k(k-2)\cdots(k-2m+2)}{k!}\, k\, z^{k-1} \\
&\qquad\qquad + \quad 2m\, e^z \sum_{k\geq 1}(-1)^{m-k}\frac{k(k-2)\cdots(k-2m+2)}{k!}\, z^{k-1} \\
&= e^z \sum_{k\geq 1}(-1)^{m+1-k}\frac{k(k-2)\cdots(k-2m+2)}{k!}\,(k-1)z^{k-1} \quad + \quad (2m-1)\, I_{m-1}(z) \\
&= z^2\, e^z \sum_{k\geq 3}(-1)^{m+1-k}\frac{k(k-2)\cdots(k-2m+2)}{k!}\,(k-1)z^{k-3} \quad + \quad (2m-1)\, I_{m-1}(z) \\
&= z^2\, e^z \sum_{k\geq 3}(-1)^{m+1-k}\frac{(k-4)\cdots(k-2m+2)}{(k-3)!}\, z^{k-3} \quad + \quad (2m-1)\, I_{m-1}(z) \\
&= z^2\, e^z \sum_{k\geq 1}(-1)^{m+1-k}\frac{(k-2)\cdots(k-2m+4)}{(k-1)!}\, z^{k-1} \quad + \quad (2m-1)\, I_{m-1}(z) \\
&= z^2\, e^z \sum_{k\geq 1}(-1)^{m+1-k}\frac{k(k-2)\cdots(k-2m+4)}{k!}\, z^{k-1} \quad + \quad (2m-1)\, I_{m-1}(z) \\
&= z^2 I_{m-2}(z) \quad + \quad (2m-1)\, I_{m-1}(z)\,.
\end{aligned}
$$

As for the initial conditions, note that 4.1 gives

$$
I_0(z) \;=\; e^z \sum_{k\geq 1}(-1)^{m+1-k}\frac{k}{k!}\, z^{k-1} \;=\; 1
$$

and

$$
\begin{aligned}
I_1(z) &= e^z \sum_{k\geq 1}(-1)^{2-k}\frac{k(k-2))}{k!}\, z^{k-1} \\
&= e^z \Big(-\frac{1(1-2)}{1!} \;+\; \sum_{k\geq 3}(-1)^{2-k}\frac{z^{k-1})}{(k-1)(k-3)!}\Big) \\
&= (1+z+\frac{z^2}{2}+\cdots)\Big(1 \;-\; \frac{z^2}{2} \;+\; \sum_{k\geq 4}(-1)^{2-k}\frac{z^{k-1})}{(k-1)(k-3)!}\Big) \\
&= 1+z \;+\; z^3 R(z)\,.
\end{aligned}
$$

But the factor $R(z)$ must vanish since we know before hand that $I_1(z)$, as defined by 3.2, is a polynomial of degree 1 in z. Thus

$$
I_1(z) \;=\; 1+z\,.
$$

as desired to complete our argument.

Our next task is to show that I.9 implies I.11. What is remarkable again is that I.9 is also equivalent to I.11. To be precise we have

Theorem 4.1

The sequence $\Psi_m(x,y)$ generated by the exponential formula

$$e^{xy} + xy\sum_{r\geq 1}\Psi_{m-1}(x,y)\frac{u^m}{m!} \;=\; e^{xy\,f(u)}\,, \qquad 4.2$$

satisfies the recursion

$$x\partial_x\Psi_{m-1}(x,y) - (2m-1)\Psi_{m-1}(x,y) \;=\; \Psi_m(x,y)\,. \qquad 4.3$$

and the initial condition

$$\Psi(x,y) \;=\; e^{xy} \qquad 4.4$$

if and only if

$$f(u) \;=\; \sqrt{1+2u} \qquad 4.5$$

Proof

Setting again as we did in section 3

$$\Psi_m(x,y) \;=\; \Psi_m(xy)$$

we see that 4.3 and 4.4 hold true if and only if we have

$$a)\quad z\Psi'_{m-1}(z) - (2m-1)\Psi_{m-1}(z) = \Psi_m(z) \quad \text{and} \qquad \text{b)} \qquad \Psi_0(z) = e^z\,. \qquad 4.6$$

This reduces us to showing that a sequence $\{\Psi_m(z)\}_{m\geq 0}$ defined by

$$e^z + z\sum_{m\geq 1}\Psi_{m-1}(z)\frac{u^m}{m!} = e^{z\,f(u)}\,, \qquad 4.7$$

with $f(u) = u + f_2u^2 + \cdots$ a formal powers series, satisfies 4.6 if and only if

$$f(u) \;=\; \sqrt{1+2u}\,.$$

To do this it is convenient to write 4.7 in the form

$$\sum_{m\geq 1}\Psi_{m-1}(z)\frac{u^m}{m!} \;=\; \frac{e^{z\,f(u)} - e^z}{z} \qquad 4.8$$

so that the operators $z\,\partial_z$ and $u\,\partial_u$ give

$$\sum_{m\geq 1} z\Psi'_{m-1}(z)\,\frac{u^m}{m!} \;=\; -z\,\frac{e^{z\,f(u)} - e^z}{z^2} \;+\; f(u)\,e^{z\,f(u)} - e^z \qquad 4.9$$

and

$$\sum_{m\geq 1} m\,\Psi_{m-1}(z)\,\frac{u^m}{m!} \;=\; u\,f'(u)\,e^{z\,f(u)}\,. \qquad 4.10$$

Now subtracting from 4.9 twice 4.10 and adding 4.8 we get

$$\sum_{m\geq 1}\big(z\Psi'_{m-1}(z)-(2m-1)\Psi_{m-1}(z)\big)\,\frac{u^m}{m!} \;=\; -z\,\frac{e^{z\,f(u)}-e^z}{z^2} \;+\; f(u)\,e^{z\,f(u)}-e^z$$

$$-\;2\,u\,f'(u)\,e^{z\,f(u)} \;+\; \frac{e^{z\,f(u)}-e^z}{z}$$

$$=\;\big(f(u)-2\,u\,f'(u)\big)e^{z\,f(u)} \;-\; e^z\,.$$

Thus 4.6 a) is equivalent to

$$e^z \;+\; \sum_{m\geq 1}\Psi_m(z)\,\frac{u^m}{m!} \;=\; \big(f(u)-2\,u\,f'(u)\big)e^{z\,f(u)}\,. \tag{4.11}$$

Note that we may also write 4.10 in the form

$$\sum_{m\geq 1}\Psi_{m-1}(z)\,\frac{u^{m-1}}{(m-1)!} \;=\; f'(u)\,e^{z\,f(u)}\,.$$

and 4.6 b) gives

$$e^z \;+\; \sum_{m\geq 1}\Psi_m(z)\,\frac{u^m}{m!} \;=\; f'(u)\,e^{z\,f(u)}\,.$$

Using this in 4.11 reduces it to

$$f'(u)\,e^{z\,f(u)} \;=\; \big(f(u)-2\,u\,f'(u)\big)e^{z\,f(u)}\,.$$

So we have 4.6 a) if and only if

$$(1+2u)f'(u) \;=\; f(u)$$

Thus, for some constant c, we must have

$$f=c\,(1+2u)^{1/2}\,,$$

but then 4.6 b) forces $c=1$, as desired.

Our final task is to derive the basic identities satisfied by the Baker-Akhiezer functions of S_2. However, before we can do that we need to establish the following somewhat surprising result.

Proposition 4.1

Let $\Psi(x,y;u,v)$ be a formal power series in its arguments such that

$$\Psi(x,y;u,v) \;=\; P(x,y;u,v)\,e^{x\,y+u\,v} \tag{4.12}$$

with $P(x,y;u,v)$ a polynomial of the form

$$P(x,y;u,v)=x^my^m \;+\; \cdots(terms\;\;of\;\;x-degree\;\;<\;\;m) \tag{4.13}$$

Then either of the two conditions

$$a)\qquad x^{2m+1}\,\big|\big(\Psi(x,y;u,v)-\Psi(-x,y;u,v)\big)\,,\qquad b)\qquad \big(\partial_x^2-\tfrac{2m}{x}\partial_x+\partial_u^2\big)\Psi \;=\; (y^2+v^2)\Psi \tag{4.14}$$

forces $P(x,y;u,v)$ to be independent of u,v.

Proof

We may write

$$\Psi(x,y;u,v) \;=\; P_{0,0}(x,y)e^{x\,y+u\,v} \;+ \sum_{r+s>0} u^r v^s P_{r,s}(x,y)\, e^{x\,y+u\,v}$$

and condition $a)$ implies that the formal power series $\Phi_{r,s}(x,y) \;=\; P_{r,s}(x,y)e^{x\,y}$ satisfies

$$x^{2m+1}\,\Big|\big(\Phi_{r,s}(x,y)-\Phi_{r,s}(-x,y)\big)\,, \qquad 4.15$$

Moreover 4.13 also implies that only $P_{0,0}(x,y)$ contains the term $x^m y^m$ and all other $P_{r,s}(x,y)$ must be of degree less than m in x. But then 4.15 and Remark 1.1 yield that $P_{r,s}(x,y) = 0$ for all $r+s>1$. This proves the theorem when $\Psi(x,y;u,v)$ satisfies condition $a)$.

Now suppose that $\Psi(x,y;u,v)$ satisfies $b)$. To begin note that for any polynomial $f(x,y;u,v)$ we have

$$\big(\partial_u - v\big)e^{x\,y+u\,v}f(x,y;u,v) \;\;=\;\; e^{x\,y+u\,v}\partial_u\, f(x,y;u,v)\,.$$

Successive applications of this identity give that

$$\big(\partial_u - v\big)^r\Psi(x,y;u,v) \;\;=\;\; e^{x\,y+u\,v}\partial_u^r\, P(x,y;u,v)\,. \qquad 4.16$$

If $P(x,y;u,v)$ were of positive degree in u then we would have an $r\geq 1$ for which $\partial_u^r\, P(x,y;u,v)$ is independent of u and does not vanish. But since the operators $\partial_x^2 - \frac{2m}{x}\partial_x + \partial_u^2$ and $\partial_u - v$ commute b) gives

$$\big(\partial_x^2 - \tfrac{2m}{x}\partial_x + \partial_u^2\big)\big(\partial_u - v\big)^r\Psi(x,y;u,v) \;\;=\;\; (y^2+v^2)\big(\partial_u - v\big)^r\Psi(x,y;u,v) \qquad 4.17$$

Recalling that $\partial_u^r\, P(x,y;u,v)$ is independent of u from 4.16 we get that

$$\begin{aligned}\partial_u^2\big(\partial_u - v\big)^r\Psi(x,y;u,v) &= \partial_u^2 e^{xy+uv}\partial_u^r P(x,y;u,v)\\ &= v^2e^{xy+uv}\,\partial_u^r P(x,y;u,v)\\ &= v^2\big(\partial_u - v\big)^r\Psi(x,y;u,v)\end{aligned} \qquad 4.18$$

and subtracting 4.18 from 4.17 gives

$$\big(\partial_x^2 - \tfrac{2m}{x}\partial_x\big)\big(\partial_u - v\big)^r\Psi(x,y;u,v) \;\;=\;\; y^2\big(\partial_u - v\big)^r\Psi(x,y;u,v)$$

and this can be rewritten as

$$\big(\partial_x^2 - \tfrac{2m}{x}\partial_x\big)e^{xy}\partial_u^r P(x,y;u,v) \;\;=\;\; y^2e^{xy}\partial_u^r P(x,y;u,v) \qquad 4.19$$

Expanding $\partial_u^r P(x,y;u,v)$ in powers of v

$$\partial_u^r P(x,y;u,v) \;\;=\;\; \sum_j f_j(x,y)\,v^j \qquad 4.20$$

by equating coefficients of powers of v in 4.19 we deduce that

$$\big(\partial_x^2 - \tfrac{2m}{x}\partial_x\big)e^{xy}f_j(x,y) \;\;=\;\; y^2e^{xy}f_j(x,y)\,. \qquad 4.21$$

Now this enters in the realm of Theorem 3.3 which assures us that the formal power series

$$\Phi_j(x,y) \;=\; f_j(x,y)e^{xy}$$

must satisfy

$$x^{2m+1}\,\big|\,\Phi_j(x,y)-\Phi_j(-x,y)\,.$$

However now we can apply Remark 1.1 again and deduce that $f_j(x,y)$ must be a polynomial of degree m in x or identically vanish. But 4.13 and 4.20 show that the degree of f_j in x is no more than $m-r-j$. Since $r\geq 1$ only the second alternative can hold true. But the vanishing of all $f_j(x,u)$ contraddicts the hypothesis that $\partial_u^r P(x,y;u,v)$ does not vanish. This contradiction proves that $P(x,y,u,v)$ is independent of u. This reduces 4.12 to

$$\Psi(x,y;u,v) \;=\; P(x,y;v)\,e^{x\,y+u\,v} \qquad 4.22$$

and condition b) becomes

$$\begin{aligned}(y^2+v^2)\Psi \;&=\; \big(\partial_x^2-\tfrac{2m}{x}\partial_x+\partial_u^2\big)\Psi\\ &=\; \big(\partial_x^2-\tfrac{2m}{x}\partial_x\big)\Psi \;+\; \partial_u^2 P(x,y;v)\,e^{x\,y+u\,v}\\ &=\; \big(\partial_x^2-\tfrac{2m}{x}\partial_x\big)\Psi \;+\; v^2\Psi\,.\end{aligned}$$

That is

$$\big(\partial_x^2-\tfrac{2m}{x}\partial_x\big)\Psi \;=\; y^2\Psi\,. \qquad 4.23$$

Now expanding $P(x,y;v)$ in powers of v

$$P(x,y;v) \;=\; P_0(x,y)+\sum_j v^j P_j(x,y)$$

gives

$$\Psi(x,y;u,v) \;=\; P_0(x,y)e^{xy+uv}+\sum_{j\geq 1} v^j P_j(x,y)\,e^{xy+uv}$$

and 4.23 translates into the equalities

$$\big(\partial_x^2-\tfrac{2m}{x}\partial_x\big)P_j(x,y)\,e^{xy} \;=\; y^2 P_j(x,y)\,e^{xy}\,. \qquad 4.24$$

Condition 4.13 assures that only $P_0(x,y)$ has degree m in x and all $P_j(x,y)$ have degrees less than m. This combined with 4.23 and Remark 1.1 forces $P_j(x,y)$ to vanish for all $j\geq 1$. Thus also $P(x,y;v)$ is independent of v and our proof is complete.

Note next that if we set

$$x=\frac{x_1-x_2}{\sqrt{2}}\,,\quad u=\frac{x_1+x_2}{\sqrt{2}}\,,\quad y=\frac{y_1-y_2}{\sqrt{2}}\,,\quad v=\frac{y_1+y_2}{\sqrt{2}}\,. \qquad 4.25$$

then

$$x_1=\frac{u+x}{\sqrt{2}}\,,\quad x_2=\frac{u-x}{\sqrt{2}}\,,\quad y_1=\frac{v+y}{\sqrt{2}}\,,\quad y_2=\frac{v-y}{\sqrt{2}}\,. \qquad 4.26$$

and

$$(a)\ x_1^2+x_2^2=x^2+u^2, \qquad (b)\ y_1^2+y_2^2=y^2+v^2 \qquad (c)\ x_1y_1+x_2y_2=xy+uv\,, \tag{4.27}$$

Morever for any formal power series $f(x_1,x_2;y_1,y_2)$ and $g(x,y;u,v)$ we have

$$\Big(\partial_{x_1}^2+\partial_{x_2}^2-\tfrac{2m}{x_1-x_2}(\partial_{x_1}-\partial_{x_2})\Big)f(x_1,x_2;y_1,y_2)= \\ =\Big(\partial_x^2+\partial_u^2-\tfrac{2m}{x}\,\partial_x\Big)f\big(\frac{u+x}{\sqrt2},\frac{u-x}{\sqrt2};\frac{v+y}{\sqrt2},\frac{v-y}{\sqrt2}\big)\Big|_{\substack{x=\frac{x_1-x_2}{\sqrt2}\ ,\ u=\frac{x_1+x_2}{\sqrt2}\\ y=\frac{y_1-y_2}{\sqrt2}\ ,\ v=\frac{y_1+y_2}{\sqrt2}}}. \tag{4.28}$$

as well as

$$\Big(\partial_x^2+\partial_u^2-\tfrac{2m}{x}\,\partial_x\Big)g(x,y;u,v)= \\ =\Big(\partial_{x_1}^2+\partial_{x_2}^2-\tfrac{2m}{x_1-x_2}(\partial_{x_1}-\partial_{x_2})\Big)g\big(\frac{x_1-x_2}{\sqrt2},\frac{x_1+x_2}{\sqrt2};\frac{y_1-y_2}{\sqrt2},\frac{y_1+y_2}{\sqrt2}\big)\Big|_{\substack{x_1=\frac{u+x}{\sqrt2}\ ,\ x_2=\frac{u-x}{\sqrt2}\\ y_1=\frac{v+y}{\sqrt2}\ ,\ y_2=\frac{v-y}{\sqrt2}}}. \tag{4.29}$$

We have now all the basic ingredients to identify the Baker-Akhiezer functions of S_2. We begin with

Theorem 4.2

If

$$\Psi_m^{S_2}(X_2,Y_2) \;=\; P_m^{S_2}(X_2,Y_2)\,e^{x_1y_1+x_2y_2} \tag{4.30}$$

and

$$P_m^{S_2}(X_2,Y_2) \;=\; 2^m\sum_{k=0}^m I_{2m-k}^k\,(-1)^{m-k}(x_1-x_2)^k(y_1-y_2)^k/2^k \tag{4.31}$$

then all the properties in I.1 hold true as well as I.3

Proof If

$$\Psi_m(x,y) \;=\; P_m(x,y)e^{xy} \tag{4.32}$$

with

$$P_m(x,y) \;=\; \sum_{k=0}^m I_{2m-k}^k\,(-1)^{m-k}x^ky^k \tag{4.33}$$

then Theorem 2.1 assures properties (A), (B), (C) are satisfied. Now from 4.30 and 4.31 and 4.27 (c) it follows that

$$\Psi_m^{S_2}(X_2,Y_2)=2^m\Psi_m(x,y)\,e^{uv}\Big|_{\substack{x=\frac{x_1-x_2}{\sqrt2}\ ,\ u=\frac{x_1+x_2}{\sqrt2}\\ y=\frac{y_1-y_2}{\sqrt2}\ ,\ v=\frac{y_1+y_2}{\sqrt2}}} \tag{4.34}$$

and

$$P_m^{S_2}(X_2,Y_2)=2^mP_m(x,y)\Big|_{\substack{x=\frac{x_1-x_2}{\sqrt2}\ ,\ u=\frac{x_1+x_2}{\sqrt2}\\ y=\frac{y_1-y_2}{\sqrt2}\ ,\ v=\frac{y_1+y_2}{\sqrt2}}} \tag{4.35}$$

Now property I.1 (2) follows from the definition in 4.31 and the fact that $I_m^m=1$. Property I.1 (3) is immediate. Property I.1 (i) follows from (B). Note that from (C) it follows that

$$\Big(\partial_x^2+\partial_u^2-\tfrac{2m}{x}\,\partial_x\Big)\Psi_m(x,y)\,e^{uv} \;=\; \big(y^2+v^2\big)\Psi_m(x,y)\,e^{uv} \tag{4.36}$$

so 4.28, 4.34, 4.36 and 4.27 give

$$\Big(\partial_{x_1}^2+\partial_{x_2}^2-\tfrac{2m}{x_1-x_2}(\partial_{x_1}-\partial_{x_2})\Big)\Psi_m^{S_2}(X_2,Y_2)=\Big(\partial_x^2+\partial_u^2-\tfrac{2m}{x}\,\partial_x\Big)2^m\Psi_m(x,y)\,e^{uv}\Big|_{\substack{x=\frac{x_1-x_2}{\sqrt{2}}\ ,\ u=\frac{x_1+x_2}{\sqrt{2}}\\ y=\frac{y_1-y_2}{\sqrt{2}}\ ,\ v=\frac{y_1+y_2}{\sqrt{2}}}}$$
$$=\big(y^2+v^2\big)2^m\Psi_m(x,y)\,e^{uv}\Big|_{\substack{x=\frac{x_1-x_2}{\sqrt{2}}\ ,\ u=\frac{x_1+x_2}{\sqrt{2}}\\ y=\frac{y_1-y_2}{\sqrt{2}}\ ,\ v=\frac{y_1+y_2}{\sqrt{2}}}}$$
$$=\big(y_1^2+y_2^2\big)\Psi_m^{S_2}(X_2,Y_2)$$

This proves (ii). It is easily seen that the change of variables in 4.25 together with 4.31 changes I.9 into I.3. This completes our proof.

To complete our treatment we need to reverse the process and show that under (1) and (2), either (ii) or (i) force I.2. We will do this by establishing that

Theorem 4.3

$$\text{I.3}\implies\text{(ii)}\implies\text{(i)}\implies\text{I.2} \qquad 4.37$$

Proof

It should be clear by now that the change of variables exhibited in the last proof can be systematically used to transfer relations involving $\Psi_m(x,y)$ and $P_m(x,y)$ into relations involving $\Psi_m^{S_2}(X_2,Y_2)$ and $P_m^{S_2}(X_2,Y_2)$ and viceversa. So in our remaining arguments we will only outline the steps needed to prove the implications in 4.37. The first step is to change variables and go from I.3 to I.9. In the second step we use Theorem 3.2 and go from I.9 to (C). In the third step by a change of variables we go from (C) to (ii). This establishes "I.3 $\implies$ (ii)". In the fourth step we start with a formal power series

$$\Phi_m(X_2,Y_2)\ =\ Q_m(X_2,Y_2)\,e^{x_1y_1+x_2y_2} \qquad 4.38$$

with $Q_m(X_2,Y_2)$ a polynomial satisfying

$$Q_m(X_2;Y_2)\ =\ (x_1-x_2)^m(y_1-y_2)^m\ +\ \cdots(\text{terms of }x\text{-degree}<m\ ) \qquad 4.39$$

and assume that we have

$$L(m)\Phi_m(X_2,Y_2)\ =\ (y_1^2+y_2^2)\Phi_m(X_2,Y_2)\,. \qquad 4.40$$

Here using the variable change in 4.26 yields the polynomial

$$P_m(x,y;u,v)\ =\ 2^{-m}Q_m(X_2,Y_2)\Big|_{\substack{x_1=\frac{u+x}{\sqrt{2}}\ ,\ x_2=\frac{u-x}{\sqrt{2}}\\ y_1=\frac{v+y}{\sqrt{2}}\ ,\ y_2=\frac{v-y}{\sqrt{2}}}} \qquad 4.41$$

and the formal power series

$$\Psi_m(x,y;u,v)\ =\ \Phi_m(X_2,Y_2)\Big|_{\substack{x_1=\frac{u+x}{\sqrt{2}}\ ,\ x_2=\frac{u-x}{\sqrt{2}}\\ y_1=\frac{v+y}{\sqrt{2}}\ ,\ y_2=\frac{v-y}{\sqrt{2}}}}\ =\ P_m(x,y;u,v)\,e^{xy+uv} \qquad 4.42$$

which satisfies the equation

$$\Big(\partial_x^2+\partial_u^2-\tfrac{2m}{x}\,\partial_x\Big)\Psi_m(x,y;u,v) \;=\; (y^2+v^2)\Psi_m(x,y;u,v)\,. \qquad 4.43$$

Since 4.39 and 4.41 yield that

$$\Psi_m(x,y;u,v) \;=\; x^m y^m \;+\; \cdots(\text{terms of } x\text{-degree} < m\ )$$

we are brought into the realm of Proposition 4.1 which assures that $\Psi_m(x,y;u,v)$ is independent of u,v. rewriting $P_m(x,y;u,v)$ to $P_m(x,y)$ reduces 4.42 to

$$\Psi_m(x,y;u,v) \;=\; P_m(x,y)\,e^{xy+uv}\,. \qquad 4.44$$

So 4.43 now gives

$$\begin{aligned}(y^2+v^2)\Psi_m(x,y;u,v) \;&=\; \Big(\partial_x^2+\partial_u^2-\tfrac{2m}{x}\,\partial_x\Big)\Psi_m(x,y;u,v)\\ &=\; \Big(\partial_x^2+\partial_u^2-\tfrac{2m}{x}\,\partial_x\Big)P_m(x,y)\,e^{xy+uv}\\ &=\; \Big(\partial_x^2-\tfrac{2m}{x}\,\partial_x\Big)P_m(x,y)\,e^{xy+uv}+v^2P_m(x,y)\,e^{xy+uv}\\ &=\; \Big(\partial_x^2-\tfrac{2m}{x}\,\partial_x\Big)\Psi_m(x,y;u,v) \;+\; v^2\Psi_m(x,y;u,v)\end{aligned}$$

or better

$$\Big(\partial_x^2-\tfrac{2m}{x}\,\partial_x\Big)\Psi_m(x,y;u,v) \;=\; y^2\Psi_m(x,y;u,v)\,.$$

Factoring out the exponential e^{uv} this reduces to

$$\Big(\partial_x^2-\tfrac{2m}{x}\,\partial_x\Big)P_m(x,y)\,e^{xy} \;=\; y^2P_m(x,y)\,e^{xy+uv}\,.$$

this brings us into the realm of Theorem 3.3 which assures that the formal series $\Psi(x,y)=P_m(x,y)\,e^{xy}$ satisfies

$$x^{2m+1}\,\big|\,\Psi(x,y)-\Psi(-x,y)\,. \qquad 4.45$$

and the change of variables in 4.25 can then be used to convert from 4.45 into

$$(x_1-x_2)^{2m+1}\,\big|\,\Big(\Phi_m(x_1,x_2;Y_2)-\Phi_m(x_2,x_1;Y_2)\Big) \qquad 4.46$$

completing the proof that

$$(ii) \implies (i)\,.$$

In the final part of this proof we start again with a formal power series

$$\Phi_m(X_2;Y_2)=Q_m(X_2;Y_2)e^{x_1y_1+x_2y_2}$$

as in the previous part but now, in addition to 4.39, we assume 4.46 rather than 4.40. As before we introduce the auxiliary polynomial $P_m(x,y;u,v)$ and formal power series $\Psi_m(x,y;u,v)$ given by 4.41 and 4.42. Here we use again Proposition 4.1 except that 4.46 now requires a use of the hypothesis in 4.14 a) rather than 4.14 b). The conclusion is the same: $P_m(x,y;u,v)$ is independent of u,v. This reduces $\Psi_m(x,y;u,v)$ of the form

$$\Psi_m(x,y;u,v) \;=\; P_m(x,y)\, e^{xy+uv} .$$

with $\Psi(x,y) = P_m(x,y)\, e^{xy}$ satisfying

$$x^{2m+1} \,\big|\, \big(\Psi(x,y) - \Psi(-x,y)\big) ,$$

Since 4.39 yields that $P_m(x,y)$ satisfies

$$P_m(x,y) \;=\; x^m y^m \;+\; \cdots (\text{terms of } x\text{-degree} < m)$$

we can use Theorem 2.1 and conclude that

$$P_m(x,y) \;=\; \sum_{k=0}^{m} (-1)^{m-k} I^{k}_{2m-k}\, (xy)^k .$$

This given, the change of variables in 4.25 proves that $Q_m(X_2;Y_2)$ must be given by the right hand side of I.2, yielding

$$\text{(i)} \implies \text{I.2}$$

and completing our task.

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