YOUNG SEMINORMAL REPRESENTATION
MURPHY ELEMENTS
AND
CONTENT EVALUATIONS

ABSTRACT

These are notes resulting from a course in Algebraic Combinatorics given at UCSD in winter 2003. The lectures where based on writings of A. Young [8], R. M. Thrall [7] and D. E. Rutherford [6], A. Jucy [3] and G, Murphy [5]. The material covers basic identities leading to the construction of Young's seminormal units. Murphy elements are used first as an aid to the construction of the entries in the seminormal matrices corresponding to the simple reflections then used to express characters and conjugacy classes. As a by-product we obtain a new way of constructing certain polynomials introduced in a recent paper by A. Goupil et Al. [2] and by Diaconis and Greene in an earlier unpublished manuscript [1]. These polynomials yield some truly remarkable formulas for the irreducible characters of the symmetric groups. More precisely, it is shown in [1] and [2] that for each partition $\gamma \vdash k$ we can construct a symmetric polynomial W_γ which evaluated at the contents of a partition $\lambda \vdash n \geq k$ yield the central character value $\omega^\lambda_{\gamma,1^{n-k}}$. The polynomials W_γ are remarkably simple when expressed in terms of the power basis. Moreover, their power basis expansion are of the form $W_\gamma = \sum_\rho c^\gamma_\rho(n)\, p_\rho$ with coefficients $c^\gamma_\rho(n)$ polynomials in n. The approach followed in [2] is quite intricate and somewhat indirect. In [1] Diaconis and Greene follow a purely combinatorial approach and derive an algorithm for constructing the polynomials W_γ. In these notes we obtain explicit closed form expressions for W_γ. Our approach is based on a method introduced by Macdonald [4] in an exercise where an explicit formula is obtained for the central character value $\omega^\lambda_{k,1^{n-k}}$.

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4.**The seminormal matrices.**

In this section we derive the entries in Young's seminormal matrices from the action of Murphy elements on the seminormal matrix units.

5.**Murphy elements and conjugacy classes.**

We start by giving two different proofs that symmetric polynomials evaluated at the Murphy elements yield class functions. This done our main goal is to obtain explicit formulas for the symmetric polynomials that yield the Characters and the Classes. Tablesof these polynomials are included in a few special cases.

1. The Young idempotents

In this section we recall some basic properties of the Young idempotents and set notation. Most of the material covered in this section is presented in full detail in the lecture notes on the "*Young's Natural Representation*" . References to these notes will be indicated by the symbol [YNR].

We shall use the French convention of depicting Ferrers diagrams as left justified rows of lattice cells of lengths decreasing from bottom to top.

Recall that a tableau T of shape $\lambda \vdash n$ is a filling of the cells of the Ferrers diagram of λ with the letters $1, 2, \ldots, n$. If the filling is row and column increasing then T is said to be "*standard*". The shape of T will be simply denoted $\lambda(T)$. Given a tableau T, we let $R(T)$ and $C(T)$ respectively denote the "*Row Group*" and "*Column Group*" of T. As in [YNR] we set

$$P(T) \;=\; \sum_{\alpha\in R(T)} \alpha\,, \qquad N(T) \;=\; \sum_{\beta\in C(T)} sign(\beta)\,\beta\,, \tag{1.1}$$

and

$$E(T) \;=\; P(T)N(T)\,. \tag{1.2}$$

Note that if T has n cells and $\sigma \in S_n$ then σT denotes the tableau obtained by replacing in T the index i by σ_i for $i = 1, \ldots, n$. It is easily seen that we have

$$R(\sigma T) \;=\; \sigma R(T)\,\sigma^{-1}\,, \qquad C(\sigma T) \;=\; \sigma C(T)\,\sigma^{-1}$$

thus

$$P(\sigma T) \;=\; \sigma P(T)\,\sigma^{-1}\,, \qquad N(\sigma T) \;=\; \sigma N(T)\,\sigma^{-1} \tag{1.3}$$

In particular if T_1 and T_2 are tableaux of the same shape and σ_{T_1,T_2} is the permutation that sends T_2 into T_1 we have the identities

$$\begin{aligned} \sigma_{T_1,T_2}P(T_2) &= P(T_1)\sigma_{T_1,T_2}\,, \\ \sigma_{T_1,T_2}N(T_2) &= N(T_1)\sigma_{T_1,T_2}\,, \\ \sigma_{T_1,T_2}E(T_2) &= E(T_1)\sigma_{T_1,T_2}\,. \end{aligned} \tag{1.4}$$

We have the following basic fact

Proposition 1.1

If T_1 and T_2 are two tableaux with n cells, then we have

$$N(T_2)P(T_1) \neq 0 \quad \text{or} \quad P(T_1)N(T_2) \neq 0 \tag{1.4}$$

only if

$$\lambda(T_2) \geq \lambda(T_1) \quad \text{(in dominance)} \tag{1.5}$$

In particular

$$\lambda(T_2) < \lambda(T_1) \quad \Longrightarrow \quad N(T_2)P(T_1) = P(T_1)N(T_2) = 0$$

Proof

Construct the diagram $T_1 \wedge T_2$ by placing in the lattice cell (i,j) the intersection of row i of T_1 with column j of T_2. Note that if one of these intersections contains two elements r and s then $R(T_1)$ and $C(T_2)$ would have the transposition (r,s) in common, but then we would have, for instance

$$N(T_2)P(T_1) \;=\; N(T_2)(r,s)P(T_1) \;=\; -N(T_2)P(T_1)$$

in plain contraddiction with 1.4. Thus 1.4 forces the cells of $T_1 \wedge T_2$ to contain at most one element. Note that if we lower these elements down their columns by eliminating the empty cells we will obtain a tableau T_3 of shape $\lambda(T_2)$, thus the number of cells in the first i rows of T_3 is given by

$$\lambda_1(T_2) + \lambda_2(T_2) + \cdots + \lambda_i(T_2)\,.$$

But all the elements of the first i rows of T_1 are in the first i rows of $T_1 \wedge T_2$ and a fortiori must all be in the first i rows of T_3. This gives the inequality

$$\lambda_1(T_2) + \lambda_2(T_2) + \cdots + \lambda_i(T_2) \;\geq\; \lambda_1(T_1) + \lambda_2(T_1) + \cdots + \lambda_i(T_1)\,.$$

Since this must hold true for all i we necessarily have 1.5 as asserted.

It will be convenient to denote the group algebra of S_n by $\mathcal{A}[S_n]$. This given, Proposition 1.1 implies the following fundamental fact

Theorem 1.1

If the tableaux T_1, T_2 have both n cells, then for any element $f \in \mathcal{A}[S_n]$ we have

$$P(T_1)fN(T_2) \neq 0 \;\; \text{or} \quad N(T_2)fP(T_1) \neq 0 \quad \Longrightarrow \quad \lambda(T_2) \geq \lambda(T_1) \tag{1.6}$$

In particular

$$\lambda(T_1) \neq \lambda(T_2) \quad \Longrightarrow \quad E(T_1)fE(T_2) \;=\; 0 \qquad \text{(for all } f \in \mathcal{A}[S_n])\,. \tag{1.7}$$

.

Proof

Note that we may write (using 1.3)

$$P(T_1)fN(T_2) \;=\; \sum_{\sigma \in S_n} f(\sigma)\, P(T_1)\sigma N(T_2) \;=\; \sum_{\sigma \in S_n} f(\sigma)\, P(T_1)N(\sigma T_2)\sigma^{-1},$$

Thus the non vanishing of the left hand side forces the non vanishing at least one term in the sum in the right hand side. But then 1.6 follows from 1.5. Of course the same will hold true if $N(T_2)fP(T_1) \neq 0$. But now note that

$$P(T_1)N(T_1)fP(T_2)N(T_2) \neq 0 \tag{1.8}$$

forces $N(T_1)fP(T_2) \neq 0$ yielding

$$\lambda(T_1) \geq \lambda(T_2)\,.$$

On the other hand, (again by 1.6) , 1.8 itself forces

$$\lambda(T_2) \geq \lambda(T_1)\,.$$

Thus only the equality $\lambda(T_1) = \lambda(T_2)$ is compatible with the non vanishing of $E(T_1)fE(T_2)$. This proves 1.7 and completes the proof.

We have the following fundamental identity whose proof may be found in [YNR].

Proposition 1.2 (Von Neuman Sandwich Lemma)

For any element $f \in \mathcal{A}(S_n)$ and any tableau T

$$P(T)\, f\, N(T) \;\;=\;\; c_T(f)\;P(T)N(T)\;\;, \tag{1.9}$$

with

$$c_T(f) \;\;=\;\; f\,N(T)P(T)\,|_\epsilon\;\;,$$

where ϵ denotes the identity permutation.

This result has the following important corollary.

Theorem 1.2

For any tableau T, the group algebra element $E(T)$ is idempotent. More precisely there is a non-vanishing constant $h(T)$ depending only on the shape of T such that

$$E(T)E(T) \;\;=\;\; h(T)\,E(T) \tag{1.10}$$

Proof

Using 1.9 with $f = N(T)P(T)$, the identity in 1.9 gives

$$E(T)E(T) \;\;=\;\; h(T)E(T)\,.$$

with

$$h(T) \;\;=\;\; N(T)P(T)N(T)P(T)\,\big|_\epsilon$$

Note that if $h(T)$ were to vanish then $E(T)$ would be nilpotent. Now a cute argument shows that that the coefficient of the identity of any nilpotent element of a group algebra necessarily vanishes. Now this immediately leads to a contraddiction since we can easily see that

$$E(T)\,\big|_\epsilon \;\;=\;\; \sum_{\alpha\in R(T)}\sum_{\beta\in C(T)} sign(\beta)\,\alpha\beta\,\big|_\epsilon \;\;=\;\; 1\,.$$

In fact the identity $\alpha\beta = \epsilon$ holds only when $\alpha = \beta = \epsilon$. We shall show later that if T has shape λ then

$$h_(T) = \frac{n!}{n_\lambda}$$

where n_λ denotes the number of standard tableaux of shape λ. But for the moment it will suffice to show that $h(T)$ depends only on $\lambda(T)$. Now this follows immediately from the identities

$$\begin{aligned} h(T) \;=\; N(T)P(T)N(T)P(T)\big|_\epsilon \;&=\; \frac{1}{n!}\sum_{\sigma\in S_n} \sigma\, N(T)P(T)N(T)P(T)\;\sigma^{-1}\big|_\epsilon \\ &=\; \frac{1}{n!}\sum_{\sigma\in S_n} N(\sigma T)P(\sigma T)N(\sigma T)P(\sigma T)\big|_\epsilon \\ &=\; \frac{1}{n!}\sum_{\lambda(T')=\lambda(T)} N(T')P(T')N(T')P(T')\big|_\epsilon \end{aligned}$$

This given, here and after we will use the symbol "h_λ" to denote $h(T)$ for any tableau T of shape λ.

In the construction of Young's Seminormal Representation we will make systematic use of Young's "*last letter order*". Given two standard tableaux T_1, T_2 of the same shape we shall say that k is the "*last letter of disagreement*" between T_1 and T_2 if the letters $k+1, k+2, \ldots, n$ occupy exatly the same positions in T_1 and T_2, but k does not. This given we shall say that

"*T_1 precedes T_2 in the Young last letter order*"

and write

$$T_1 <_{LL} T_2 \qquad 1.11$$

if and only if the last letter of disagrement is *higher* in T_1 than in T_2.

If T is a standard tableau we shall denote by $T(k)$ the tableau obtained by removing from T the letters larger than k together with their cells. Note that if $\lambda(T_1) = \lambda(T_2)$ and $k+1$ is the last letter of disagreement between T_1 and T_2 then $\lambda\big(T_1(k+1)\big) = \lambda\big(T_2(k+1)\big)$ and 1.11 holds true if and only if

$$\lambda(T_1(k)) > \lambda(T_2(k)) \qquad \text{(in dominance)} \qquad 1.12$$

For $f \in \mathcal{A}(S_n)$ let us set

$$\downarrow f \;=\; \sum_{\sigma\in S_n} f(\sigma)\,\sigma^{-1} \qquad 1.13$$

We shall refer to this operation as "*flipping f*" There are interesting identities connected with the operation of passing from T to $T(k)$, they may be stated as follows

Proposition 1.3

For any tableau T of shape $\lambda \vdash n$ and $k \leq n$ we have two elements $p_k(T)$ and $n_k(T)$ such that

$$\begin{aligned} &a)\;\; p_k(T)P(T(k)) \;=\; P(T) \;=\; P(T(k))\downarrow p_k(T)\,, \\ &b)\;\; n_k(T)N(T(k)) \;=\; N(T) \;=\; N(T(k))\downarrow n_k(T)\,. \end{aligned} \qquad 1.13$$

Proof

Note that the second equality in 1.13 a) follows by flipping both sides of the first equality. The same holds true for 1.13 b). So in each case we need only show the first equality. Moreover since we can pass from T to $T(k)$ by successive removals of the largest letter, we can see that we need only show our equalities in the case $k = n-1$. To this end let us suppose that the row of T that contains the letter n consists of the letters

$$a_1 \ , \ a_2 \ , \ \ldots \ , \ a_r \ , \ n \, .$$

In this case denoting by

$$(a_1, a_2, \ldots, a_r, n) \quad \text{and} \quad (a_1, a_2, \ldots, a_r)$$

the formal sums of all the permutations of S_n that leave fixed the elements of the complements of $\{a_1, a_2, \ldots, a_r, n\}$ and $\{a_1, a_2, \ldots, a_r\}$ respectively, we have, by left coset decomposition

$$(a_1, a_2, \ldots, a_r, n) \;=\; \big(\epsilon + (a_1, n) + (a_2, n) + \cdots + (a_r, n)\big)(a_1, a_2, \ldots, a_r)\, .$$

Multiplying this identity by the contributions to $P(T)$ coming from the other rows of T yields

$$P(T) \;=\; \big(\epsilon + (a_1, n) + (a_2, n) + \cdots + (a_r, n)\big)P(T(n-1))$$

which is precisely the first equality in 1.13 a) for $k = n-1$.

For 1.13 b) we can proceed in a similar way. Indeed let us suppose that the column of T that contains the letter n consists of the letters

$$b_1 \ , \ b_2 \ , \ \ldots \ , \ b_s \ , \ n \, .$$

Denoting by

$$(b_1, b_2, \ldots, b_s, n)' \quad \text{and} \quad (b_1, b_2, \ldots, a_s)'$$

the formal sums of all the signed permutations of S_n that leave fixed the elements of the complements of $\{b_1, b_2, \ldots, b_s, n\}$ and $\{b_1, b_2, \ldots, b_s\}$ respectively, we have, by left coset decomposition

$$(b_1, b_2, \ldots, b_s, n)' \;=\; \big(\epsilon - (b_1, n) - (b_2, n) - \cdots - (b_s, n)\big)(b_1, b_2, \ldots, b_r)'\, .$$

Multiplying this identity by the contributions to $N(T)$ coming from the other columns of T yields

$$N(T) \;=\; \big(\epsilon - (b_1, n) - (b_2, n) - \cdots - (b_r, n)\big)N(T(n-1))$$

which is precisely the first equality in 1.13 b) for $k = n-1$. This completes our proof.

The following result will play a crucial role here.

Theorem 1.3

For two standard tableau T_1 and T_2 of the same shape we have

$$N(T_1)P(T_2) \neq 0 \quad \Longrightarrow \quad T_1 <_{LL} T_2 \qquad 1.13$$

Proof

Let $k+1$ be the last letter of disagrement between T_1 and T_2. Using Proposition 1.3 we derive the factorization

$$N(T_1)P(T_2) \;=\; n_k(T_1)\, N(T_1(k))P(T_2(k)) \downarrow p_k(T_2)$$

Thus

$$N(T_1)P(T_2) \neq 0 \quad \Longrightarrow \quad N(T_1(k))P(T_2(k)) \neq 0$$

and Proposition 1.1 gives

$$\lambda((T_1(k)) > \lambda(T_2(k))$$

However we have seen that this holds true if and only if

$$T_1 <_{LL} T_2\,.$$

This completes our argument.

2. Young's seminormal units

We have shown in the Representation Theory notes (here and after referred to by the symbol [RT]) that the group algebra $\mathcal{A}(G)$ of a finite group G has a basis

$$\big\{\{e_{ij}^{\lambda}\}_{i,j=1}^{n_\lambda}\big\}_{\lambda\in\Lambda} \tag{2.1}$$

with the property that

$$e_{ij}^{\lambda} e_{rs}^{\mu} \;=\; \begin{cases} 0 & \text{if } \lambda \neq \mu\,, \\ 0 & \text{if } \lambda = \mu\,, \text{but } j \neq r \\ e_{is}^{\mu} & \text{if } \lambda = \mu \text{ and } j = r\,. \end{cases} \tag{2.2}$$

Moreover, we have also derived the identities

$$e_{ij}^{\lambda}\big|_{\epsilon} \;=\; \begin{cases} 0 & \text{if } i \neq j\,, \\ & \\ 1/h_\lambda & \text{if } i = j\,. \end{cases} \qquad\qquad (\text{with } h_\lambda = |G|/n_\lambda) \tag{2.3}$$

From these identities it follows that we have the expansions

$$f \;=\; \sum_{\lambda\in\Lambda} h_\lambda \sum_{i,j=1}^{n_\lambda} f e_{ji}^{\lambda}\big|_{\epsilon}\, e_{ij}^{\lambda}\,, \qquad\qquad (\text{for all } f \in \mathcal{A}(G))\,. \tag{2.4}$$

This basis allows us to construct a complete set of representatives of irreducible representations of G. These are simply given by the collection $\{A^\lambda\}_{\lambda\in\Lambda}$ obtained by setting

$$A^\lambda(\sigma) = \|a_{ij}^{\lambda}(\sigma))\|_{i,j=1}^{n_\lambda} \tag{2.5}$$

with

$$a_{ij}^{\lambda}(\sigma) \;=\; h_\lambda e_{ji}^{\lambda}\big|_{\sigma^{-1}} \qquad\qquad (\text{ for all } \sigma \in G) \tag{2.6}$$

In fact, from 2.4 and 2.6 we derive the expansion

$$\sigma \;=\; \sum_{\lambda\in\Lambda}\sum_{i,j=1}^{n_\lambda} a_{ij}^{\lambda}(\sigma)\, e_{ij}^{\lambda}\,. \qquad 2.7$$

and the relations in 2.3 immediately yield that

$$\sigma e_{r,s}^{\mu} \;=\; \sum_{i=1}^{n_\mu} e_{i,s}^{\mu}\, a_{i,r}^{\mu}(\sigma) \qquad 2.8$$

or in matrix form

$$\sigma\big\langle e_{i,s}^{\mu}\big\rangle_{1\le i\le n_\mu} \;=\; \big\langle e_{i,s}^{\mu}\big\rangle_{1\le i\le n_\mu} A^{\mu}(\sigma) \qquad 2.9$$

Now we see from 2.6 that the representation A^λ is orthogonal (that is $a_{ij}(\sigma^{-1})=a_{ji}(\sigma)$) if and only if we have

$$e_{ji}^{\lambda}(\sigma) \;=\; e_{ij}^{\lambda}(\sigma^{-1}) \qquad (\text{ for all } \sigma\in G) \qquad 2.10$$

and this, in compact form may be simply be rewritten as

$$\downarrow e_{ij}^{\lambda} \;=\; e_{ji}^{\lambda} \qquad 2.11$$

When Young set himself the task of finding a complete set of irreducible orthogonal representations of S_n, his point of departure (see [8] QSA VI) was the construction of units e_{ij}^{λ} satisfying conditions 2.2 and 2.3 together with the additional condition

$$\downarrow e_{ij}^{\lambda} \;=\; \frac{d_i^{\lambda}}{d_j^{\lambda}}\, e_{ji}^{\lambda}\,. \qquad 2.12$$

These constructs have come to be referred to as "*Young's seminormal units*". It is easily seen that setting

$$f_{ij}^{\lambda} \;=\; \sqrt{\frac{d_j^{\lambda}}{d_i^{\lambda}}}\; e_{ij}^{\lambda} \qquad 2.13$$

the orthogonality condition in 2.11 will be satisfied by the f_{ij}^{λ} and the desired orthogonal representations can then be readily obtained.

Surprisingly Young's seminormal units can be written down with a minimal amount of additional notation from the material which is already in our possession. To begin, we let

$$T_1^{\lambda}\,,\; T_2^{\lambda}\,,\;\ldots\;,\; T_{n_\lambda}^{\lambda}$$

denote the standard tableaux of shape λ in Young's last letter order. Moreover the permutation that sends T_j^λ onto T_i^λ will be denoted σ_{ij}^{λ}. In symbols

$$\sigma_{ij}^{\lambda}\, T_j^{\lambda} \;=\; T_i^{\lambda} \qquad 2.14$$

Finally, for a tableau T with n cells, it will be convenient, to use the abreviations

$$T(n-1)=\overline{T}\,,\quad T(n-2)=\overline{\overline{T}}\,,\quad T(n-3)=\overline{\overline{\overline{T}}}$$

This given, Young's seminormal units are simply given by the formulas

$$e_{ij}^{\lambda} \;=\; e(\overline{T}_i^{\lambda})\frac{P(T_i^{\lambda})\,\sigma_{ij}^{\lambda}\,N(T_j^{\lambda})}{h_\lambda}e(\overline{T}_j^{\lambda})\,, \qquad 2.15$$

where for a given standard tableau T of shape λ the group algebra element $e(T)$ is recursively defined by setting

$$e(T) \;=\; \begin{cases} \epsilon & \text{if } T=[1]\,, \\ e(\overline{T})\frac{P(T)N(T)}{h_\lambda}e(\overline{T}) & \text{otherwise.} \end{cases} \qquad 2.16$$

The remainder of this section is dedicated to proving that these units satisfy the identities in 2.2, 2.3 and 2.12. But before we can carry this out we need to derive a few identities.

Our basic tool is provided by the following

Proposition 2.1

For any standard tableau T we have

$$\begin{aligned} a)\quad & E(T)e(\overline{T})E(T) \;=\; h(T)\,E(T) \\ b)\quad & E(T)e(T)E(T) \;=\; h(T)\,E(T) \\ c)\quad & e(T)E(T)e(T) \;=\; h(T)\,e(T) \end{aligned} \qquad 2.17$$

where we should set $h(T)=h_\lambda$ when $\lambda(T)=\lambda$.

To prove this we need two auxiliary identities which are of independent interest.

Lemma 2.1

$$\begin{aligned} a)\quad & e(T)P(T)N(T)\big|_{\epsilon} \;=\; 1\,, \\ b)\quad & e(\overline{T})P(T)N(T)\big|_{\epsilon} \;=\; 1\,. \end{aligned} \qquad 2.18$$

Proof

We proceed by induction on the size of T. So assume that 2.18 a) is true for any tableau with $n-1$ cells and let T be a tableau with n cells. Note that the induction hypothesis then implies that

$$e(\overline{T})P(\overline{T})N(\overline{T})\big|_{\epsilon} \;=\; 1\,. \qquad 2.19$$

Now let the elements of T that are in the same row and column as n be given as in the proof of Proposition 1.3. We may then write

$$\sigma\,P(T)N(T)\big|_{\epsilon}=P(\overline{T})\big(\epsilon+(a_1,n)+\cdots+(a_r,n)\big)\big(\epsilon-(b_1,n)-\cdots-(b_s,n)\big)N(\overline{T})\big|_{\sigma^{-1}}\,. \qquad 2.20$$

Note that for any pair i,j, the product of the two cycles $(a_i,n)(b_j,n)$ is the 3-cycle (a_i,n,b_j) which is not in S_{n-1}, and since all of the permutations in $P(\overline{T})$ or $N(\overline{T})$ are in S_{n-1}, the only terms in 2.20

that will yield permutations in S_{n-1} are those produced by the identity term ϵ. Thus we necessarily have

$$\sigma\, P(T)N(T)\,\big|_\epsilon = P(\overline{T})N(\overline{T})\,\big|_{\sigma^{-1}} = \sigma\, P(\overline{T})N(\overline{T})\,\big|_\epsilon . \qquad \text{(for all } \sigma\in S_{n-1}) .$$

Multiplying this relation by $e(\overline{T})\,\big|_\sigma$ and summing over $\sigma\in S_{n-1}$ we get

$$e(\overline{T})\, P(T)N(T)\,\big|_\epsilon \;=\; e(\overline{T})\, P(\overline{T})N(\overline{T})\,\big|_\epsilon .$$

and 2.19 gives

$$e(\overline{T})\, P(T)N(T)\,\big|_\epsilon \;=\; 1 ,$$

proving 2.18 b). Now 2.16 gives

$$\begin{aligned}
h(T)e(T)P(T)N(T)\,\big|_\epsilon &= e(\overline{T})P(T)N(T)e(\overline{T})P(T)N(T)\,\big|_\epsilon \\
\text{(by Prop. 1.2)} &= e(\overline{T})\Big(N(T)e(\overline{T})P(T)N(T)P(T)\,\big|_\epsilon\Big)P(T)N(T)\,\big|_\epsilon \\
\text{(by 2.18 b)} &= \Big(e(\overline{T})P(T)N(T)P(T)N(T)\,\big|_\epsilon\Big) \\
\text{(by 1.10)} &= h(T)\, e(\overline{T})P(T)N(T)\,\big|_\epsilon \\
\text{(by 2.18 b)} &= h(T) .
\end{aligned}$$

This proves

$$e(T)P(T)N(T)\,\big|_\epsilon \;=\; 1 .$$

and completes the induction.

Proof of Proposition 2.1

To begin we have

$$\begin{aligned}
E(T)e(\overline{T})E(T) &= P(T)N(T)e(\overline{T})P(T)N(T) \\
\text{(by Prop. 1.2)} &= N(T)e(\overline{T})P(T)N(T)P(T)\big|_\epsilon P(T)N(T) \\
&= e(\overline{T})P(T)N(T)P(T)N(T)\big|_\epsilon P(T)N(T) \\
\text{(by 1.10)} &= h(T)e(\overline{T})P(T)N(T)\big|_\epsilon P(T)N(T) \\
\text{(by 2.18 b))} &= h(T)\, P(T)N(T) \;=\; h(T)E(T) .
\end{aligned}$$

This proves 2.17 a).

Next we have

$$\begin{aligned}
E(T)e(T)E(T) &= P(T)N(T)e(T)P(T)N(T) \\
\text{(by Prop. 1.2)} &= N(T)e(T)P(T)N(T)P(T)\big|_\epsilon P(T)N(T) \\
&= e(T)P(T)N(T)P(T)N(T)\big|_\epsilon P(T)N(T) \\
\text{(by 1.10)} &= h(T)e(T)P(T)N(T)\big|_\epsilon P(T)N(T) \\
\text{(by 2.18 a))} &= h(T)\, P(T)N(T) = h(T)E(T) .
\end{aligned}$$

This proves 2.17 b).

Finally from the definition in 2.16 we get

$$\begin{aligned} e(T)E(T)e(T) &= e(\overline{T})\frac{E(T)}{h(T)}e(\overline{T})E(T)e(T) \\ \text{(by 2.17 a))} &= e(\overline{T})E(T)e(T) \\ \text{(by 2.16)} &= e(\overline{T})E(T)e(\overline{T})\frac{E(T)}{h(T)}e(\overline{T}) \\ \text{(by 2.17 a))} &= e(\overline{T})E(T)e(\overline{T}) \\ \text{(by 2.16)} &= h(T)\, e(T)\,. \end{aligned}$$

This proves 2.17 c).

We are now ready to establish the basic properties of Young's seminormal units. We begin with a result which is an immediate consequence of the definition.

Theorem 2.1

For any standard tableau T the group algebra element $e(T)$ is idempotent.

Proof

For $T = [1]$ this is true by definition, we can thus proceed by induction on the number of cells of T. Now we have

$$\begin{aligned} e(T)e(T) &= e(\overline{T})\frac{E(T)}{h(T)}\, e(\overline{T})\, e(\overline{T})\, \frac{E(T)}{h(T)}\, e(\overline{T}) \\ \text{(by induction)} &= e(\overline{T})\frac{E(T)}{h(T)}\, e(\overline{T})\, \frac{E(T)}{h(T)} e(\overline{T}) \\ \text{(by 2.17 a))} &= e(\overline{T})\frac{E(T)}{h(T)} e(\overline{T}) \;=\; e(T)\,. \end{aligned}$$

Q.E.D.

These idempotents are orthogonal, more precisely we have

Theorem 2.2

If T_1 and T_2 are standard tableaux with n cells then

$$e(T_1)\, e(T_2) \;=\; \begin{cases} 0 & \textit{if } \lambda(T_1) \neq \lambda(T_2) \\ 0 & \textit{if } \lambda(T_1) = \lambda(T_2) \textit{ but } T_1 \neq T_2 \\ e(T_1) & \textit{if } T_1 = T_2 \end{cases} \qquad 2.20$$

Proof

Since by definition

$$e(T_1)\, e(T_2) \;=\; e(\overline{T}_1)\frac{E(T_1)}{h(T_1)}\, e(\overline{T}_1)\, e(\overline{T}_2)\frac{E(T_2)}{h(T_2)}\, e(\overline{T}_2)\,, \qquad 2.21$$

the first equality is an immediate consequence of 1.7. The last equality is a restatement of Theorem 2.1. We are left to prove the second equality. Since for $n = 1, 2$ there is nothing to prove, we shall proceed by induction on n and assume its validity for $n-1$. This given we have the following sequence of implications

$$\begin{aligned} & e(T_1)\, e(T_2) \neq 0 \\ \text{(by 2.21)} \quad & \Downarrow \\ & e(\overline{T}_1)\, e(\overline{T}_2) \neq 0 \\ \text{(by induction)} \quad & \Downarrow \\ & \overline{T}_1 = \overline{T}_2 \\ \text{(when } \lambda(T_1)=\lambda(T_2)) \quad & \Downarrow \\ & T_1 = T_2 \end{aligned}$$

This proves our assertion.

Theorem 2.3

Young's seminormal units statisfy the identities

$$e^{\lambda}_{ij} e^{\mu}_{rs} \;=\; \begin{cases} 0 & \text{if } \lambda \neq \mu\,, \\ 0 & \text{if } \lambda = \mu\,, \text{but } j \neq r \\ e^{\mu}_{is} & \text{if } \lambda = \mu \text{ and } j = r\,. \end{cases} \tag{2.22}$$

Proof

From the definition in 2.15 we derive that

$$a) \quad e^{\lambda}_{ij} \;=\; e(\overline{T}^{\lambda}_i)\, \sigma^{\lambda}_{ij}\, \frac{E(T^{\lambda}_j)}{h_\lambda}\, e(\overline{T}^{\lambda}_j)\,, \qquad b) \quad e^{\mu}_{rs} \;=\; e(\overline{T}^{\mu}_r)\, \frac{E(T^{\mu}_r)}{h_\mu}\, \sigma^{\mu}_{rs}\, e(\overline{T}^{\mu}_s)\,. \tag{2.23}$$

Thus the first equality in 2.1 is an immediate consequence of 1.7. In the case $\mu = \lambda$, 2.33 a) and b) give

$$e^{\lambda}_{ij}\; e^{\lambda}_{rs} \;=\; e(\overline{T}^{\lambda}_i)\, \sigma^{\lambda}_{ij}\, \frac{E(T^{\lambda}_j)}{h_\lambda}\, e(\overline{T}^{\lambda}_j) e(\overline{T}^{\lambda}_r)\, \frac{E(T^{\mu}_r)}{h_\lambda}\, \sigma^{\lambda}_{rs} e(\overline{T}^{\lambda}_s)\,, \tag{2.24}$$

Now if $r \neq j$, Theorem 2.2 gives $e(\overline{T}^{\lambda}_j) e(\overline{T}^{\lambda}_r) = 0$, proving the second case of 2.22. Finally for $\lambda = \mu$ and $j = r$, 2.24 becomes

$$\begin{aligned} e^{\lambda}_{ij}\; e^{\lambda}_{js} \;&=\; e(\overline{T}^{\lambda}_i)\, \sigma^{\lambda}_{ij}\, \frac{E(T^{\lambda}_j)}{h_\lambda}\, e(\overline{T}^{\lambda}_j)\, e(\overline{T}^{\lambda}_j) \frac{E(T^{\mu}_j)}{h_\lambda}\, \sigma^{\lambda}_{js}\, e(\overline{T}^{\lambda}_s)\,, \\ \text{(by Theorem 2.1)} \;&=\; e(\overline{T}^{\lambda}_i)\, \sigma^{\lambda}_{ij}\, \frac{E(T^{\lambda}_j)}{h_\lambda}\, e(\overline{T}^{\lambda}_j)\, \frac{E(T^{\mu}_j)}{h_\lambda}\, \sigma^{\lambda}_{js}\, e(\overline{T}^{\lambda}_s)\,, \\ \big(\text{by 2.17 a)}\big) \;&=\; e(\overline{T}^{\lambda}_i)\, \sigma^{\lambda}_{ij}\, \frac{E(T^{\lambda}_j)}{h_\lambda}\, \sigma^{\lambda}_{js}\, e(\overline{T}^{\lambda}_s)\,, \\ \text{(by 1.4)} \;&=\; e(\overline{T}^{\lambda}_i)\, \frac{P(T^{\lambda}_i)\, \sigma^{\lambda}_{is}\, N(T^{\lambda}_s)}{h_\lambda}\, e(\overline{T}^{\lambda}_s) \;=\; e^{\lambda}_{is}\,. \end{aligned} \tag{2.25}$$

This proves the last equality in 2.22 and completes our argument.

Note that we also have

Theorem 2.4

$$e^{\lambda}_{ij}\big|_{\epsilon} \;=\; \begin{cases} 0 & \text{if } i\neq j\,,\\ 1/h_\lambda & \text{if } i=j\,. \end{cases} \qquad 2.26$$

Proof

The definition in 2.15 gives

$$\begin{aligned} e^{\lambda}_{ij}\big|_{\epsilon} &= e(\overline{T}^{\lambda}_i)\,\frac{P(T^{\lambda}_i)\,\sigma^{\lambda}_{ij}\,N(T^{\lambda}_j)}{h_\lambda}\,e(\overline{T}^{\lambda}_j)\Big|_{\epsilon}\\ &= e(\overline{T}^{\lambda}_j)\,e(\overline{T}^{\lambda}_i)\,\frac{P(T^{\lambda}_i)\,\sigma^{\lambda}_{ij}\,N(T^{\lambda}_j)}{h_\lambda}\Big|_{\epsilon} \end{aligned}$$

and the first case of 2.26 follows from Theorem 2.2. But for $i=j$ this becomes

$$\begin{aligned} e^{\lambda}_{ii}\big|_{\epsilon} &= e(\overline{T}^{\lambda}_i)\,e(\overline{T}^{\lambda}_i)\,\frac{P(T^{\lambda}_i)\,N(T^{\lambda}_i)}{h_\lambda}\Big|_{\epsilon}\\ \text{(by Theorem 2.1)}\quad &= \frac{1}{h_\lambda}\,e(\overline{T}^{\lambda}_i)\,P(T^{\lambda}_i)\,N(T^{\lambda}_i)\Big|_{\epsilon}\\ \text{(by 2.18 b))}\quad &= \frac{1}{h_\lambda} \end{aligned}$$

This completes our proof.

From the identities in 2.22 and formula 2.9, it follows that, for $\lambda\vdash n$, the character of the representation resulting from the action of S_n on a linear span

$$\mathcal{L}\big[e^{\lambda}_{1s}\,,\,e^{\lambda}_{2s}\,,\,\ldots\,,\,e^{\lambda}_{n_\lambda s}\big]$$

is the same as the character of the action of S_n on the left ideal $\mathcal{A}(S_n)e(T^{\lambda}_s)$.

From this observation we derive the following important fact.

Proposition 2.2

The character of the action of S_n on $\mathcal{L}\big[e^{\lambda}_{1s}\,,\,e^{\lambda}_{2s}\,,\,\ldots\,,\,e^{\lambda}_{n_\lambda s}\big]$ depends only on λ and it is given by the formula

$$\chi^{\lambda} \;=\; \sum_{T\in St(\lambda)} \frac{P(T)N(T)}{h_\lambda}$$

where "$T\in St(\lambda)$" is to indicate that the sum is over all tableaux of shape λ.

Proof

It is shown in the [RT] notes that if $\mathbf{e}$ is an idempotent of the group algebra $\mathcal{A}(G)$ then the character χ^{e} of the left multiplication action of G on the ideal $\mathcal{A}(G)\mathbf{e}$ is given by the formula

$$\chi^{e} \;=\; \sum_{\sigma\in G}\sigma\,\mathbf{e}\,\sigma^{-1}$$

Taking $T = T_s^\lambda$, from this formula we obtain that

$$\begin{aligned} \chi^\lambda &= \sum_{\sigma\in G} \sigma\, e(T)\, \sigma^{-1} = \sum_{\sigma\in S_n} \sigma\, e(\overline{T})\, \frac{P(T)N(T)}{h_\lambda}\, e(\overline{T})\, \sigma^{-1} \\ \text{(Circular rearrangements are OK)} &= \sum_{\sigma\in S_n} \sigma\ e(\overline{T}) e(\overline{T})\, \frac{P(T)N(T)}{h_\lambda}\, \sigma^{-1} \\ \text{(by Theorem 2.1)} &= \sum_{\sigma\in S_n} \sigma\ e(\overline{T})\, \frac{P(T)N(T)}{h_\lambda}\, \sigma^{-1} \\ &= \sum_{\sigma\in S_n} \sigma\, \frac{N(T)\, e(\overline{T})\, P(T)}{h_\lambda}\, \sigma^{-1} \\ \text{(by Von Neuman lemma)} &= \Big(e(\overline{T}) P(T) N(T) \big|_\epsilon \Big) \sum_{\sigma\in S_n} \sigma\, \frac{N(T)P(T)}{h_\lambda}\, \sigma^{-1}\,. \end{aligned}$$

and 2.18 b) gives (by 1.3)

$$\chi^\lambda = \sum_{\sigma\in S_n} \frac{N(\sigma T)P(\sigma T)}{h_\lambda}\,.$$

This proves the Theorem since σT describes all tableaux of shape λ, as σ varies in S_n.

Remark 2.1

It is easily derived from the identities in 2.22 that the set $\big\{\{e_{ij}^\lambda\}_{ij=1}^{n_\lambda}\big\}_{\lambda\vdash n}$ is independent. Furthermore, we proved in [YNR] that the numbers of standard tableaux n_λ satisfy the identity

$$\sum_{\lambda\vdash n} n_\lambda^2 = n!\,.$$

Thus $\big\{\{e_{ij}^\lambda\}_{ij=1}^{n_\lambda}\big\}_{\lambda\vdash n}$ is an independent subset of $\mathcal{A}(S_n)$ of cardinality equal to the dimension of $\mathcal{A}(S_n)$. So it must be a basis. From this it is easy to derive that the expansion of any element $f \in \mathcal{A}(S_n)$ in terms of the Young's seminormal units is given by formula 2.4. We terminate this section with two important applications of this formula.

We begin with the following beautiful identity of Alfred Young.

Theorem 2.5

For any standard tableau T

$$e(\overline{T}) = \sum_{S} \chi(\overline{S} = \overline{T})\, e(S) \qquad 2.27$$

where the sum is over all standard tableaux S yielding $\overline{T}$ upon removal of n .

Proof

Using formula 2.4 for $f = e(\overline{T})$ and Young's units we get, (if T has n cells)

$$e(\overline{T}) = \sum_{\lambda\vdash n} h_\lambda \sum_{i,j=1}^{n_\lambda} e(\overline{T})\, e_{ji}^\lambda \Big|_\epsilon\, e_{ij}^\lambda \qquad 2.28$$

Now we have

$$
\begin{aligned}
e(\overline{T})\, e^\lambda_{ji}\Big|_\epsilon &= e(\overline{T})e(\overline{T}^\lambda_j)\frac{P(T^\lambda_j)\sigma^\lambda_{ji}N(T^\lambda_i)}{h\lambda}\, e(\overline{T}^\lambda_i)\Big|_\epsilon \\
&= e(\overline{T}^\lambda_i)e(\overline{T})e(\overline{T}^\lambda_j)\frac{P(T^\lambda_j)\sigma^\lambda_{ji}N(T^\lambda_i)}{h\lambda}\Big|_\epsilon ,.
\end{aligned}
\qquad 2.29
$$

and we immediately derive from Theorem 2.2 that this term doesn't vanish only if

$$
\overline{T}^\lambda_i \;=\; \overline{T} \;=\; \overline{T}^\lambda_j\,. \qquad 2.30
$$

In particular this forces $i = j$, and 2.29 becomes

$$
\begin{aligned}
e(\overline{T})\, e^\lambda_{ii}\Big|_\epsilon &= e(\overline{T}^\lambda_i)e(\overline{T})e(\overline{T}^\lambda_i)\frac{P(T^\lambda_i)N(T^\lambda_i)}{h_\lambda}\Big|_\epsilon \\
\text{(by Theorem 2.1)} &= e(\overline{T}^\lambda_i)\frac{P(T^\lambda_j)N(T^\lambda_i)}{h_\lambda}\Big|_\epsilon \\
\text{(by 2.18 b))} &= \frac{1}{h_\lambda}\,.
\end{aligned}
\qquad 2.31
$$

Using 2.30 and 2.31 reduces 2.28 to

$$
e(\overline{T}) \;=\; \sum_\lambda \sum_{i=1}^{n_\lambda} \chi\big(\overline{T}^\lambda_i = \overline{T}\big)\, e^\lambda_{ii}\,. \qquad 2.32
$$

Since

$$
e^\lambda_{ii} \;=\; e(\overline{T}^\lambda_i)\frac{P(T^\lambda_i)N(T^\lambda_i)}{h_\lambda}\, e(\overline{T}^\lambda_i) \;=\; e(T^\lambda_i)
$$

formula 2.32 is only another way of writing 2.27.

A immediate corollary of Theorem 2.5 is the identification of the constant h_λ.

Proposition 2.3

$$
h_\lambda \;=\; \frac{n!}{n_\lambda} \qquad 2.33
$$

Proof

Equating coefficients of the identity on both sides of 2.27, and using 2.26 we get

$$
\frac{1}{h(\overline{T})} \;=\; \sum_S \chi(\overline{S} = \overline{T})\,\frac{1}{h(S)}\,.
$$

Assuming that T has n cells, and that $\lambda(\overline{T}) = \mu$ we may rewrite this identity in the form

$$
\frac{1}{h_\mu} \;=\; \sum_\lambda \chi(\mu\to\lambda)\,\frac{1}{h_\lambda}\,, \qquad 2.34
$$

where the symbol "$\mu\to\lambda$" is to express that λ is obtained by adding a cell to μ. Multiplying both sides of 2.33 by $n!$ and setting

$$
k_\mu \;=\; \frac{(n-1)!}{h_\mu}\,, \qquad k_\lambda \;=\; \frac{n!}{h_\lambda}
$$

converts 2.34 into the identity

$$n\ k_\mu \;=\; \sum_\lambda \chi(\mu\to\lambda)\, k_\lambda\,.$$

which is precisely the recursion statisfied by the number of standard tableaux. From this it easily follows that we must have

$$k_\lambda \;=\; n_\lambda \qquad\qquad \forall\ \lambda$$

as desired.

The seminormality of the Young units is based on the following remarkable fact.

Theorem 2.6

For any standard tableaux T we have

$$\downarrow e(T) \;=\; e(T)\,. \tag{2.35}$$

Proof

Since 2.35 is obviously true for $T=[1]$ we can proceed by induction on the number of cells of T. This given we have

$$\downarrow e(T) \;=\; e(\overline{T})\frac{N(T)P(T)}{h(T)}e(\overline{T})$$

Using 2.27 this may be rewritten as

$$\downarrow e(T) \;=\; \sum_{\overline{T}_1=\overline{T}}\ \sum_{\overline{T}_2=\overline{T}} e(T_1)\frac{N(T)P(T)}{h(T)}e(T_2) \tag{2.36}$$

Since

$$e(T_1)\frac{N(T)P(T)}{h(T)}e(T_2) \;=\; e(\overline{T}_1)\frac{E(T_1)}{h(T_1)}e(\overline{T}_1)\ \frac{N(T)P(T)}{h(T)}\ e(\overline{T}_2)\frac{E(T_2)}{h(T_2)}e(\overline{T}_2)$$

from Theorem 1.1 we derive that T_1, T_2 and T must all have the same shape. But then the conditions $\overline{T}_1=\overline{T}=\overline{T}_2$ force them all to be the same, converting 2.36 to

$$\begin{aligned}
\downarrow e(T) &= e(\overline{T})\frac{E(T)}{h(T)}e(\overline{T})\ \frac{N(T)P(T)}{h(T)}\ e(\overline{T})\frac{E(T)}{h(T)}e(\overline{T})\\
&= e(\overline{T})\frac{P(T)N(T)}{h(T)}e(\overline{T})\ \frac{N(T)P(T)}{h(T)}\ e(\overline{T})\frac{P(T)N(T)}{h(T)}e(\overline{T})\\
\text{(by 1.9 used twice)} &= \frac{ab}{h(T)^3}\, e(\overline{T})P(T)N(T)\ \ P(T)N(T)e(\overline{T})\\
\text{(by 1.10)} &= \frac{ab}{h(T)^2}\, e(\overline{T})P(T)N(T)e(\overline{T}) \;=\; \frac{ab}{h(T)}\, e(T)
\end{aligned} \tag{2.37}$$

where

$$a \;=\; N(T)e(\overline{T})N(T)P(T)\Big|_\epsilon \quad \text{and} \quad b \;=\; e(\overline{T})P(T)N(T)P(T)\Big|_\epsilon\,.$$

To avoid computing these constants we simply observe that $\downarrow e(T)$ and $e(T)$ must have the same coefficient of the identity, and since this coefficient must be $1/h(T)$ by 2.26, the identity in 2.37 forces

$$ab \;=\; h(T)$$

and proves 2.35.

We are finally ready to prove seminormality.

Theorem 2.7

For each partition λ we have constants

$$d_1^\lambda \ , \ d_2^\lambda \ , \ \ldots \ , \ d_{n_\lambda}^\lambda \tag{2.38}$$

giving

$$\downarrow e_{ij}^\lambda \ = \ \frac{d_i^\lambda}{d_j^\lambda}\ e_{ji}^\lambda \qquad \text{(for all } 1 \leq i, j \leq n_\lambda) \,. \tag{2.39}$$

Proof

Since 2.39 may also be written as

$$\downarrow e_{ij}^\lambda \ = \ \frac{d_i^\lambda / d_1^\lambda}{d_j^\lambda / d_1^\lambda}\ e_{ji}^\lambda \,,$$

there is no loss in assuming that

$$d_1^\lambda \ = \ 1 \,.$$

This given, we need only show that for some constants d_j we have

$$\downarrow e_{1j} \ = \ \frac{1}{d_j}\, e_{j1} \qquad \text{(for } 1 \leq j \leq n_\lambda), \tag{2.40}$$

where to lighten our notation we shall for a moment omit the superscript λ. In fact, 2.40 implies

$$\downarrow e_{i1} \ = \ d_i\, e_{1i} \qquad \text{(for } 1 \leq i \leq n_\lambda),$$

and then 2.22 gives

$$\downarrow e_{ij} \ = \ \downarrow (e_{i1} e_{1j}) \ = \ (\downarrow e_{1j})(\downarrow e_{i1}) \ = \ \frac{d_i}{d_j}\, e_{j1}\, e_{1i} \ = \ \frac{d_i}{d_j}\, e_{ji} \,,$$

proving 2.39. So let us then prove 2.40. To this end we use the expansion formula 2.4 and get

$$\downarrow e_{1j} \ = \ \sum_{r=1}^{n_\lambda} \sum_{s=1}^{n_\lambda} \Big((\downarrow e_{1j}) e_{sr} \big|_\epsilon \Big)\, e_{rs} \tag{2.41}$$

However, note that Theorem 2.6 gives

$$\begin{aligned}(\downarrow e_{1j}) e_{sr} \big|_\epsilon \ &= \ e(\overline{T}_j) \frac{N(T_j)P(T_j)}{h_\lambda}\ \sigma_{j1} e(\overline{T}_1) e(\overline{T}_s) \frac{N(T_s)P(T_s)}{h_\lambda}\ \sigma_{sr} e(\overline{T}_r) \Big|_\epsilon \\ &= \ \frac{N(T_j)P(T_j)}{h_\lambda}\ \sigma_{j1} e(\overline{T}_1) e(\overline{T}_s) \frac{N(T_s)P(T_s)}{h_\lambda}\ \sigma_{sr} e(\overline{T}_r) e(\overline{T}_j) \Big|_\epsilon\end{aligned}$$

and the non vanishing of the two products

$$e(\overline{T}_1)e(\overline{T}_s) \quad \text{and} \quad e(\overline{T}_r)e(\overline{T}_j)$$

forces $s = 1$ and $r = j$ reducing 2.41 to

$$\downarrow e_{1j} \;=\; \Big((\downarrow e_{1j}) e_{1j}\big|_{\epsilon} \Big)\, e_{j1}$$

and this proves 2.40 with

$$\frac{1}{d_j} \;=\; (\downarrow e_{1j}) e_{1j}\big|_{\epsilon}\,.$$

This completes the proof and this section. The actual nature of the constants in 2.38 will come to the surface in the next section.

3. The Murphy elements

A remarkable set of group algebra elements was shown by Murphy [5] to play an elegant role in the study of representations of S_n. It develops that these elements considerably simplify manipulations with Young seminormal units. Their definition is quite simple. We set

$$m_k \;=\; (1,k) \,+\, (2,k) \,+\, (3,k) \,+\, \cdots \,+\, (k-1,k) \qquad \text{(for } k = 2, 3, \ldots, n\text{)} \tag{3.1}$$

These elements generate a commutative subalgebra of $\mathcal{A}(S_n)$. In fact we have the following basic relations.

Theorem 3.1

Letting s_h *denote the transposition* $(h, h+1)$

$$\begin{aligned} &a) \quad m_h\, m_k \;=\; m_k\, m_h \qquad \text{(for } 2 \le h \le k \le n\text{)}\,,\\ &b) \quad s_h\, m_k\, s_h \;=\; m_k \qquad \text{(for } h \ne k, k-1\text{)}\,,\\ &c) \quad s_k\, m_k\, s_k \;=\; m_{k+1} - s_k\,.\\ &d) \quad s_{k-1}\, m_k\, s_{k-1} \;=\; m_{k-1} + s_k\,. \end{aligned} \tag{3.2}$$

Proof

Note first that in $\mathcal{A}(S_3)$ we have

$$(1,2)\big((1,2)+(1,3)+(2,3)\big) \;=\; \big((1,2)+(1,3)+(2,3)\big)(1,2)$$

and this immediately implies

$$m_2\, m_3 \;=\; m_3\, m_2\,.$$

This relation will clearly remain valid in $\mathcal{A}(S_n)$ for all $n \ge 3$. So we may proceed by induction and suppose that

$$m_2\;,\; m_3\;,\; \ldots\;,\; m_{n-1}$$

have been shown to commute in $\mathcal{A}(S_{n-1})$. Since they will necessarily commute also in $\mathcal{A}(S_n)$. and

$$C_2 \;=\; m_2 + m_3 + \cdots + m_n \tag{3.3}$$

is a conjugacy class, we deduce that for all $2 \leq k \leq n-1$ we have

$$\begin{aligned} m_2\, m_k \; + \cdots + m_{n-1}\, m_k + m_k\, m_n &= m_k\,(m_2 + m_3 + \cdots + m_n) \\ &= (m_2 + m_3 + \cdots + m_n)\, m_k \\ &= m_2\, m_k + \cdots + m_{n-1}\, m_k + m_n\, m_k \end{aligned}$$

This yields

$$m_k\, m_n \;=\; m_n\, m_k \qquad \text{(for all } 2 \leq k \leq n-1)$$

and proves 3.2 a). Now note that 3.2 b) is trivial when $h > k$. On the other hand, when $h < k$, conjugation of m_k by s_h only interchanges the terms (h, k) and $(h+1, k)$ in m_k. Thus 3.2 b) must hold true for all $h \neq k$ precisely as asserted. Finally, we see that we also have

$$s_k\Big((1,k)+(2,k)+\cdots+(k-1,k)\Big)s_k = ((1,k+1)+(2,k+1)+\cdots+(k-1,k+1)-m_{k+1} \;-\; (k,k+1)\,.$$

This proves 3.2 c). The proof is now complete since 3.2 d) immediately follows from 3.2 c) upon replacing k by $k-1$.

What is truly remarkable is that the Murphy elements have the Young seminormal units e^λ_{ij} as their common eigenvectors. This is an immediate consequence of the following identity.

Theorem 3.2

For every standard tableaux T of shape $\lambda \vdash n$ we have

$$C_2\, e(T) \;=\; \big(n(\lambda') - n(\lambda)\big)\, e(T)\,, \tag{3.4}$$

where for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ we set

$$n(\lambda) \;=\; \sum_{i=1}^{k} (i-1)\,\lambda_i \tag{3.5}$$

and λ' denotes the conjugate of λ

Proof

Since C_2 commutes with every element of $\mathcal{A}(S_n)$ we derive that

$$C_2\, e(T) \;=\; e(\overline{T})\frac{P(T)C_2N(T)}{h_\lambda}e(\overline{T})\,, \tag{3.6}$$

and Von Neuman's lemma then yields

$$P(T)C_2N(T) \;=\; C_2N(T)P(T)\big|_\epsilon\; E(T)\,.$$

Substituting this in 3.6 we get

$$\begin{aligned} C_2\, e(T) &= \Big(C_2 N(T)P(T)\,\big|_\epsilon\Big)\; e(\overline{T})\frac{P(T)N(T)}{h_\lambda}e(\overline{T}) \\ &= \Big(C_2 N(T)P(T)\,\big|_\epsilon\Big)\; e(T)\,. \end{aligned}$$

It remains to prove

$$C_2 N(T)P(T)\,\big|_\epsilon \;=\; n(\lambda') - n(\lambda)\,. \tag{3.7}$$

Now the definition in 1.1 gives

$$C_2 N(T)P(T)\,\big|_\epsilon \;=\; \sum_{1\le i<j\le n}\;\sum_{\alpha\in R(T)}\;\sum_{\beta\in C(T)} sign(\beta)\chi\big(\beta\alpha=(i,j)\big)\,. \tag{3.8}$$

However, it should be apparent that unless the indices i,j are in the same row or column of T it is impossible to interchange their positions in T by a row permutation followed by a column permutation without messing up the positions of the other entries in T. Thus the only way we can have the equality $\beta\alpha=(i,j)$ is $\beta=(i,j)$ or $\alpha=(i,j)$. This reduces the evaluation of the right hand side of 3.8 to counting transpositions in $C(T)$ and $R(T)$. Now if

$$\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_k) \quad\text{and}\quad \lambda'=(\lambda'_1,\lambda'_2,\ldots,\lambda'_h)$$

then the number of transpositions in $C(T)$ and $R(T)$ are respectively given by

$$\sum_{i=1}^h \binom{\lambda'_i}{2} \quad\text{and}\quad \sum_{i=1}^k \binom{\lambda_i}{2}\,.$$

This reduces 3.8 to

$$C_2 N(T)P(T)\,\big|_\epsilon \;=\; \sum_{i=1}^k \binom{\lambda_i}{2} \;-\; \sum_{i=1}^h \binom{\lambda'_i}{2}\,,$$

and it easily seen that this is only another way of writing the equality in 3.7. Our proof of 3.5 is thus complete.

Remark 3.1

We should mention that the identity in 3.7 implies a classical identity (see [4]) satisfied by the characters of S_n. To see this note that since conjugation by an elemnt $\sigma\in S_n$ does not change the coefficient of the identity, formula 3.7 may also be rewritten as

$$\frac{h_\lambda}{n!}\sum_{\sigma\in S_n} C_2\,\sigma\,\frac{N(T)P(T)}{h_\lambda}\,\sigma^{-1}\,\big|_\epsilon \;=\; n(\lambda')-n(\lambda)\,.$$

But then Proposition 2.1 gives

$$\frac{h_\lambda}{n!}\;C_2\,\chi^\lambda\,\big|_\epsilon \;=\; n(\lambda')-n(\lambda)\,.$$

Since $n!/h_\lambda = n_\lambda$ we finally obtain that

$$\frac{|C_2|}{n_\lambda}\ \chi^\lambda_{21^{n-2}} \;=\; n(\lambda') - n(\lambda)\,, \tag{3.9}$$

where $\chi^\lambda_{21^{n-2}}$ gives the value of χ^λ at the conjugacy class C_2 and $|C_2|$ gives the cardinality of C_2.

It is customary to call the difference "$j-i$" the "*content* " of the lattice cell (i,j). For instance in the figure below we display the Ferrers diagram of the partition $(5,3,3)$ with its cells filled by their contents.

$$\begin{array}{|c|c|c|cc} \hline -2 & -1 & 0 & & \\ \hline -1 & 0 & 1 & & \\ \hline 0 & 1 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \tag{3.10}$$

For a given tableau T with n cells and an integer $2 \leq k \leq n$ let us denote by $c_T(k)$ the content of the cell that contains k in T. This given it is easy to see that for any tableau T of shape λ we necessarily have

$$\sum_{k=2}^{n} c_T(k) \;=\; n(\lambda') - n(\lambda)\,. \tag{3.11}$$

This simple observation causes Theorem 3.2 to have the following truly remarkable corollary.

Theorem 3.3

For every standard tableau T with n cells we have for $2 \leq k \leq n$

$$\begin{aligned} a)\quad m_k\, e(T) &= c_T(k)\, e(T)\,, \\ b)\quad e(T)\, m_k &= c_T(k)\, e(T)\,. \end{aligned} \tag{3.12}$$

Proof

Note that since by Theorem 2.6 $e(T)$ is self flipping and m_k is trivially so, 3.12 b) can be obtained by flipping both sides of 3.12 a). So we need only prove the latter. From the definition in 2.16 it follows that

$$e\Big(\begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array}\Big) = \epsilon - (1,2) \qquad \text{and} \qquad e\big(\begin{array}{|c|c|}\hline 1&2\\ \hline\end{array}\big) = \epsilon + (1,2)\,.$$

Thus we see that

$$m_2\, e\Big(\begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array}\Big) = -e\Big(\begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array}\Big) \qquad \text{and} \qquad m_2\, e\big(\begin{array}{|c|c|}\hline 1&2\\ \hline\end{array}\big) = e\big(\begin{array}{|c|c|}\hline 1&2\\ \hline\end{array}\big)\,.$$

This verifies 3.12 a) for $n=2$. So we may proceed by induction and suppose that 3.12 a) has been verified up to $n-1$. In particular if T is any standard tableau with n cells we will necessarily have

$$m_k\, e(\overline{T}) = c_{\overline{T}}(k) e(\overline{T}) = c_T(k) e(\overline{T}) \qquad \text{(for } 2 \leq k \leq n-1)\,. \tag{3.13}$$

Thus

$$\begin{aligned} m_k\, e(T) &= m_k\, e(\overline{T})\, \frac{P(T)N(T)}{h(T)}\, e(\overline{T}) \\ \text{(by 3.13)} &= c_T(k)\, e(\overline{T})\, \frac{P(T)N(T)}{h(T)}\, e(\overline{T}) = c_T(k)\, e(T)\,. \end{aligned} \tag{3.14}$$

This proves 3.12 for for $2 \leq k \leq n-1$. But for $k = n$ we may use 3.4 with $n(\lambda') - n(\lambda)$ given by 3.11 and get

$$(m_2 + m_3 + \cdots + m_n)\, e(T) \;=\; (c_T(2) + c_T(3) + \cdots + c_T(n))\, e(T)\,.$$

Subtracting the identities in 3.14 for $2 \leq k \leq n-1$ yields

$$m_n\, e(T) \;=\; c_T(n)\, e(T)\,.$$

completing the induction and the proof.

A beautiful consequence of this result is a completely explicit formula for Young's seminormal units. To present this development we need a few preliminary observations. To begin note that if $P(x_2, x_3, \ldots, x_n)$ is any polynomial in its arguments and T is any standard tableau, then from Theorem 3.3 it follows that

$$P(m_2, m_3, \ldots, m_n)\, e(T) \;=\; P(c_T(2), c_T(3), \ldots, c_T(n))\, e(T)$$

now we may construct $P(x_2, x_3, \ldots, x_n)$ in such manner that $P(c_T(2), c_T(3), \ldots, c_T(n)) = 1$ while at the same time the operator $P(m_2, m_3, \ldots, m_n)$ kills all the other seminormal idempotents. To give a precise and efficient construction of such a polynomial we need notation.

Let us recall that the "*addable cells* " a partition μ of $n-1$ are the cells we may add to the Ferrers diagram of μ to obtain of the Ferrers diagram of a partition of n. The collection of contents of the addable cells of μ will be simply denoted "$\mathcal{AC}_\mu$". For instance it is easily seen from figure 3.10 that

$$\mathcal{AC}_{533} \;=\; \{-3, 2, 5\}\,.$$

This given we have

Theorem 3.4

For any standard tableau T we recursively define a polynomial $P_T(x_2, x_3, \ldots, x_n)$ by setting

$$P_T(x_2, x_3, \ldots, x_n) \;=\; P_{\overline{T}}(x_2, x_3, \ldots, x_{n-1}) \prod_{\substack{c \in \mathcal{AC}_{\lambda(\overline{T})} \\ c \neq c_T(n)}} \frac{x_n - c}{c_T(n) - c}\,, \qquad (\textit{with } P_{[1]} = 1) \qquad 3.16$$

Then

$$P_T(m_2, m_3, \ldots, m_n)\, e(S) \;=\; \begin{cases} 0 & \text{if } S \text{ is standard and } S \neq T\,, \\ e(T) & \text{if } S = T \end{cases} \qquad 3.17$$

Proof

For $T = [1]$ there is nothing to prove. We can thus proceed by induction and assume 3.17 to be valid for all for tableaux with $n-1$ cells. Note that the induction hypothesis immediately implies that the expression

$$P_{\overline{T}}(m_2, m_3, \ldots, m_{n-1})\, e(\overline{S})$$

fails to vanish only if and only if $\overline{S} = \overline{T}$ and in that case we have

$$P_{\overline{T}}(m_2, m_3, \ldots, m_{n-1})\, e(\overline{T}) \;=\; 1$$

This also proves the first case of 3.17 when $\overline{S} \neq \overline{T}$. So we are left with the case $\overline{S} = \overline{T}$. Now from from 3.16 it follows that for $\overline{S} = \overline{T}$

$$\begin{aligned} P_T(m_2, m_3, \ldots, m_n)\, e(S) \;&=\; \Big(\prod_{\substack{c \in \mathcal{AC}_{\lambda(\overline{T})} \\ c \neq c_T(n)}} \frac{x_k - c}{c_T(n) - c} \Big) e(S) \\ (\text{ by 3.12 a)}) \;&=\; \Big(\prod_{\substack{c \in \mathcal{AC}_{\lambda(\overline{T})} \\ c \neq c_T(n)}} \frac{c_S(n) - c}{c_T(n) - c} \Big) e(S) \end{aligned}$$

This may also be written as

$$P_T(m_2, m_3, \ldots, m_n)\, e(S) \;=\; \Big(\prod_{\substack{c \in \mathcal{AC}_{\mu} \\ c \neq c_T(n)}} \frac{c_S(n) - c}{c_T(n) - c} \Big) e(S) \qquad 3.18$$

where for convenience we have set

$$\mu \;=\; \lambda(\overline{S}) = \lambda(\overline{T})$$

Note further that the equality $\overline{S} = \overline{T}$ forces $c_S(n) \in \mathcal{AC}_\mu$. So for the right hand side of 3.18 not to vanish we must have $c_S(n) = c_T(n)$. But this holds true if and only if $S = T$. In this case we have

$$\prod_{\substack{c \in \mathcal{AC}_{\lambda(\overline{T})} \\ c \neq c_T(n)}} \frac{c_S(n) - c}{c_T(n) - c} \;=\; \prod_{\substack{c \in \mathcal{AC}_{\lambda(\overline{T})} \\ c \neq c_T(n)}} \frac{c_T(n) - c}{c_T(n) - c} \;=\; 1$$

and 3.18 reduces to

$$P_T(m_2, m_3, \ldots, m_n)\, e(T) \;=\; e(T)\,,$$

completing the proof of the Theorem.

We can now prove the following remarkable fact

Theorem 3.5

For any standard tableau T we have

$$e(T) \;=\; P_T(m_2, m_3, \ldots, m_n)\,. \qquad 3.19$$

Proof

Assume that

$$T = T_r^\mu\,. \qquad 3.20$$

Then

$$\begin{aligned} P_T(m_2,m_3,\ldots,m_n)e^\lambda_{ji}\,\Big|_\epsilon &= P_T(m_2,m_3,\ldots,m_n)e^\lambda_{jj}e^\lambda_{ji}\,e^\lambda_{ii}\,\Big|_\epsilon \\ &= P_T(m_2,m_3,\ldots,m_n)e(T^\lambda_j)e^\lambda_{ji}\,e(T^\lambda_i)\,\Big|_\epsilon \\ &= e(T^\lambda_j)e^\lambda_{ji}\,e(T^\lambda_i)\,P_T(m_2,m_3,\ldots,m_n)\Big|_\epsilon \end{aligned} \qquad 3.21$$

Now, by 3.20 and Theorem 3.4, we have

$$P_T(m_2,m_3,\ldots,m_n)e(T^\lambda_j)\neq 0 \quad\Longrightarrow\quad \lambda=\mu \ \text{ and } \ j=r \qquad 3.22$$

Similarly, in view of 3.12 b) we also derive that

$$e(T^\lambda_i)\,P_T(m_2,m_3,\ldots,m_n)\neq 0 \quad\Longrightarrow\quad \lambda=\mu \ \text{ and } \ i=r \qquad 3.23$$

Using 3.21, 3.22 and 3.23 the expansion in 2.4 with $f=P_T(m_2,m_3,\ldots,m_n)$ reduces to

$$\begin{aligned} P_T(m_2,m_3,\ldots,m_n) &= h_\mu\,P_T(m_2,m_3,\ldots,m_n)e(T^\mu_r)\,\Big|_\epsilon e(T^\mu_r) \\ \text{(by 3.20)} &= h_\mu\,P_T(m_2,m_3,\ldots,m_n)e(T)\,\Big|_\epsilon e(T) \\ \text{(by Theorem 3.4)} &= h_\mu\,e(T)\,\Big|_\epsilon e(T) \\ \text{(by 2.26)} &= e(T) \end{aligned}$$

completing the proof of the Theorem.

It might be good at this point to exhibit a few instances of the identity in 3.19. For the tableaux with three cells Theorem 3.4 gives

$$e\Big(\,\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\,\Big)=(m_2+1)(m_3+1)/6\,,\quad e\Big(\,\begin{array}{|c|c}\hline 3&\\\hline 1&\multicolumn{1}{c|}{2}\\\hline\end{array}\,\Big)=(m_2+1)(m_3-2)/6$$

$$e\Big(\,\begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\,\Big)=-(m_2-1)(m_3-1)/6\,,\quad e\Big(\,\begin{array}{|c|c}\hline 2&\\\hline 1&\multicolumn{1}{c|}{3}\\\hline\end{array}\,\Big)=-(m_2-1)(m_3+2)/6$$

We should also note that we have

$$e\Big(\,\begin{array}{|c|c}\hline 5&\\\hline 2&\multicolumn{1}{c|}{4}\\\hline 1&\multicolumn{1}{c|}{3}\\\hline\end{array}\,\Big) \;=\; -(m_2-1)(m_3+2)(m_4-2)(m_4+2)(m_5-2)/96\,.$$

These expressions are easily derived if we take the view that the successive linear factors must be selected to kick each additional label into the position that it occupies in the target tableau.

4. The seminormal matrices

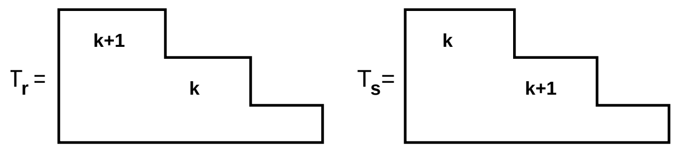
In this section we shall work out explicit formulas for the seminormal matrices corresponding to simple reflections for any given partition μ. Since we shall keep μ fixed throughout our derivation, to simplify the displays we will sometimes omit the superscript μ. So we assume that

$$T_1\,,T_2\,,\ldots\,,T_{n_\mu} \qquad 4.1$$

are the standard tableaux of shape μ in the Young Last Letter Order. We pick a simple transposition $s_k=(k,k+1)$ (for $1\le k\le n-1$) and proceed to construct all the entries of the seminormal matrix $A^\mu(s_k)$. Suppose first that for a pair $1\le r<s\le n_\mu$ we have

$$s_k\,T_r \;=\; T_s\,. \qquad 4.2$$

Note that since T_r and T_s only differ in the positions of k and $k+1$, the last letter of disagreement between T_r and T_s is necessarily $k+1$. Moreover, since $r < s \Rightarrow T_r <_{LL} T_s$ it follows that $k+1$ is higher in T_r than in T_s. Clearly k and $k+1$ cannot be in the same row or column of T_r for then 4.2 would force T_s not to be standard. Furthermore two successive integers can never be in the same diagonal of any standard tableau. In conclusion we see that these two tableaux can only be as indicated in the figure below



4.3

This given our first step is to compute the image by s_k of the seminormal idempotent $e_{rr} = e(T_r)$. To this end we use the expansion formula in 2.4 and start by writing

$$s_k e(T_r) \;=\; \sum_\lambda h_\lambda \sum_{i,j=1}^{n_\lambda} \big(s_k e(T_r)\, e^\lambda_{ji}\Big|_\epsilon\big)\, e^\lambda_{ij}\,. \tag{4.4}$$

However, since by our conventions $e(T_r) = e^\mu_{rr}$ a use of Theorem 2.3 quickly reduces 4.4 to

$$\begin{aligned} s_k e(T_r) &= h_\mu \sum_{i=1}^{n_\mu} \big(s_k e(T_r)\, e^\mu_{ri}\Big|_\epsilon\big)\, e^\mu_{ir} \\ &= h_\mu \sum_{i=1}^{n_\mu} \big(s_k\, e^\mu_{ri}\Big|_\epsilon\big)\, e^\mu_{ir} \\ &= h_\mu \sum_{i=1}^{n_\mu} e^\mu_{ri}(s_k)\, e^\mu_{ir}\,. \end{aligned} \tag{4.5}$$

Now it develops that this expansion can be reduced drammatically further by use of Murphy elements. To this end set

$$\Pi \;=\; \prod_{\substack{h=2\\ h\neq k,k+1}}^{n} \;\prod_{\substack{c=c_1\\ c\neq c_{T_r}(h)}}^{c_2} \frac{(m_h - c)}{(c_{T_r}(h) - c)} \tag{4.6}$$

where c_1 and c_2 respectively denote the minimum and the maximum of the contents of the cells of the diagram of μ. Note that since T_r and T_s only differ in the positions of k and $k+1$ we will have

$$c_{T_r}(h) = c_{T_s}(h) \qquad \text{(for all } h \neq k, k+1)\,. \tag{4.7}$$

It follows from 4.6 and 3.12 a) that

$$\Pi\, e(T_i) \;=\; \begin{cases} 0 & \text{if } i \neq r, s, \\ e(T_r) & \text{if } i = r, \\ e(T_s) & \text{if } i = s. \end{cases} \tag{4.8}$$

In fact, the last two cases of 4.8 are immediate consequences of 4.6 and 4.7. On the other hand we see from 4.6 that

$$\Pi\, e(T_i) \neq 0 \implies c_{T_i}(h) = c_{T_r}(h) \quad \forall \quad h \neq k, k+1\,.$$

Since standard tableaux increase along diagonals these equalities force T_i to be identical with T_r except for the positions of k and $k+1$. In other words the non-vanishing of $\Pi e(T_i)$ forces $T_i = T_r$ or $T_i = T_s$. This proves the first case of 4.8.

Note next that, since there is no occurrence of m_k and m_{k+1} in the right hand side of 4.6, it follows from Theorem 3.1 that

$$\Pi\, s_k \;=\; s_k\,\Pi \tag{4.9}$$

This given, we get

$$\begin{aligned} s_k\, e(T_r) &= s_k\, \Pi\, e(T_r) \\ \text{(by 4.9)} &= \Pi\, s_k e(T_r) \\ \text{(by 4.5)} &= h_\mu \sum_{i=1}^{n_\mu} e^\mu_{ri}(s_k)\, \Pi\, e^\mu_{ir} \\ \text{(by 2.22)} &= h_\mu \sum_{i=1}^{n_\mu} e^\mu_{ri}(s_k)\, \Pi\, e^\mu_{ii}\, e^\mu_{ir} \\ \text{(by 4.9)} &= h_\mu\, e^\mu_{rr}(s_k)\, e^\mu_{rr} \;+\; h_\mu\, e^\mu_{rs}(s_k)\, e^\mu_{sr}\,. \end{aligned}$$

This may be rewritten as

$$s_k\, e^\mu_{rr} \;=\; a\, e^\mu_{rr} \;+\; b\, e^\mu_{sr}\,, \tag{4.10}$$

where a and b are constants we shall soon determine.

To begin we multiply both sides of 4.10 by the Murphy element m_k and get (using 3.2 c))

$$(s_k\, m_{k+1} - 1)\, e^\mu_{rr} \;=\; a\, m_k\, e^\mu_{rr} \;+\; b\, m_k\, e^\mu_{sr}\,,$$

and 3.12 a) gives

$$c_{T_r}(k+1)\, s_k\, e^\mu_{rr} - e^\mu_{rr} \;=\; a\, c_{T_r}(k)\, e^\mu_{rr} \;+\; b\, c_{T_r}(k+1)\, e^\mu_{sr}\,,$$

Subtracting from this 4.10 multiplied by $c_{T_r}(k+1)$ yields

$$-e^\mu_{rr} \;=\; a\,(c_{T_r}(k) - c_{T_r}(k+1))\, e^\mu_{rr}\,.$$

Thus

$$a = \frac{1}{c_{T_r}(k+1) - c_{T_r}(k)}\,. \tag{4.11}$$

Our next step is to multiply 4.10 by s_k. This gives

$$\begin{aligned} e^\mu_{rr} &= a\, s_k\, e^\mu_{rr} \;+\; b\, s_k\, e^\mu_{sr}\,. \\ \text{(by 4.10)} &= a(a\, e^\mu_{rr} \;+\; b\, e^\mu_{sr}\,) \;+\; b\, s_k\, e^\mu_{sr}\,. \\ &= a^2\, e^\mu_{rr} \;+\; ab\, e^\mu_{sr} \;+\; b\, s_k\, e^\mu_{sr}\,. \end{aligned} \tag{4.12}$$

Note that $c_{T_r}(k+1)-c_{T_r}(k)=1$ forces k and $k+1$ to be in the same row of T_r and $c_{T_r}(k+1)-c_{T_r}(k)=-1$ forces k and $k+1$ to be in the same column. Since these two alternatives have been excluded we can't have $a^2=1$. But then 4.12 shows that we can't have $b=0$. This allows us to extract $s_k\, e^{\mu}_{sr}$ out of 4.12 and obtain

$$s_k\, e^{\mu}_{sr} \;=\; \tfrac{(1-a^2)}{b}\, e^{\mu}_{rr} \;-\; a\, e^{\mu}_{sr}\,. \qquad 4.13$$

To determine b we need a further consequence of Theorem 1.3 which is quite interesting in its own right. This may be stated as follows.

Proposition 4.1

For two standard tableaux T_1 T_2 of the same shape we have

$$\begin{array}{ll} a) & E(T_2)\, e(T_1) \neq 0 \;\Longrightarrow\; T_2 <_{LL} T_1\,, \\ b) & e(T_1)\, E(T_2) \neq 0 \;\Longrightarrow\; T_1 <_{LL} T_2\,. \end{array} \qquad 4.12$$

Proof

Let $k+1$ be the last letter of disagreement between T_1 and T_2. In view of the recursive definition of the seminormal unit $e(T_2)$ given in 2.16 we may write $e(T_1)$ in the form

$$e(T_1) \;=\; e\big(T_1(k-1)\big)P\big(T_1(k)\big)\, R_1 \qquad 4.13$$

where R_1 is a residual factor of no concern. In the same vein, using 1.13 b) we may write

$$E(T_2) \;=\; P(T_2)\, n_k(T_2) N\big(T_2(k)\big)\,. \qquad 4.14$$

Using 4.13 and 4.14 we see that

$$E(T_2)\, e(T_1) \neq 0 \;\Longrightarrow\; N\big(T_2(k)\big)e\big(T_1(k-1)\big)P\big(T_1(k)\big) \neq 0\,.$$

Thus from Proposition 1.1 it follows that

$$E(T_2)\, e(T_1) \neq 0 \;\Longrightarrow\; \lambda\big(T_2(k)\big) \;\geq\; \lambda\big(T_1(k)\big) \qquad \text{(in dominance)}$$

and 4.12 a) necessarily follows from the definition of Young's Last Letter Order. The proof of 4.12 b) is entirely analogous.

As a corollary of 4.12 we can now immediatly derive the following surprising result.

Proposition 4.1

The constant b appearing in 4.10 and 4.13 is plainly and simply equal to 1.

Proof

Note that since $e^{\mu}_{sr}=e^{\mu}_{ss}e^{\mu}_{sr}e^{\mu}_{rr}$ we may write

$$\begin{aligned} h_\mu\, e^{\mu}_{sr} &= e(\overline{T}_s)\frac{E(T_s)}{h_\mu}e(\overline{T}_s)\Big(e(\overline{T}_s)E(T_s)\sigma^{\mu}_{sr}e(\overline{T}_r)\Big)e(\overline{T}_r)\frac{E(T_r)}{h_\mu}e(\overline{T}_r) \\ &= e(\overline{T}_s)\frac{E(T_s)}{h_\mu}\Big(e(\overline{T}_s)E(T_s)\sigma^{\mu}_{sr}e(\overline{T}_r)\Big)\frac{E(T_r)}{h_\mu}e(\overline{T}_r) \\ &= e(T_s)E(T_s)\sigma^{\mu}_{sr}e(T_r) \end{aligned}$$

and since by assumption we have $s_k T_r = T_s$ we can set $\sigma^\mu_{sr} = s_k$ in this last identity and obtain

$$\begin{aligned} h_\mu \, e^\mu_{sr} &= e(T_s)E(T_s)s_k \, e(T_r) \\ \text{(by 4.10)} &= e(T_s)E(T_s)\left(a \, e^\mu_{rr} + b \, e^\mu_{sr}\right) \\ &= a \, e(T_s)E(T_s)e(T_r) \ + \ b \, e(T_s)E(T_s)e^\mu_{sr} \\ &= a \, e(T_s)E(T_s)e(T_r) \ + \ b \, e(T_s)E(T_s)\Big(e(T_s)\frac{E(T_s)}{h_\mu}s_k \, e(T_r)\Big) \\ \text{(Using 2.17 c))} &= a \, e(T_s)E(T_s)e(T_r) \ + \ b \, e(T_s)E(T_s)s_k \, e(T_r) \\ &= a \, e(T_s)E(T_s)e(T_r) \ + \ b \, h_\mu \, e^\mu_{sr} \, . \end{aligned} \qquad 4.15$$

Now note that we cannot have

$$E(T_s)e(T_r) \neq 0$$

for otherwise our Proposition 4.1 would give $s < r$ which contraddicts our original assumptions. Thus 4.15 necessarily reduces to

$$h_\mu \, e^\mu_{sr} \ = \ b \, h_\mu \, e^\mu_{sr}$$

which forces

$$b \ = \ 1$$

and completes our proof.

To continue our construction of the seminormal matrix corresponding to the simple transposition s_k we need to compute the image by s_k of the idempotent $e(T_r)$ when k and $k+1$ are in the same row or column. Our point of departure, also in these cases is formula 4.5. Moreover, since now the tableau $s_k \, T_r$ is no longer standard, from 4.6 we derive that

$$\Pi \, e(T_i) \ = \ \begin{cases} 0 & \text{if } i \neq r, \\ e(T_r) & \text{if } i = r. \end{cases} \qquad 4.16$$

This given, multiplying both sides of 4.5 by Π we derive that

$$\begin{aligned} s_k e(T_r) \ = \ s_k \Pi \, e(T_r) \ = \ \Pi \, s_k e(T_r) &= h_\mu \sum_{i=1}^{n_\mu} e^\mu_{ri}(s_k) \, \Pi \, e^\mu_{ir} \\ &= h_\mu \sum_{i=1}^{n_\mu} e^\mu_{ri}(s_k) \, \Pi \, e^\mu_{ii} \, e^\mu_{ir} \\ \text{(by 4.16)} &= h_\mu \, e^\mu_{rr}(s_k) e^\mu_{rr} \\ &= a \, e(T_r) \, . \end{aligned} \qquad 4.17$$

with

$$a = h_\mu \, e^\mu_{rr}(s_k) \, . \qquad 4.18$$

To determine a we multiply both sides of 4.18 by the Murphy element m_k and use 3.2 c) to get

$$\begin{aligned} a\, c_{T_r}(k)\, e^{\mu}_{rr} &= m_k\, s_k e^{\mu}_{rr} \\ &= s_k m_{k+1}\, e(T_r) \,-\, e(T_r) \\ &= c_{T_r}(k+1)\, s_k\, e(T_r) \,-\, e(T_r) \\ \text{(by 4.18)} &= c_{T_r}(k+1)\, a\, e(T_r) \,-\, e(T_r)\,, \end{aligned}$$

and this gives

$$a \;=\; \frac{1}{c_{T_r}(k+1) - c_{T_r}(k)} \;=\; \begin{cases} \;\;1 & \text{if } k \text{ and } k+1 \text{ are in the same row of } T_r\,, \\ \\ -1 & \text{if } k \text{ and } k+1 \text{ are in the same column.} \end{cases} \qquad 4.19$$

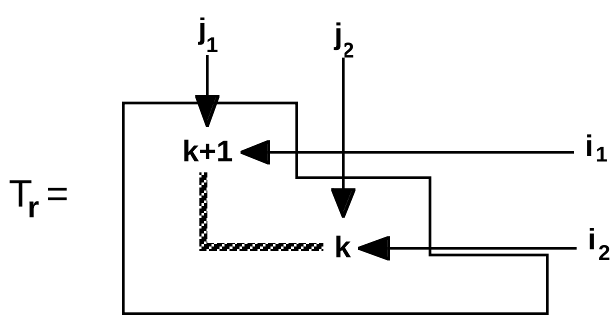
To state our final result in a compact form we need one further observation concerning the case when k and $k+1$ are not in the same row or column of T_r. Then if $k+1$ is in cell (i_1, j_1) and k is in cell (i_2, j_2) we may write

$$\begin{aligned} c_{T_r}(k) - c_{T_r}(k+1) &= (j_2 - i_2) \;-\; (j_1 - i_1) \\ &= j_2 - j_1 \,+\, i_1 - i_2 \end{aligned}$$

When $k+1$ is in a higher cell than k in T_r, as it was assumed at the beginning of this section, then the quantity

$$\pi_k(T_r) \;=\; c_{T_r}(k) - c_{T_r}(k+1) \;=\; -1/a \qquad 4.20$$

gives the "*taxi-cab distance*" between k and $k+1$. That is the length of any path that joins $k+1$ to k by EAST and SOUTH steps. See figure below where we have drawn such a path by a dashed line.



This given, the identities proved in this section yield the following fundamental result

Theorem 4.2

Let

$$T_1^\mu \ , \ T_2^\mu \ , \ \ldots \ , \ T_{n_\mu}^\mu$$

be the standard tableaux of shape $\mu \vdash n$ in Young's Last Letter Order, then for any pair $1 \le r, i \le n_\mu$ and $1 \le k < n$ we have

$$s_k\, e_{ru}^\mu \;=\; \begin{cases} e_{ru}^\mu & \text{if } k \text{ and } k+1 \text{ are in the same row of } T_r^\mu \\ -e_{ru}^\mu & \text{if } k \text{ and } k+1 \text{ are in the same column of } T_r^\mu \\ -\dfrac{1}{\pi_k(T_r^\mu)}\, e_{ru}^\mu + e_{su}^\mu & \text{if } T_s^\mu = s_k T_r^\mu \text{ and } k+1 \text{ is higher than } k \text{ in } T_r^\mu \\ \dfrac{1}{\pi_k(T_r^\mu)}\, e_{ru}^\mu + \Big(1 - \dfrac{1}{\pi_k^2(T_r^\mu)}\Big)\, e_{su}^\mu & \text{if } T_s^\mu = s_k T_r^\mu \text{ and } k \text{ is higher than } k+1 \text{ in } T_r^\mu \end{cases} \qquad 4.21$$

Proof

Given 4.19, the first two cases are simply a restatement of 4.17 right multiplied by e_{ru}^μ. Note next that if we combine 4.10, 4.11, 4.20 and Proposition 4.1 we obtain

$$s_k\, e_{rr}^\mu \;=\; -\frac{1}{\pi_k(T_r^\mu)}\, e_{rr}^\mu \;+\; e_{sr}^\mu \qquad 4.22$$

This gives the third case after right multiplication by e_{ru}^μ. Similarly, 4.13 gives

$$s_k\, e_{sr}^\mu \;=\; \Big(1 - \frac{1}{\pi_k^2(T_r^\mu)}\Big)\, e_{rr}^\mu \;+\; \frac{1}{\pi_k(T_r^\mu)}\, e_{sr}^\mu \qquad 4.23$$

Since $\pi_k(T_r^\mu) = \pi_k(T_s^\mu)$, the fourth case of 4.21 is obtained by an interchange of r and s followed by right multiplication by e_{su}^μ. This completes the proof.

From 4.21 we obtain a rather simple recipe for constructing the seminormal representation matrix $A^\mu(s_k)$. More precisely we have

Theorem 4.3

If $A^\mu(s_k)$ is the matrix yielding the action of the simple transposition $s_k = (k, k+1)$ on the basis $\langle e_{1,u}^\mu, e_{2,u}^\mu, \ldots, e_{n_\mu,u}^\mu \rangle$, that is

$$s_k \langle e_{1,u}^\mu, e_{2,u}^\mu, \ldots, e_{n_\mu,u}^\mu \rangle \;=\; \langle e_{1,u}^\mu, e_{2,u}^\mu, \ldots, e_{n_\mu,u}^\mu \rangle A^\mu(s_k) \qquad 4.24$$

then $A^\mu(s_k) = \| a_{ij}^\mu(s_k) \|_{i,j=1}^{n_\mu}$ with

$$a_{rr}(s_k) \;=\; \begin{cases} \dfrac{-1}{\pi_k(T_r^\mu)} & \text{if } k+1 \text{ is higher than } k \text{ in } T_r^\mu, \\ \dfrac{1}{\pi_k(T_r^\mu)} & \text{if } k \text{ is higher than } k+1 \text{ in } T_r^\mu, \end{cases} \qquad 4.25$$

where $\pi_k(T_r^\mu)$ is the taxi-cab distance between k and $k+1$ in T_r^μ. Moreover for $i \neq j$ we have

$$a_{ij}(s_k) \;=\; \begin{cases} 0 & \text{if } s_kT_i^\mu \neq T_j^\mu, \\[2ex] \dfrac{1}{1-\pi_k^2(T_i^\mu)} & \text{if } s_kT_i^\mu = T_j^\mu \text{ and } i<j, \\[2ex] 1 & \text{if } s_kT_i^\mu = T_j^\mu \text{ and } i>j. \end{cases} \qquad 4.26$$

Proof

The equation in 4.23 simply says that for each $1 \leq j \leq n_\mu$ we have

$$s_k e_{ju}^\mu \;=\; \sum_{i=1}^{n_\mu} e_{iu}^\mu \, a_{ij}^\mu(s_k)$$

Thus the identities in 2.25 and 2.26 are obtained by equating coefficients of the units e_{iu}^μ on both sides of 4.21.

We are now in a position to obtain explicit expressions for the factors d_i^μ occurring in 2.39. Our starting point is the following auxiliary identity.

Proposition 4.2

Let $T_s^\mu = s_k\, T_r^\mu$ with $r<s$ then

$$d_s^\mu/d_r^\mu \;=\; \big(1 - 1/\pi_k^2(T_r)\big)\,. \qquad 4.27$$

Proof

Equating coefficients of the identity in 4.23 and using 2.26 gives

$$s_k\, e_{sr}^\mu\,\Big|_\epsilon \;=\; \Big(1 - \frac{1}{\pi_k^2(T_r)}\Big)\frac{1}{h_\mu}\,. \qquad 4.28$$

On the other hand we have

$$\begin{aligned} s_k\, e_{sr}^\mu\,\Big|_\epsilon \;&=\; \downarrow\!(s_k\, e_{sr}^\mu)\,\Big|_\epsilon \\ &=\; (\downarrow\! e_{sr}^\mu)\, s_k\,\Big|_\epsilon \\ (\text{ by 2.39})\quad &=\; \frac{d_s^\mu}{d_r^\mu}\; e_{rs}^\mu\, s_k\,\Big|_\epsilon \\ &=\; \frac{d_s^\mu}{d_r^\mu}\; s_k\, e_{rs}^\mu\,\Big|_\epsilon \\ (\text{ by 4.22 right multiplied by } e_{rs}^\mu)\quad &=\; \frac{d_s^\mu}{d_r^\mu}\big(-1/\pi_k(T_r^\mu)\, e_{rs}^\mu \;+\; e_{ss}^\mu\big)\Big|_\epsilon \;=\; \frac{d_s^\mu}{d_r^\mu}\frac{1}{h_\mu}\,. \end{aligned} \qquad 4.29$$

and 4.27 follows by combining 4.28 and 4.29.

To complete the construction of these factors we need one further result.

Proposition 4.3

For any λ and any $1 \leq s \leq n_\lambda$ we can join T_1^λ to T_s^λ by a chain of tableaux

$$T_1^\lambda = T_{i_1}^\lambda \rightarrow T_{i_2}^\lambda \rightarrow \cdots \rightarrow T_{i_{m-1}}^\lambda \rightarrow T_{i_m}^\lambda = T_s^\lambda \qquad 4.30$$

with the following two basic properties

$$\begin{aligned}&\text{(a) } T_{i_{r-1}}^\lambda <_{LL} T_{i_r}^\lambda \text{ for all } 1 \leq r \leq m-1,\\ &\text{(b) } T_{i_r}^\lambda = s_{k_r} T_{i_{r-1}}^\lambda \text{ with } s_{k_r} = (k_r, k_r+1).\end{aligned} \qquad 4.31$$

Proof

The assertion is trivial for $\lambda \vdash 2$. So, by induction, let us assume we have proved the result for any $\mu \vdash n-1$. Let then $\lambda \vdash n$ and $1 \leq s \leq n_\lambda$. To construct the chain in 4.30 we need some notation. To begin let c_s be the lattice cell that contains n in T_s^λ and let c_1 be the lattice cell that contains n in T_1^λ. We should note that c_1 is necessarily be the highest corner of the diagram of λ. Now set $\lambda(\overline{T}_s^\lambda) = \mu$. Using the induction hypothesis we construct a chain for $\overline{T}_s^\lambda$:

$$T_1^\mu = T_{j_1}^\mu \rightarrow T_{j_2}^\mu \rightarrow \cdots \rightarrow T_{j_{m-1}}^\mu \rightarrow T_{j_m}^\mu = \overline{T}_s^\lambda \,. \qquad 4.32$$

This given, let $T_{i_r}^\lambda$, for $1 \leq r \leq m$, be the tableau obtained by adding n to $T_{j_r}^\mu$ in the cell c_s. If it happens that $c_1 = c_s$ then the tableaux $T_{i_r}^\lambda$ will be satisfy 4.30 and 4.31 and we are done. If c_s is a lower corner of the diagram of λ then the chain

$$T_{i_1}^\lambda \rightarrow T_{i_2}^\lambda \rightarrow \cdots \rightarrow T_{i_{m-1}}^\lambda \rightarrow T_{i_m}^\lambda = \overline{T}_s^\lambda \,. \qquad 4.33$$

will satisfy 4.31 a) and b). Now we only need to construct a chain that joins T_1^λ to $T_{i_1}^\lambda$. The crucial observation is that when $c_s \neq c_1$ then c_1 is both the highest corner of the diagrams of μ and λ. Thus in T_1^μ the label $n-1$ is necessarily in c_1. In particular $T_{i_1}^\lambda$ has n in c_s and $n-1$ in c_1. This given, the tableau $s_{n-1}T_{i_1}^\lambda$ will have n in c_1 and $n-1$ in c_s. Thus

$$s_{n-1}T_{i_1}^\lambda \;<_{LL}\; T_{i_1}^\lambda \,.$$

This reduces us to the previous case since now n is the highest corner of λ. Thus the constrution of the desired chain can now be carried out by prepending the chain in 4.33 with the chain that joins T_1^λ to $s_{n-1}T_{i_1}^\lambda$. This completes the induction and the proof.

Combining the last two propositions we obtain

Theorem 4.4

If

$$T_1^\lambda = T_{i_1}^\lambda \rightarrow T_{i_2}^\lambda \rightarrow \cdots \rightarrow T_{i_{m-1}}^\lambda \rightarrow T_{i_m}^\lambda = T_s^\lambda$$

is any chain satisfying 4.31 a) and b) then

$$d_s^\lambda \;=\; \prod_{r=1}^{m-1} \big(1 - \pi_{k_r}^2(T_{i_r}^\mu)\big) \qquad 4.34$$

Proof

Clearly, 4.34 follows by succesive applications of 4.27.

5. Murphy elements and conjugacy classes

We should point out that the name "*Murphy elements*" may be a bit unfair. These elements first appeared in a 1971 paper of A. Jucys. In a 1980 paper Murphy's rediscovers these elements (presumably) independently of Jucys work. Murphy's fundamental contribution is discovering their connection with Young's seminormal units. However, in a remarkable 1972 paper Jucys goes on to show that that every class function of S_n may be expressed as a symmetric polynomial in $m_2, m_3, \ldots, m_n$. Jucys' development was basically existential. The purpose of this section is to reestablish Jucys' results in a constructive manner and obtain explicit expressions for the characters of S_n as well as the conjugacy classes. We should mention that such a constructive approach was adopted in a joint paper by P. Diaconis and C. Greene [1] where a number of explicit formulas were derived in special cases. We also solve here a number of problems posed in the Diaconis-Greene paper.

To begin we shall derive two separate proofs of the following basic fact.

Theorem 5.1

If $Q(y_2, y_3, \ldots, y_n)$ is a symmetric polynomial in its arguments then the group algebra element

$$Q(m_2, m_3, \ldots, m_n) \tag{5.1}$$

is a class function of S_n.

1st Proof

The result follows if we prove that

$$\sigma\, Q(m_2, m_3, \ldots, m_n)\, \sigma^{-1} \;=\; Q(m_2, m_3, \ldots, m_n) \qquad (\forall\, \sigma \in S_n) \tag{5.2}$$

Since the simple transpositions $s_1, s_2, \ldots, s_{n-1}$ generate S_n we need only check the identities

$$s_k\, Q(m_2, m_3, \ldots, m_n)\, s_k \;=\; Q(m_2, m_3, \ldots, m_n) \qquad (\forall\, k = 1, 2, \ldots, n-1) \tag{5.3}$$

Moreover, since every symmetric polynomial in $y_2, y_3, \ldots, y_n$ is a polynomial in the elementary symmetric functions

$$e_1(y_2, y_3, \ldots, y_n)\,,\;\; e_2(y_2, y_3, \ldots, y_n)\,, \ldots,\; e_{n-1}(y_2, y_3, \ldots, y_n)\,, \tag{5.4}$$

and products of class functions are class function we need only verify 5.2 when Q is one of the elementaries in 5.4.

Now we can do this all at once by showing 5.3 for

$$Q(y_2, y_3, \ldots, y_n) \;=\; \sum_{r=1}^{n-1} t^r\, e_r(y_2, y_3, \ldots, y_n) \;=\; \prod_{h=2}^{n} (1 + t\, y_h)$$

To this end note that the relations in 3.2 b) give us

$$s_k\Big(\prod_{h=2}^{n}(1+t\,m_h)\Big)\,s_k \;=\; \Big(\prod_{\substack{h=2\\ h\neq k,k-1}}^{n}(1+t\,m_h)\Big)\,s_k\,(1+t\,m_k)\,(1+t\,m_{k+1})s_k\,. \qquad 5.5$$

But from 3.2 c) and d) we then obtain

$$\begin{aligned} s_k\,(1+t\,m_k)\,(1+t\,m_{k+1})s_k &= s_k\,(1+t\,m_k)\,s_k\;s_k\,(1+t\,m_{k+1})s_k\\ &= (1+t\,m_{k+1}-t\,s_k)\;(1+t\,m_k+t\,s_k)\\ &= (1+t\,m_{k+1}-t\,s_k+t\,m_k+t\,s_k+t^2(m_{k+1}-s_k)(m_k+s_k)\\ &= (1+t\,m_{k+1}+t\,m_k+t^2(m_{k+1}m_k+m_{k+1}s_k-s_k\,m_k-1) \end{aligned} \qquad 5.6$$

Now note that right multiplication of 3.2 c) by s_k also gives

$$m_{k+1}s_k-s_k\,m_k-1 \;=\; 0$$

Substituting this in 5.6 reduces it to

$$\begin{aligned} s_k\,(1+t\,m_k)\,(1+t\,m_{k+1})s_k &= (1+t\,m_{k+1}+t\,m_k+t^2(m_{k+1}m_k)\\ &= (1+t\,m_k)\,(1+t\,m_{k+1}) \end{aligned}$$

Using this relation in 5.5 finally yields the desired identity

$$s_k\Big(\prod_{h=2}^{n}(1+t\,m_h)\Big)\,s_k \;=\; \Big(\prod_{h=2}^{n}(1+t\,m_h)\Big)$$

completing our first proof.

We have seen that the seminormal units $e_(T)$ may be expressed as polynomials in the Murphy elements. We need to do the reverse here and express the Murphy elements in terms of the seminormal units. More precisely

Theorem 5.2

For $k\leq n$ we have

$$m_k \;=\; \sum_{T\,\in ST(n)} c_T(k)\,e(T) \qquad 5.7$$

where the symbol "$T\in ST(n)$" is to indicate that the sum is to be carried out over all standard tableaux with n cells.

Proof

Formula 2.4 with $f=m_k$ gives

$$m_k \;=\; \sum_{\lambda\vdash n} h_\lambda \sum_{i,j=1}^{n_\lambda} m_k\,e^\lambda_{ji}\,\Big|_\epsilon\,e^\lambda_{ij} \qquad 5.8$$

However from the relations in 2.22 we derive that

$$\begin{aligned} m_k\, e^\lambda_{ji} &= m_k\, e^\lambda_{jj} e^\lambda_{ji} \\ \text{(by 3.12 a))} &= c_{T^\lambda_j}(k)\, e^\lambda_{jj} e^\lambda_{ji} \;=\; c_{T^\lambda_j}(k)\, e^\lambda_{ji}\,, \end{aligned}$$

and 2.26 gives

$$m_k\, e^\lambda_{ji}\Big|_\epsilon \;=\; c_{T^\lambda_j}(k)\, e^\lambda_{ji}\Big|_\epsilon \;=\; \begin{cases} 0 & \text{if } i \neq j \\ c_{T^\lambda_j}(k)\,/h_\lambda & \text{if } i = j \end{cases}$$

Using this in 5.8 reduces it to

$$m_k \;=\; \sum_{\lambda\vdash n} \sum_{j=1}^{n_\lambda} c_{T^\lambda_j}(k)\, e^\lambda_{jj}\,.$$

This completes our proof since this identity is simply another way of writing 5.7.

We are now ready to give our

2$^{\text{nd}}$ Proof of Theorem 5.1.

From 5.7 and the orthogonality relations in 2.20 we derive that

$$m_{k_1} m_{k_2} \cdots m_{k_l} \;=\; \sum_{T\,\in St(n)} c_T(k_1)\, c_T(k_2)\, \cdots\, c_T(k_l)\, e(T)\,.$$

It thus follows that for any polynomial $Q(y_2, y_2, \ldots, y_n)$ we must also have

$$Q(m_2 m_3 \cdots m_n) \;=\; \sum_{T\,\in St(n)} Q\big(c_T(2), c_T(3), \cdots, c_T(n)\big)\, e(T)\,. \qquad 5.9$$

Now if $Q(y_2, y_2, \ldots, y_n)$ is a symmetric function in its arguments, the order in which the quantities $c_T(2), c_T(3), \cdots, c_T(n)$ are substituted in $Q(y_2, y_2, \ldots, y_n)$ is immaterial and by grouping terms where $Q\big(c_T(2), c_T(3), \cdots, c_T(n)\big)$ takes the same value, formula 5.9 may be rewritten in the form

$$Q(m_2 m_3 \cdots m_n) \;=\; \sum_{\lambda\vdash n} Q\big(c_\lambda(2), c_\lambda(3), \cdots, c_\lambda(n)\big) \sum_{T\in ST(\lambda)} e(T)\,. \qquad 5.10$$

where the quantities $c_\lambda(2), c_\lambda(3), \cdots, c_\lambda(n)$ represent the contents of the cells of the diagram of λ in some preferred order and the symbol "$T \in ST(\lambda)$" represents that the sum is to be carried out over all standard tableaux of shape λ. Now 5.10 would complete the proof once we realize that the summand

$$U^\lambda \;=\; \sum_{T\in ST(\lambda)} e(T) \;=\; \sum_{i=1}^{n_\lambda} e^\lambda_{ii} \qquad 5.11$$

is none other than the character χ^λ. However, all we need here is to show that the U^λ are class functions, and for that we need only verify that they commute with all the seminormal units e^μ_{rs}. However this is an immediate consequence of the relations in 2.22.

To show the converse of Theorem 5.1 we need only show that the irreducible characters χ^λ may be expressed as symmetric polynomials in the Murphy elements. To see where this leads us, suppose that for each partition λ we have a symmetric polynomial

$$Q_\lambda(x_2, x_3, \ldots, x_n) \tag{5.12}$$

such that

$$\chi^\lambda \;=\; Q_\lambda(m_2, m_3, \ldots, m_n) \tag{5.13}$$

Now it follows immediately from Proposition 2.2 and 2.33 that

$$\chi^\lambda \;=\; h_\lambda \sum_{i=1}^{n_\lambda} e_{ii}^\lambda\,. \tag{5.14}$$

Using the identities in 2.22 we then immediately derive that

$$\chi^\lambda\,\chi^\mu \;=\; \begin{cases} 0 & \text{if } \mu \neq \lambda, \\ h_\lambda\,\chi^\lambda & \text{if } \mu = \lambda\ . \end{cases} \tag{5.15}$$

These are the well known orthogonality relations satisfied by the irreducible characters. Combining 5.13 with 5.15 we derive that we must have

$$Q_\lambda(m_2, m_3, \ldots, m_n)\,\chi^\mu \;=\; \begin{cases} 0 & \text{if } \mu \neq \lambda\ , \\ h_\lambda\,\chi^\lambda & \text{if } \mu = \lambda\ . \end{cases} \tag{5.16}$$

However, from 3.12 a) and the symmetry of Q_λ it follows (as in the 2^{nd} proof of Theorem 5.1) that

$$Q_\lambda(m_2, m_3, \ldots, m_n)\,e_{ii}^\mu \;=\; Q_\lambda(c_\mu(2), c_\mu(3), \ldots, c_\mu(n))\,e_{ii}^\mu$$

and thus we also have

$$Q_\lambda(m_2, m_3, \ldots, m_n)\,\chi^\mu \;=\; Q_\lambda(c_\mu(2), c_\mu(3), \ldots, c_\mu(n))\,\chi^\mu \tag{5.17}$$

Comparing with 5.16 we derive that the polynomial Q_λ must satisfy the identities

$$Q_\lambda(c_\mu(2), c_\mu(3), \ldots, c_\mu(n)) \;=\; \begin{cases} 0 & \text{if } \mu \neq \lambda\ , \\ h_\lambda & \text{if } \mu = \lambda\ . \end{cases} \tag{5.17}$$

Experimenting with special cases shows that these equations do not uniquely determine Q_λ. Nevertheless, all we need here is a systematic way of constructing one particular solution for each λ. Now it develops that we may in fact, produce Q_λ in terms of a symmetric polynomial which first appears in an exercise of Macdonald (see [4] ex. 5, 6 and 7 p. 117).

Let us recall that the symbol $(x)_a$ represents the "*lower factorial polynomial*" that is

$$(x)_a \;=\; x(x-1)(x-2)\cdots(x-a+1)$$

This given, our starting point is the following remarkable result.

Proposition 5.1

For a given integer vector $a = (a_1 > a_2 > \ldots > a_n \geq 0)$ *set*

$$\Xi_a(y) = \Xi_a(y_1, y_2, \ldots, y_n) \;=\; \frac{det\big\|(y_i)_{a_j}\big\|_{1\leq i,j\leq n}}{\prod_{1\leq i<j\leq n}(y_i - y_j)}\,. \qquad 5.18$$

then for all $b = (b_1 > b_2 > \ldots > b_n \geq 0)$ *we have*

$$\Xi_a[\,b\,] \;=\; \begin{cases} \dfrac{a_1!a_2!\cdots a_n!}{\prod_{1\leq i<j\leq n}(a_i - a_j)} & \text{if } \; b = a\,, \\ \\ 0 & \text{if } \; |b| \leq |a| \;\;\&\;\; b \neq a\,.\;(\dagger) \end{cases} \qquad 5.19$$

Proof

By the definition of determinant we have

$$det\big\|(y_i)_{a_j}\big\|_{1\leq i,j\leq n} \;=\; \sum_{\sigma\in S_n} sign(\sigma)\,(y_1)_{a_{\sigma_1}}(y_2)_{a_{\sigma_2}}\cdots(y_n)_{a_{\sigma_n}} \qquad 5.20$$

Now note that for an integer x we have

$$(x)_a \;=\; \frac{x!}{(x-a)!}$$

Thus with the replacements $y_i \to b_i$ we may rewrite 5.20 in the form

$$det\big\|(b_i)_{a_j}\big\|_{1\leq i,j\leq n} \;=\; b_1!b_2!\cdots b_n! \sum_{\sigma\in S_n} sign(\sigma)\,\frac{1}{(b_1 - a_{\sigma_1})!(b_2 - a_{\sigma_2})!\cdots(b_n - a_{\sigma_n})!} \qquad 5.21$$

Now for the summand corresponding to σ to survive we must have

$$b_1 \geq a_{\sigma_1}\;,\; b_2 \geq a_{\sigma_2}\;,\;\ldots\;,\; b_n \geq a_{\sigma_n}\;,$$

and this gives

$$b_1 + b_2 + \cdots + b_n \geq a_1 + a_2 + \cdots + a_n$$

Thus when $|b| \leq |a|$ all terms in 5.20 do vanish unless $b = a$. But in this case only the identity term fails to vanish, and 5.21 reduces to

$$det\big\|(a_i)_{a_j}\big\|_{1\leq i,j\leq n} \;=\; a_1!a_2!\cdots a_n!\,.$$

This implies 5.19 precisely as stated.

(†) Recall that for a vector $y = (y_1, y_2, \ldots, y_n)$ we set $|y| = y_1 + y_2 + \cdots + y_n$

The identities in 5.19 strongly suggest that we should be able to use the polynomial $\Xi_a(y)$ in the construction of Q_λ. This is prescisely what we will do. However to carry this out we need some notation and a few auxiliary facts.

For a pair of integers $m, k \geq 0$ set

$$\Sigma_k(m) \; = \; \sum_{s=1}^{m} i^k \,, \tag{5.22}$$

with the understanding that

$$\Sigma_0(m) \; = \; m\,. \tag{5.23}$$

The following result is well known

Proposition 5.2

The sequence of polynomials $R_k(x)$ with generating function

$$\sum_{k\geq 0} \frac{u^k}{k!} R_k(x) \; = \; e^u \, \frac{e^{ux}-1}{e^u-1} \tag{5.24}$$

yields all the integer sums in 5.22. More precisely we have

$$\Sigma_k(m) = R_k(m) \qquad\qquad \forall\, k,m \geq 0 \tag{5.25}$$

Proof

We have

$$\begin{aligned}
\sum_{k\geq 0} \frac{u^k}{k!} \Sigma_k(m) \; &= \; \sum_{k\geq 0} \frac{u^k}{k!} \sum_{i=1}^{m} i^k \\
&= \; \sum_{i=1}^{m} \sum_{k\geq 0} \frac{(iu)^k}{k!} \; = \; \sum_{i=1}^{m} e^{i\,u} \\
&= \; e^u \; + \; e^{2\,u} \; + \; \cdots \; + \; e^{m\,u} \\
&= \; e^u \, \frac{e^{m\,u}-1}{e^u-1}\,.
\end{aligned}$$

This proves 5.25.

It will be good to see here a few of these polynomials. We should mention that they were computed with MAPLE directly from 5.24:

$$\begin{gathered}
R_1(x) = \frac{1}{2} x(x+1) \\
R_2(x) = \frac{1}{6} x(x+1)(2x+1) \\
R_3(x) = \frac{1}{4} x^2(x+1)^2 \\
R_4(x) = \frac{1}{30} x(2x+1)(x+1)(3x^2+3x-1)
\end{gathered}$$

$$R_5(x) = \frac{1}{12}x^2(2x^2 + +2x - 1)(x+1)^2$$
$$R_6(x) = \frac{1}{42}x(2x+1)(x+1)(3x^4 + 6x^3 - 3x + 1)$$

If $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0)$ is a partition we set $l(\lambda) = m$ and call it the "*length of* λ". This given if $l(\lambda) \leq n$ we set

$$\lambda(n) \;=\; \big(\lambda_1(n), \lambda_2(n), \ldots, \lambda_n(n)\big)$$

with

$$\lambda_i(n) \;=\; \begin{cases} \lambda_i + n - 1 & \text{if } 1 \leq i \leq l(\lambda) \\ n - i & \text{if } l(\lambda) < i \leq n \end{cases}$$

It will also be convenient here and after to denote by "$\mathcal{C}(\lambda)$" the sequence of contents of the diagram of a partition λ, in some order.

Recalling that we denote by $p_k(x_1, x_2, \ldots, x_n)$ the "power sum" symmetric function

$$p_k(x) \;=\; p_k(x_1, x_2, \ldots, x_n) \;=\; x_1^k + x_2^k + \cdots + x_n^k$$

we have the following basic fact

Theorem 5.3

The symmetric polynomial

$$\pi_{n,k}(x) = R_k(n-1) + \sum_{s=1}^{k} \binom{k}{s} \big(n^s - (n-1)^s\big) p_{k-s}(x) \qquad 5.26$$

yields the identities

$$p_k[\lambda(n)] = \pi_{n,k}[\mathcal{C}(\lambda)] \qquad \textit{for all } \lambda \vdash n \qquad 5.27$$

Proof

If $l(\lambda) < n$ set

$$\lambda_i \;=\; 0 \qquad \text{for all } i > l(\lambda)\,.$$

From the definition of contents we then get that

$$\begin{aligned} p_k[\mathcal{C}(\lambda)] \;&=\; \sum_{i=1}^{n} \sum_{j=1}^{\lambda_i} (j-i)^k \\ &=\; \sum_{i=1}^{n} \sum_{j=1}^{\lambda_i} \sum_{a=0}^{k} \binom{k}{a} (-i)^{k-a} j^a \quad . \qquad 5.28 \\ &=\; \sum_{i=1}^{n} \sum_{a=0}^{k} \binom{k}{a} (-i)^{k-a} R_a(\lambda_i) \end{aligned}$$

Note that this formula remains valid for $k=0$ provided we adopt the convention of setting

$$p_o[\mathcal{C}(\lambda)] \;=\; n \qquad \text{for all } \lambda \vdash n\,. \tag{5.29}$$

In fact, setting $k=0$, the last line of 5.28 reduces to

$$\sum_{i=1}^{n} R_o(\lambda_i)$$

and 5.23 gives

$$\sum_{i=1}^{n} R_o(\lambda_i) \;=\; \sum_{i=1}^{n} \lambda_i \;=\; n\,.$$

This given, we derive that

$$\begin{aligned}
\sum_{k\geq 0} \frac{p_k[\mathcal{C}(\lambda)]}{k!}\, u^k \;&=\; \sum_{i=1}^{n}\sum_{a\geq 0} \frac{R_a(\lambda_i)}{a!}\, u^a \sum_{k\geq a} \frac{(-i)^{k-a}}{(k-a)!}\, u^{k-a} \\
\text{(using 5.25)} \;&=\; \sum_{i=1}^{n} \frac{e^u}{e^u-1}\big(e^{\lambda_i\, u}-1\big)\, e^{-i\, u} \\
&=\; \frac{1}{1-e^{-u}} \sum_{i=1}^{n} \big(e^{(\lambda_i-i)\, u} - e^{-i\, u}\big) \\
&=\; \frac{e^{-n\, u}}{1-e^{-u}} \sum_{i=1}^{n} \big(e^{(\lambda_i+n-i)\, u} - e^{(n-i)u}\big) \\
&=\; \frac{e^{-n\, u}}{1-e^{-u}} \sum_{i=1}^{n}\sum_{k\geq 1} \big(\lambda_i^k(n) - (n-i)^k\big)\frac{u^k}{k!} \\
&=\; \frac{e^{-n\, u}}{1-e^{-u}} \sum_{k\geq 1} \big(p_k[\lambda(n)] - R_k(n-1)\big)\frac{u^k}{k!}\,.
\end{aligned} \tag{5.30}$$

Now this may be inverted to

$$\sum_{k\geq 1} p_k[\lambda(n)]\, \frac{u^k}{k!} \;=\; \sum_{k\geq 1} R_k(n-1)\big)\frac{u^k}{k!} \;+\; \big(e^{n\, u} - e^{(n-1)u}\big)\sum_{k\geq 0} p_k[\mathcal{C}(\lambda)]\frac{u^k}{k!}\,.$$

Equating coefficients of u^k in this equality gives

$$p_k[\lambda(n)] = R_k(n-1) + \sum_{s=1}^{k}\binom{k}{s}\big(n^s-(n-1)^s\big)p_{k-s}[\mathcal{C}(\lambda)]\,.$$

and this proves 5.27 with $\pi_{n,k}$ given by 5.26, precisely as asserted.

This result has the following immediate corollary.

Theorem 5.4

Let P be a symmetric polynomial in $(x_1,\ldots,x_n)$ and let

$$P = H(p_1,p_2,\cdots,p_n) \;=\; \sum_\rho c_\rho\, p_\rho$$

be its expansion in terms of the power symmetric function basis. Then the symmetric polynomial

$$\pi_n P \;=\; H(p_1,p_2,\cdots,p_n)\Big|_{p_k\rightarrow \pi_{n,k}}$$

yields the identities

$$\big(\pi_n P\big)\big[\mathcal{C}(\lambda)\big] \;=\; P\big[\lambda(n)\big] \qquad\qquad \forall \lambda \vdash n$$

Now we need only one more fact to obtain our polynomial Q_λ, Namely the following classical formula.

Lemma 5.1

The number of standard tableaux of shape $\lambda \vdash n$ is given by the ratio

$$f_\lambda \;=\; \frac{n!}{\lambda_1(n)!\lambda_2(n)!\cdots\lambda_n(n)!}\prod_{1\leq i<j\leq n}(\lambda_i(n)-\lambda_j(n)) \qquad 5.31$$

In particular we get that

$$h_\lambda \;=\; \frac{n!}{f_\lambda} \;=\; \frac{\lambda_1(n)!\lambda_2(n)!\cdots\lambda_n(n)!}{\prod_{1\leq i<j\leq n}(\lambda_i(n)-\lambda_j(n))} \qquad 5.32$$

Proof

It is well known (see [4] p. 114) that for any $\rho \vdash n$ we have the Schur function expansion

$$p_\rho \;=\; \sum_{\lambda\vdash n}\chi^\lambda_\rho\, S_\lambda$$

since we also have

$$S_\lambda(x_1,\ldots,x_n) \;=\; \frac{det\,\big\|x_i^{\lambda_j+n-j}\big\|_{i,j=1}^n}{det\,\big\|x_i^{n-j}\big\|_{i,j=1}^n}$$

we see that

$$\chi^\lambda_\rho \;=\; p_\rho\, det\,\big\|x_i^{n-j}\big\|_{i,j=1}^n\Big|_{x_1^{\lambda_1(n)}x_2^{\lambda_2(n)}\cdots x_n^{\lambda_n(n)}}\,. \qquad 5.33$$

In the particular case that $\rho = 1^n$ this gives

$$\begin{aligned} f_\lambda &= \sum_{\sigma\in S_n} sign(\sigma) \sum_{\alpha_1+\alpha_2+\cdots+\alpha_n=n} \frac{n!}{\alpha_1!\alpha_2!\cdots\alpha_n!}\, x_1^{\alpha_1+n-\sigma_1} x_2^{\alpha_2+n-\sigma_2}\cdots x_n^{\alpha_n+n-\sigma_n}\Big|_{x_1^{\lambda_1(n)}x_2^{\lambda_2(n)}\cdots x_n^{\lambda_n(n)}} . \\ &= n! \sum_{\sigma\in S_n} sign(\sigma)\, \frac{1}{(\lambda_1(n)+\sigma_1-n)!(\lambda_2(n)+\sigma_2-n)!\cdots(\lambda_n(n)+\sigma_n-n)!} \\ &= \frac{n!}{\lambda_1(n)!\lambda_2(n)!\cdots\lambda_n(n)!} \sum_{\sigma\in S_n} sign(\sigma)\, \big(\lambda_1(n)\big)_{n-\sigma_1} \big(\lambda_2(n)\big)_{n-\sigma_2}\cdots\big(\lambda_n(n)\big)_{n-\sigma_n} \\ &= \frac{n!}{\lambda_1(n)!\lambda_2(n)!\cdots\lambda_n(n)!}\, det\left\|\big(\lambda_i(n)\big)_{n-j}\right\|_{i,j=1}^n \end{aligned} \qquad 5.34$$

This yields the desired formula 5.31 since the determinant in 5.34 can be reduced by simple column manipulations to the Vandermonde determinant evaluated at $\lambda_1(n), \lambda_2(n), \ldots, \lambda_n(n)$,.

We can now finally state the main result of this section

Theorem 5.5

For $\lambda \vdash n$ set

$$Q_\lambda[y_1, y_2, \ldots, y_n] \;=\; \pi_n\, \Xi_{\lambda(n)}\,[y_1, y_2, \ldots, y_n\,] \qquad 5.35$$

then for $\mu \vdash n$ we have

$$Q_\lambda[\mathcal{C}(\mu)] \;=\; \begin{cases} h_\lambda & \text{if } \mu = \lambda \\ 0 & \text{if } \mu \neq \lambda \end{cases} . \qquad 5.36$$

In particular we must also have

$$\chi^\lambda \;=\; Q_\lambda[0, m_2, \ldots, m_n] \qquad 5.37$$

Proof

Since the numerator in 5.18 is clearly an alternating polynomial in $y_1, y_2, \ldots, y_n$, the Vandermonde determinant factors out and the ratio evaluates to a symmetric polynomial. We can thus apply Theorem 5.4 and derive that

$$Q_\lambda[\mathcal{C}(\mu)] \;=\; \Xi_{\lambda(n)}\big[\mu(n)\big] .$$

Since

$$\sum_{i=1}^n \lambda_i(n) \;=\; \binom{n}{2} \;+\; n \;=\; \sum_{i=1}^n \mu_i(n)$$

we can apply Proposition 5.1 and derive that

$$Q_\lambda[\mathcal{C}(\mu)] \;=\; \begin{cases} \frac{\lambda_1(n)!\lambda_2(n)!\cdots\lambda_n(n)!}{\prod_{1\le i<j\le n}\big(\lambda_i(n)-\lambda_j(n)\big)} & \text{if } \mu \neq \lambda\ , \\ 0 & \text{if } \mu = \lambda\ . \end{cases}$$

and 5.36 immediately follows from the identity in 5.32.

Now note that combining 5.36 with 3.12 we get for any standard tableau T of shape μ

$$Q_\lambda[0,m_2,\ldots,m_n]\,e(T) \;=\; Q_\lambda(\mathcal{C}(\mu))\,e(T) \;=\; \begin{cases} h_\lambda\, e(T) & \text{if } \mu=\lambda \\ 0 & \text{if } \mu\neq\lambda\,. \end{cases} \qquad 5.38$$

Since

$$\chi^\mu \;=\; \sum_{\lambda(T)=\mu} e(T)$$

from 5.38 we get that

$$Q_\lambda[0,m_2,\ldots,m_n]\,\chi^\mu \;=\; \begin{cases} h_\lambda\,\chi^\lambda & \text{if } \mu=\lambda\ , \\ 0 & \text{if } \mu\neq\lambda\ . \end{cases}$$

Subtracting from this result the well known identities

$$\chi^\lambda\,\chi^\mu \;=\; \begin{cases} h_\lambda\,\chi^\lambda & \text{if } \mu=\lambda\ , \\ 0 & \text{if } \mu\neq\lambda\ . \end{cases}$$

yields

$$\Big(Q_\lambda[0,m_2,\ldots,m_n] \;-\; \chi^\lambda\Big)\,\chi^\mu \;=\; 0\,, \qquad \text{for all } \mu\vdash n\,. \qquad 5.39$$

Since Theorem 5.1 guarantees that the group algebra element

$$Q_\lambda[0,m_2,\ldots,m_n] \;-\; \chi^\lambda$$

is a class function, the identities in 5.39 are sufficient to guarantee that this difference must identically vanish. This completes our proof.

It develops that these polynomials are quite remarkable in their relative simplicity. To begin with, computer experimentation reveals that the polynomial $\Xi_{\lambda(n)}[y_1,y_2,\ldots,y_n]$ may be quite monstruous even for small partitions. Of course that should be expected. When expressed in terms of the power basis it may still take a few lines of print even for $\lambda\vdash 4$. However, surprisingly, the replacement $p_k\rightarrow\pi_{n,k}$ produces drammatic simplifications so that the resulting polynomials end up containing only a few terms. For instance for the partitions of 4 we obtain

$$\begin{aligned} Q_{1111} &= 1+4p_1-p_2/2+p_1^2/2-p_3 \\ Q_{211} &= -3-6p_1+3p_2/2-p_1^2/2+p_3 \\ Q_{22} &= 20++p_1-4p_2 \\ Q_{31} &= -3+6p_1+3p_2/2-p_1^2/2-p_3 \\ Q_{4} &= 1-4p_1-p_2/2+p_1^2/2+p_3 \end{aligned}$$

Even for partitions of 6 these polynomials contain only a few terms. For instance we get

$$Q_{321} \;=\; 161-22p_2-8p_1^2-p_2^2+6p_4\,.$$

Theorem 5.5 has two corollaries that are worth stating at this point.

Theorem 5.6

The polynomial

$$H_n[y_1, y_2, \ldots, y_n] \;=\; \sum_{\lambda \vdash n} Q_\lambda[y_1, y_2, \ldots, y_n]$$

satisfies the identities

$$H_n[\mathcal{C}(\mu)] \;=\; h_\mu \qquad \textit{for all } \mu \vdash n \tag{5.40}$$

Proof

This is an immediate consequence of 5.36.

Again these polynomials are surprisingly simple. For instance up to $n = 6$ we obtain

$$\begin{aligned} H_3(y) &= 1 + p_2 \\ H_4(y) &= 16 - 2p_2 + p_1^2 \\ H_5(y) &= p_4 - 6p_2 - p_1^2 + 46 \\ H_6(y) &= -9p_4 + 2p_2^2 + 14p_2 + 12p_1^2 + 11 \end{aligned}$$

Theorem 5.7

For $\rho \vdash n$ set

$$\mathcal{Z}_\rho[y_1, y_2, \ldots, y_n] \;=\; \sum_{\lambda \vdash n} Q_\lambda[y_1, y_2, \ldots, y_n]\, \chi^\lambda_\rho / z_\rho \tag{5.41}$$

where for $\rho = 1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}$ as customary we set

$$z_\rho = 1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}\; \alpha_1! \alpha_2! \cdots \alpha_n! \tag{5.42}$$

then

$$\mathcal{Z}_\rho[\mathcal{C}(\mu)] \;=\; h_\mu\, \chi^\mu_\rho / z_\rho \tag{5.43}$$

and the conjugacy class C_ρ (as an element of the group algebra of S_n) is given by the formula

$$C_\rho \;=\; \mathcal{Z}_\rho[0, m_2, \ldots, m_n] \tag{5.44}$$

Proof

The identity in 5.43 follows immediately from 5.36. Next, recall that the class function C_ρ, in terms of the characters, has the expansion

$$C_\rho \;=\; \sum_{\lambda \vdash n} \chi^\lambda\, \chi^\lambda_\rho / z_\rho$$

Thus 5.44 is an immediate consequence of 5.37.

We give below a few samples we obtained from formula 5.41

$$\mathcal{Z}_{[2,2]}(y) \;=\; \tfrac{1}{2}p_1^2 - \tfrac{3}{2}p_2$$

$$\mathcal{Z}_{[3,2]}(y) \;=\; 12p_1 + p_1p_2 - 4p_3$$

$$\mathcal{Z}_{[3,3]}(y) \;=\; -30 + 6p_2 + 3p_1^2 + 6p_2 + \tfrac{1}{2}p_2^2 - \tfrac{5}{2}p_4$$

In a recent paper [2] A. Goupil et al, endevour to express the central parameter

$$\omega_\rho^\lambda \;=\; \chi_\rho^\lambda\, h_\lambda / z_\rho \qquad 5.45$$

as polynomial depending on ρ evaluated at $C(\lambda)$. The results they obtain are interesting in the present context since their polynomials yield alternate versions of the polynomails $\mathcal{Z}_\rho$.

More precisely one of their results may be stated as follows

Theorem 5.7 (Goupil et al [2])

For each partition of n of the form

$$\gamma, 1^{n-r} \qquad \textit{with } \gamma \vdash r$$

we can construct a symmetric polynomial

$$G_\gamma \;=\; \sum_{|\rho|\le r} c_\rho^\gamma(n)\, p_\rho \qquad \textit{with } c_\rho(n) \in \mathbb{Q}[n] \qquad 5.46$$

satisfying

$$G_\gamma[\mathcal{C}(\lambda)] \;=\; \sum_{|\rho|\le r} c_\rho^\gamma(n)\, p_\rho[\mathcal{C}(\lambda)] \;=\; \omega^\lambda_{\gamma,1^{n-r}} \qquad \text{(for all } \lambda\vdash n) \qquad 5.47$$

The polynomials G_γ constructed by Goupil et al are also relatively simple when expressed in terms of power sums. The proof of Theorem 5.7 by Goupil et al is algorithmic, but the resulting algorithm is of considerable complexity. Our aim here is to obtain an alternate algorithm.

It develops that the crucial idea in carrying this out stems from a calculation initiated by Macdonald in the same previously quoted exercise (†). In fact, the aim of Macdonald in that exercise is to obtain a formula for $\omega^\lambda_{k,1^{n-k}}$. Extending Macdonald's idea to its most general form naturally leads us to the following remarkable result.

(†) Ex. 7, page 117 of [4].

Theorem 5.8

Let γ be a partition of m all of whose parts are > 1 and set

$$\Psi_{\gamma,n}(y) \;=\; \frac{1}{z_\gamma}\sum_{\alpha\vdash r}\chi_\gamma^\alpha\;\Xi_{\alpha(n)}(y) \tag{5.48}$$

then the polynomial

$$\Phi_{\gamma,n}(y) \;=\; \pi_n\,\Psi_{\gamma,n}(y) \tag{5.49}$$

yields the identities

$$\Phi_{\gamma,n}\big[\mathcal{C}(\lambda)\big] \;=\; w^\lambda_{\gamma,1^{n-r}} \qquad\qquad (\forall\;\lambda\vdash n) \tag{5.50}$$

Proof

From the well known formula

$$\chi^\lambda_\rho \;=\; p_\rho\,\det\|x_i^{n-j}\|_{i,j=1}^n\Big|_{x_1^{\lambda_1+n-1}x_2^{\lambda_2+n-2}\cdots x_n^{\lambda_n+n-n}}$$

using our notation, we get for $\rho=\gamma,1^{n-r}$

$$\chi^\lambda_{\gamma,1^{n-r}} \;=\; p_1^{n-r}p_\gamma\,\det\|x_i^{n-j}\|_{i,j=1}^n\Big|_{x_1^{\lambda_1(n)}x_2^{\lambda_2(n)}\cdots x_n^{\lambda_n(n)}}$$

Expanding p_γ in terms of Schur functions we get

$$\begin{aligned}
\chi^\lambda_{\gamma,1^{n-r}} &= \sum_{\alpha\vdash r}\chi_\gamma^\alpha\,p_1^{n-r}S_\alpha(x_1,\ldots,x_n)\Delta_n(x_1,\ldots,x_n)\Big|_{x_1^{\lambda_1(n)}x_2^{\lambda_2(n)}\cdots x_n^{\lambda_n(n)}}\\
&= \sum_{\alpha\vdash r}\chi_\gamma^\alpha\,p_1^{n-r}det\,\big\|x_i^{\alpha_j(n)}\big\|_{i,j=1}^n\,\Big|_{x_1^{\lambda_1(n)}x_2^{\lambda_2(n)}\cdots x_n^{\lambda_n(n)}}\\
&= \sum_{\alpha\vdash r}\chi_\gamma^\alpha\,p_1^{n-r}\sum_{\sigma\in S_n}sign(\sigma)\,x_1^{\alpha_{\sigma_1}(n)}x_2^{\alpha_{\sigma_2}(n)}\cdots x_n^{\alpha_{\sigma_n}(n)}\,\Big|_{x_1^{\lambda_1(n)}x_2^{\lambda_2(n)}\cdots x_n^{\lambda_n(n)}}
\end{aligned}$$

Following Macdonald we use the multinomial expansion of p_1^{n-r} and obtain

$$\begin{aligned}
\chi^\lambda_{\gamma,1^{n-r}} &= \sum_{\alpha\vdash r}\chi_\gamma^\alpha\sum_{\sigma\in S_n}sign(\sigma)\sum_{p_1+p_2+\cdots+p_n=n-r}\frac{(n-r)!}{p_1!p_2!\cdots p_n!}\,x_1^{\alpha_{\sigma_1}(n)+p_1}x_2^{\alpha_{\sigma_2}(n)+p_2}\cdots x_n^{\alpha_{\sigma_n}(n)+p_n}\,\Big|_{x_1^{\lambda_1(n)}x_2^{\lambda_2(n)}\cdots x_n^{\lambda_n(n)}}\\
&= (n-r)!\sum_{\alpha\vdash r}\chi_\gamma^\alpha\sum_{\sigma\in S_n}sign(\sigma)\;\frac{1}{(\lambda_1(n)-\alpha_{\sigma_1}(n)!(\lambda_2(n)-\alpha_{\sigma_2}(n)!\cdots(\lambda_n(n)-\alpha_{\sigma_n}(n)!}
\end{aligned}$$

Since all the parts of γ are >1 we have that $z_{\gamma,1^{n-r}}=z_\gamma(n-r)!$ thus 5.45 with $\rho=\gamma,1^{n-r}$ and 5.32 give

$$\omega^\lambda_{\gamma,1^{n-r}} \;=\; \chi^\lambda_{\gamma,1^{n-r}}\frac{h_\lambda}{z_\gamma(n-r)!} \;=\; \frac{\chi^\lambda_{\gamma,1^{n-r}}}{z_\gamma(n-r)!}\,\frac{\lambda_1(n)!\lambda_2(n)!\cdots\lambda_n(n)!}{\prod_{1\le i<j\le n}(\lambda_i(n)!-\lambda_j(n)!)}$$

Using this in our previous relation we obtain

$$\begin{aligned} w^{\lambda}_{\gamma,1^{n-r}} &= \frac{1}{z_\gamma \prod_{1\le i<j\le n}(\lambda_i(n)!-\lambda_j(n)!)} \sum_{\alpha\vdash r} \chi^\alpha_\gamma \sum_{\sigma\in S_n} sign(\sigma)\, \frac{\lambda_1(n)!\lambda_2(n)!\cdots\lambda_n(n)!}{(\lambda_1(n)-\alpha_{\sigma_1}(n)!(\lambda_2(n)-\alpha_{\sigma_2}(n)!\cdots(\lambda_n(n)-\alpha_{\sigma_n}(n)!} \\ &= \frac{1}{z_\gamma}\sum_{\alpha\vdash r}\chi^\alpha_\gamma \frac{\sum_{\sigma\in S_n} sign(\sigma)\,(\lambda_1(n))_{\alpha_{\sigma_1}}(\lambda_2(n))_{\alpha_{\sigma_2}}\cdots(\lambda_n(n))_{\alpha_{\sigma_n}}}{\prod_{1\le i<j\le n}(\lambda_i(n)!-\lambda_j(n)!)} \\ &= \frac{1}{z_\gamma}\sum_{\alpha\vdash r}\chi^\alpha_\gamma\ \Xi_{\alpha(n)}\big[\lambda(n)\big]\,. \qquad 5.51 \end{aligned}$$

Now, from 5.49 and Theorem 5.4 we get that

$$\Phi_{\gamma,n}\big[\mathcal{C}(\lambda)\big] \;=\; \Psi_{\gamma,n}\big[\mu(\lambda)\big]$$

and 5.51 together with the definition in 5.48 gives 5.50 and completes the proof of the theorem.

Formulas 5.48 and 5.49 combined are not sufficient to render explicit the dependence on n. To achieve this and obtain expansions similar to 5.47 for the polynomials $\Phi_{\gamma,n}$ we need one further step. This is provided by the following basic identity.

Proposition 5.3

For $\alpha\vdash m<n$ *let*

$$R_\alpha[x_1,x_2,\cdots,x_n;n] \;=\; \Xi_{\alpha(n)}[x(n)], \qquad 5.52$$

where for convenience we have set

$$x(n)=\big(x_1(n),x_2(n),\ldots,x_n(n)\big) \qquad (with\ x_i(n)=x_i+n-i)\,,$$

then

$$R_\alpha[x_1,x_2,\cdots,x_n;n]\Big|_{x_n=0} \;=\; R_\alpha[x_1,x_2,\cdots,x_{n-1};n-1] \qquad 5.53$$

Proof

Note that if $a_j\ge 1$ we may write

$$(y_i)_{a_j} \;=\; y_i\,(y_i-1)_{a_j-1} \qquad \text{for } i=1,\ldots,n\,.$$

Thus

$$\det\big\|(y_i)_{a_j}\big\|^n_{i,j=1}\Big|_{\substack{y_n=0\\ a_n=0}} \;=\; y_1y_2\cdots y_{n-1}\det\begin{bmatrix} (y_1-1)_{a_1-1} & \cdots & (y_1-1)_{a_{n-1}-1} & 1\\ (y_2-1)_{a_1-1} & \cdots & (y_2-1)_{a_{n-1}-1} & 1\\ \vdots & \vdots & \vdots & \vdots\\ (y_{n-1}-1)_{a_1-1} & \cdots & (y_{n-1}-1)_{a_{n-1}-1} & 1\\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

Thus from the definition in 5.18 we get that

$$\Xi_a[y_1,y_2,\ldots,y_n\Big|_{\substack{y_n=0\\a_n=0}} \;=\; \det\begin{bmatrix} (y_1-1)_{a_1-1} & \cdots & (y_1-1)_{a_{n-1}-1}\\ (y_2-1)_{a_1-1} & \cdots & (y_2-1)_{a_{n-1}-1}\\ \vdots & \cdots & \vdots\\ (y_{n-1}-1)_{a_1-1} & \cdots & (y_{n-1}-1)_{a_{n-1}-1}\end{bmatrix}\Big/\prod_{1\le i<j\le n-1}(y_i-y_j) \qquad 5.54$$

Note that for $l(\alpha)<n$ we have $\alpha_n(n)=0$ and $\alpha_i(n)\ge 1$ for all $i<n$. Moreover, we also have

$$x_i(n)-1 \;=\; x_i+n-i-1 \;=\; x_i(n-1) \qquad (\text{for } i=1,\ldots,n-1)$$

and

$$\alpha_i(n)-1 \;=\; \alpha_i+n-i-1 \;=\; \alpha_i(n-1) \qquad (\text{for } i=1,\ldots,n-1)$$

Thus, setting $y_i=x_i(n)$ and $a_i=\alpha_i(n)$, in 5.54 immediately gives 5.53 precisely as asserted.

Proposition 5.3 has the following remarkable corollary

Proposition 5.4

If $\alpha\vdash m$ then for any $n\ge m$ we have the power basis expansion

$$\Xi_{\alpha(n)}(y_1,y_2,\ldots,y_n) \;=\; \sum_{|\rho|\le m} c^\alpha_\rho(n)\,p_\rho(y_1,y_2,\ldots,y_n)\,. \qquad 5.55$$

This given, define the coefficients $d^\alpha_\rho(n)$ through the equation

$$\sum_{|\rho|\le m} c^\alpha_\rho(n)\,p_\rho\Big|_{p_k\to R_k(n-1)+\sum_{s=1}^k\binom{k}{s}(n^s-(n-1)^s)q_s} \;=\; \sum_{|\rho|\le m} d^\alpha_\rho(n)\,q_\rho \qquad 5.56$$

where $q_1,q_2,\ldots,q_m$ are indeterminates and for $\rho=1^{k_1}2^{k_2}\cdots m^{k_m}$ we set

$$q_\rho \;=\; q_1^{k_1}q_2^{k_2}\cdots q_m^{k_m}$$

then for all $n\ge m$ we have

$$d^\alpha_\rho(n) \;=\; d^\alpha_\rho(m) \qquad (\text{ for all } |\rho|\le m\,). \qquad 5.57$$

Proof

It is easily seen from the definition in 5.18 that Ξ_α is a polynomial of degree

$$a_1+a_2+\cdots+a_n-\binom{n}{2}$$

thus it follows that, when $\alpha\vdash m$, the polynomial in the left hand side of 5.55 is of degree m. Thus the power sum expansion of $\Xi_{\alpha(n)}(y_1,y_2,\ldots,y_n)$ must necessarily be of the form given in 5.55.

For convenience set for all $k\ge 1$

$$q_k(x_1,x_2,\ldots,x_n) \;=\; \sum_{i=1}^n\sum_{s=0}^k\binom{k}{s}(-i)^{k-a}\,R_a(x_i) \qquad 5.58$$

Note that by combining 5.26, 5.27 and 5.28 we derive that,

$$p_k\big(x_1(n), x_2(n), \ldots, x_n(n)\big) \;=\; R_k(n-1) + \sum_{s=1}^{k} \binom{k}{s}(n^s - (n-1)^s) q_s(x_1, x_2, \ldots, x_n) \qquad 5.59$$

Then it follows from the definition in 5.52 that

$$\begin{aligned} R_\alpha[x_1, x_2, \ldots, x_n\,; n\,] &= \sum_{|\rho|\le m} c^\alpha_\rho(n)\, p_\rho(x_1(n), x_2(n), \ldots, x_n(n))\,. \\ \text{(by 5.56)} &= \sum_{|\rho|\le m} d^\alpha_\rho(n)\, q_\rho(x_1, x_2, \ldots, x_n) \end{aligned} \qquad 5.60$$

where for $\rho = 1^{k_1}2^{k_2}\cdots m^{k_m}$ we set

$$q_\rho(x_1, x_2, \ldots, x_n) \;=\; q_1^{k_1}(x_1, x_2, \ldots, x_n) q_2^{k_2}(x_1, x_2, \ldots, x_n)\cdots q_m^{k_m}(x_1, x_2, \ldots, x_n)$$

Note next that Proposition 5.3 gives

$$\begin{aligned} R_\alpha[x_1, x_2, \ldots, x_n\,; n\,]\Big|_{x_{m+1}=x_{m+2}=\cdots=x_n=0} &= R_\alpha[x_1, x_2, \ldots, x_m\,; m\,] \\ \text{(by 5.60 for } m=n) &= \sum_{|\rho|\le m} d^\alpha_\rho(m)\, q_\rho(x_1, x_2, \ldots, x_m) \end{aligned} \qquad 5.61$$

On the other hand 6.60 itself gives

$$R_\alpha[x_1, x_2, \ldots, x_n\,; n\,]\Big|_{x_{m+1}=x_{m+2}=\cdots=x_n=0} = \sum_{|\rho|\le m} d^\alpha_\rho(n)\, q_\rho(x_1, x_2, \ldots, x_n)\Big|_{x_{m+1}=x_{m+2}=\cdots=x_n=0}. \qquad 5.62$$

But we plainly see from 5.58 that for all $k \ge 0$ we have

$$q_k(x_1, x_2, \ldots, x_n)\Big|_{x_{m+1}=x_{m+2}=\cdots=x_n=0} \;=\; q_k(x_1, x_2, \ldots, x_m)\,. \qquad 5.63$$

Combining 5.61, 5.62 and 5.63 gives

$$\sum_{|\rho|\le m} d^\alpha_\rho(m)\, q_\rho(x_1, x_2, \ldots, x_m) \;=\; \sum_{|\rho|\le m} d^\alpha_\rho(n)\, q_\rho(x_1, x_2, \ldots, x_m)$$

and this forces 5.57, completing the proof.

Theorem 5.9

Let γ be a partition of m all of whose parts are >1 and set

$$W_\gamma(p_1, p_2, \ldots, p_m\,;\, n) \;=\; \frac{1}{z_\gamma}\sum_{\alpha\vdash m} \chi^\alpha_\gamma \sum_{|\rho|\le m} d^\alpha_\rho(m)\, p_\rho\Big|_{p_0\to n} \qquad 5.64$$

then for all $\lambda\vdash n$ we have

$$W_\gamma(p_1, p_2, \ldots, p_m\,;\, n)\Big|_{p_k\to p_k[\,\mathcal{C}(\lambda)]} \;=\; \omega_{\gamma,1^{n-m}} \qquad 5.65$$

In particular the conjugacy class $C_{\gamma,1^{n-m}} \in \mathcal{A}(S_n)$ is given by the formula

$$C_{\gamma,1^{n-m}} \;=\; W_\gamma(p_1,p_2,\ldots,p_m\,;\,n)\Big|_{p_k\rightarrow p_k[m_2,m_2,\ldots,m_n]}\,. \qquad 5.66$$

Proof

It follows from Proposition 5.4 that, for $\gamma\vdash m$, the polynomial

$$\Phi_{\gamma,n} \;=\; \pi_n\,\Psi_{\gamma,n}$$

defined in 5.49, may also be written in the form

$$\Phi_{\gamma,n} \;=\; \frac{1}{z_\gamma}\sum_{\alpha\vdash m}\chi^\alpha_\gamma\sum_{|\rho|\le m} d^\alpha_\rho(m)\,p_\rho$$

We should note however that the definition 5.56 yields that some powers of q_0 will necessarily occur in the expansion on the right hand side of 5.56. Since for any $\lambda\vdash n$, we have $p_0[\mathcal{C}(\lambda)]=n$, we see that in making the replacement $q_k{\rightarrow}p_k[\mathcal{C}(\lambda)]$, the variable q_0 will necessarily be replaced by n. Thus 5.65 follows by combining Theorems 5.8 and 5.9. Formula 5.66 can then be derived from 5.65 the same way we derived 5.37 from 5.36.

The reader may find interesting to observe the remarkable simplicity of some of our polynomials W_γ given in the tables which follow.

$$\mathbf{W}_2=\mathbf{p}_1$$

$$\mathbf{W}_3=1/2\,n\,(n-1)+\mathbf{p}_2$$

$$\begin{aligned}\mathbf{W}_4&=-\,(-3+2\,n)\,\mathbf{p}_1+\mathbf{p}_3\\ \mathbf{W}_{22}&=1/2\,n\,(n-1)+1/2\,{\mathbf{p}_1}^2-3/2\,\mathbf{p}_2\end{aligned}$$

$$\begin{aligned}\mathbf{W}_5&=1/6\,n\,(n-1)\,(5\,n-19)+\mathbf{p}_4-(-10+3\,n)\,\mathbf{p}_2-2\,{\mathbf{p}_1}^2\\ \mathbf{W}_{32}&=-1/2\,\left(16-13\,n+n^2\right)\mathbf{p}_1-4\,\mathbf{p}_3+\mathbf{p}_2\,\mathbf{p}_1\end{aligned}$$

$$\begin{aligned}\mathbf{W}_6&=2\,(3\,n-4)\,(n-5)\,\mathbf{p}_1+\mathbf{p}_5-(-25+4\,n)\,\mathbf{p}_3-6\,\mathbf{p}_2\,\mathbf{p}_1\\ \mathbf{W}_{42}&=-4/3\,n\,(n-1)\,(2\,n-7)-5\,\mathbf{p}_4+(-35+12\,n)\,\mathbf{p}_2+\mathbf{p}_3\,\mathbf{p}_1-(-11+2\,n)\,{\mathbf{p}_1}^2\\ \mathbf{W}_{33}&=1/8\,n\,(n-1)\left(n^2-13\,n+34\right)-5/2\,\mathbf{p}_4+1/2\,{\mathbf{p}_2}^2-1/2\,(n-3)\,(n-10)\,\mathbf{p}_2+3\,{\mathbf{p}_1}^2\\ \mathbf{W}_{222}&=1/2\,\left(10-9\,n+n^2\right)\mathbf{p}_1+10/3\,\mathbf{p}_3+1/6\,{\mathbf{p}_1}^3-3/2\,\mathbf{p}_2\,\mathbf{p}_1\end{aligned}$$

$$\begin{aligned}
\mathbf{W}_7 &= -1/24\, n\, (n-1) \left(49\, n^2 - 609\, n + 1502\right) - 9/2\, \mathbf{p}_2{}^2 - 8\, \mathbf{p}_3 \mathbf{p}_1 + 2\, (-36 + 7\, n)\, \mathbf{p}_1{}^2 \\
&\quad + 1/2\, \left(504 + 21\, n^2 - 241\, n\right) \mathbf{p}_2 - 5/2\, (2\, n - 21)\, \mathbf{p}_4 + \mathbf{p}_6 \\
\mathbf{W}_{52} &= 10\, (-11 + 2\, n)\, \mathbf{p}_3 + 1/6\, \left(-174\, n^2 + 889\, n - 864 + 5\, n^3\right) \mathbf{p}_1 \\
&\quad - (3\, n - 40)\, \mathbf{p}_2 \mathbf{p}_1 + \mathbf{p}_4 \mathbf{p}_1 - 6\, \mathbf{p}_5 - 2\, \mathbf{p}_1{}^3 \\
\mathbf{W}_{43} &= -1/2\, (n-5)\, (n-36)\, \mathbf{p}_3 + 1/2\, \left(-240 - 53\, n^2 + 251\, n + 2\, n^3\right) \mathbf{p}_1 \\
&\quad - (-27 + 2\, n)\, \mathbf{p}_2 \mathbf{p}_1 - 6\, \mathbf{p}_5 + \mathbf{p}_3 \mathbf{p}_2 \\
\mathbf{W}_{322} &= -1/4\, (n-1) \left(n^2 - 25\, n + 72\right) n + 1/2\, \mathbf{p}_2 \mathbf{p}_1{}^2 - 3/2\, \mathbf{p}_2{}^2 - 4\, \mathbf{p}_3 \mathbf{p}_1 \\
&\quad - 1/4\, \left(n^2 - 25\, n + 104\right) \mathbf{p}_1{}^2 + 5/4\, \left(n^2 - 25\, n + 60\right) \mathbf{p}_2 + 15\, \mathbf{p}_4
\end{aligned}$$

$$\begin{aligned}
\mathbf{W}_8 &= 1/3\, \left(48\, n^2 + 3283 - 918\, n\right) \mathbf{p}_3 - 4/3\, \left(-264\, n^2 - 945 + 1058\, n + 16\, n^3\right) \mathbf{p}_1 \\
&\quad + 2\, (24\, n - 197)\, \mathbf{p}_2 \mathbf{p}_1 - 2\, (-49 + 3\, n)\, \mathbf{p}_5 + \frac{32}{3}\, \mathbf{p}_1{}^3 - 12\, \mathbf{p}_3 \mathbf{p}_2 + \mathbf{p}_7 - 10\, \mathbf{p}_4 \mathbf{p}_1 \\
\mathbf{W}_{62} &= 3\, n\, (n-1) \left(3\, n^2 - 35\, n + 82\right) - 6\, \mathbf{p}_2 \mathbf{p}_1{}^2 + 27\, \mathbf{p}_2{}^2 - (4\, n - 73)\, \mathbf{p}_3 \mathbf{p}_1 \\
&\quad + \left(6\, n^2 - 110\, n + 379\right) \mathbf{p}_1{}^2 + \mathbf{p}_5 \mathbf{p}_1 - \left(-564\, n + 1099 + 54\, n^2\right) \mathbf{p}_2 + 10\, (-28 + 3\, n)\, \mathbf{p}_4 - 7\, \mathbf{p}_6 \\
\mathbf{W}_{53} &= -1/24\, n\, (n-1) \left(10\, n^3 - 273\, n^2 + 2243\, n - 4770\right) - 2\, \mathbf{p}_2 \mathbf{p}_1{}^2 - 1/2\, (6\, n - 65)\, \mathbf{p}_2{}^2 + \mathbf{p}_4 \mathbf{p}_2 + 40\, \mathbf{p}_3 \mathbf{p}_1 \\
&\quad + \left(n^2 - 61\, n + 260\right) \mathbf{p}_1{}^2 + 7/3\, \left(-372 + n^3 + 206\, n - 27\, n^2\right) \mathbf{p}_2 - 1/2\, \left(455 + n^2 - 61\, n\right) \mathbf{p}_4 - 7\, \mathbf{p}_6 \\
\mathbf{W}_{44} &= 2/3\, n\, (n-1) \left(6\, n^2 - 62\, n + 139\right) + 9\, \mathbf{p}_2{}^2 + 1/2\, \mathbf{p}_3{}^2 - (2\, n - 23)\, \mathbf{p}_3 \mathbf{p}_1 \\
&\quad + 1/2\, \left(4\, n^2 + 267 - 76\, n\right) \mathbf{p}_1{}^2 - 3/2\, (8\, n - 21)\, (2\, n - 13)\, \mathbf{p}_2 + 15\, (-7 + n)\, \mathbf{p}_4 - 7/2\, \mathbf{p}_6 \\
\mathbf{W}_{422} &= 1/2\, \left(560 - 121\, n + n^2\right) \mathbf{p}_3 - 1/6\, \left(-471\, n^2 - 1890 + 22\, n^3 + 2069\, n\right) \mathbf{p}_1 + 1/2\, (30\, n - 247)\, \mathbf{p}_2 \mathbf{p}_1 \\
&\quad + 21\, \mathbf{p}_5 - 1/2\, (-19 + 2\, n)\, \mathbf{p}_1{}^3 + 1/2\, \mathbf{p}_3 \mathbf{p}_1{}^2 - 5\, \mathbf{p}_4 \mathbf{p}_1 - 3/2\, \mathbf{p}_3 \mathbf{p}_2 \\
\mathbf{W}_{332} &= \left(2\, n^2 - 62\, n + 245\right) \mathbf{p}_3 + 1/8\, \left(2240 - 2490\, n + n^4 + 607\, n^2 - 38\, n^3\right) \mathbf{p}_1 - 5/2\, \mathbf{p}_4 \mathbf{p}_1 \\
&\quad - 1/2\, \left(-25\, n + n^2 + 190\right) \mathbf{p}_2 \mathbf{p}_1 + 1/2\, \mathbf{p}_1 \mathbf{p}_2{}^2 + 21\, \mathbf{p}_5 + 3\, \mathbf{p}_1{}^3 - 4\, \mathbf{p}_3 \mathbf{p}_2 \\
\mathbf{W}_{2222} &= 1/24\, n\, (n-1) \left(3\, n^2 - 67\, n + 182\right) + 1/24\, \mathbf{p}_1{}^4 - 3/4\, \mathbf{p}_2 \mathbf{p}_1{}^2 + \frac{9}{8}\, \mathbf{p}_2{}^2 + 10/3\, \mathbf{p}_3 \mathbf{p}_1 \\
&\quad + 1/4\, \left(n^2 - 17\, n + 56\right) \mathbf{p}_1{}^2 - 1/4\, \left(140 - 63\, n + 3\, n^2\right) \mathbf{p}_2 - \frac{35}{4}\, \mathbf{p}_4
\end{aligned}$$

These notes would not be complete without the evaluation of the elementary symmetric functions at the Murphy elements. This can be stated as follows

Theorem 5.10

For $s = 1, 2, \ldots, n$ we have

$$e_s(m_2, m_3, \ldots, m_n) = \sum_{l(\rho) = n - s} \mathcal{C}_\rho \qquad 6.66$$

Proof

Remarkably, this identity is equivalent to a formula giving the principal specialization of Schur functions. This is shown by a purely combinatorial argument by Diaconis and Greene in [1].

We shall reverse the cart here and derive it from the Schur function identity. To be precise it is shown in [4] (ex. 4 p. 45) that for $\lambda \vdash n$ and for all $N \geq n$ we have

$$S_\lambda(x_1, \ldots x_N)\Big|_{x_1=\cdots=x_N=1} \;=\; \frac{1}{h_\lambda} \prod_{(i,j)\in\lambda} \big(N + j - i\big) \qquad 6.67$$

On the other hand, Frobenius formula gives

$$S_\lambda(x_1, \ldots x_N)\Big|_{x_1=\cdots=x_N=1} \;=\; \sum_{\rho\vdash n} \frac{\chi^\lambda_\rho}{z_\rho}\, p_\rho(x_1, \ldots x_N)\Big|_{x_1=\cdots=x_N=1} \;=\; \sum_{\rho\vdash n} \frac{\chi^\lambda_\rho}{z_\rho}\, N^{\,l(\rho)}$$

Clearly the validity of 6.67 and 6.68 for all N implies the polynomial equality

$$\sum_{\rho\vdash n} \frac{\chi^\lambda_\rho}{z_\rho}\, t^{\,l(\rho)} \;=\; \frac{1}{h_\lambda} \prod_{(i,j)\in\lambda} \big(t + j - i\big)$$

Thus it follows from Theorem 3.3 that

$$\prod_{k=2}^{n} (t + m_k)\, \chi^\lambda \;=\; \Big(\sum_{\rho\vdash n} \frac{\chi^\lambda_\rho}{z_\rho}\, h_\lambda\, t^{\,l(\rho)} \Big) \chi^\lambda$$

and equating coefficients of t^{n-s} we finally obtain

$$\begin{aligned} e_s(m_2, m_3, \ldots, m_n)\, \chi^\lambda \;&=\; \Big(\sum_{l(\rho)=n-s} \frac{\chi^\lambda_\rho}{z_\rho}\, h_\lambda \Big) \chi^\lambda \\ &=\; \Big(\sum_{l(\rho)=n-s} \mathcal{C}_\rho \Big) \chi^\lambda \end{aligned}$$

and the validity of this for all $\lambda \vdash n$ proves 6.66.

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