# The non-degeneracy of the bilinear form of m-Quasi-Invariants

by

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**ABSTRACT**

We give here a new proof of the non-degeneracy of the fundamental bilinear form for $S_n$-$m$-Quasi-Invariants and for $m$-Quasi-Invariants of classical Weyl groups. We also indicate how our approach can be extended to other Coxeter groups. This bilinear form plays a crucial role in the original proof [6] that $m$-Quasi-Invariants are a free module over the invariants as well as in all subsequent proofs [3], [11]. However, in previous literature this non-degeneracy was stated and used without proof with reference to some deep results of Opdam [15] on shift-differential operators. This result hinges on the validity of a deceptively simple identity on Dunkl operators which, at least in the $S_n$ case, begs for an elementary painless proof. An elementary but by all means not painless proof of this identity can be found in a paper of Dunkl and Hanlon [5]. Our proof here is not elementary but hopefully it should be painless and informative.

## Introduction

In the present context the $S_n$ Dunkl operator $\nabla_i(m)$ is written in the form

$$\nabla_i(m) \;=\; \partial_{x_i} - m \sum_{j=1, j\neq i}^{n} \tfrac{1}{x_i - x_j}(1 - s_{ij}) \qquad \text{I.1}$$

where "$\partial_{x_i}$" is ordinary partial differentiation with respect to $x_i$ and "$s_{ij}$" denotes the transposition that interchanges $x_i$ and $x_j$. These operators as well as their analogous counterpart for other reflection groups have truly remarkable properties. In fact, they have a surprising variety of properties in common with ordinary differentiation. In particular, they act on polynomials in $x_1, x_2, \ldots, x_n$ and statisfy the commutativity relations

$$\nabla_i(m)\nabla_j(m) \;=\; \nabla_j(m)\nabla_i(m) \qquad (\,\forall \quad 1 \leq i < i \leq n) \qquad \text{I.2}$$

That means that for any polynomial $P(x_1, x_2, \ldots, x_n)$ the operator

$$P\big(\nabla(m)\big) \;=\; P\big(\nabla_1(m), \nabla_2(m), \ldots, \nabla_n(m)\big) \qquad \text{I.3}$$

is well defined. The following identity is part of the collection of identities for Coxeter groups that are the main object of this paper:

$$\Pi_n\big(\nabla(m)\big)\Pi_n(x) \;=\; n!(-1)^{\binom{n}{2}} \prod_{1\leq i<j\leq n} (mj - i) \qquad \text{I.4}$$

where

$$\Pi_n(x) \;=\; \prod_{1\leq i<j\leq n} (x_i - x_j) \qquad \text{I.5}$$

is the familiar Vandermonde determinant. Although I.4 can be easily conjectured by computer experimentation, efforts at producing an elementary proof of it quickly lead to surprising technical difficulties. Nevertheless Dunkl and Hanlon in [5] were able to provide a brute force derivation of I.4 as well as a considerably more general version of it. Our attempts at deciphering the Dunkl-Hanlon proof persuaded us to seek for other paths. In doing so we quickly learned that I.4 may be also derived from

(1) *the Theory of Double Affine Hecke Algebras,*
(2) *the Theory of Macdonald polynomials,*
(3) *the Theory of Jack polynomials.*

For (1), (2) and (3) we are respectively grateful to I. Cherednik, I. M. Macdonald and Luc Lapointe who personally provided us with a surprisingly detailed outline of the arguments. It develops that in each case I.4 quickly follows from basic identities of each theory. However, in each case, the effort at developing the basics of the corresponding theory, although certainly worthwhile from a general education standpoint, turned out to be quite disproportionate to our ultimate goal.

The breakthrough that led us to a more economical path to I.4 came from a paper of Zeilberger [16] whose principal goal was an attempt at a WZ evaluation of the classical Mehta integral. In this attempt Zeilberger unwittigly ties up his evaluation to a very simple identity implicitely involving Dunkl operators. Our basic contribution here is to show that the Zeilberger identity is in fact equivalent to I.4. thus obtaining I.4 as an elementary consequence of the Mehta integral.

Our presentation consists of five sections. In the first section we rederive the Zeilberger identity and show its equivalence to I.4. This section uses a number of identities that may be well known to experts in the area. For them the resulting proof of I.4 may be complete. However, our presentation is aimed at a more general audience. Since detailed proofs of many of these identities are difficult to find in the literature, we feel compelled to include additional sections to cover what is customarily omitted or briefly sketched. This given, in the second section we derive all the needed basic identities on Dunkl operators. We carry this out in the general Weyl group setting and show how the arguments of section 1 extend to this more general case. The third section contains a complete proof of the Selberg integral itself including many usually omitted details. In the forth section we give a detailed derivation of the Mehta integral from the Selberg integral. We also include there a proof of the Macdonald-Metha identities for $B_n$ and $D_n$. These identities were first proved by Regev [14] who derived them from the Selberg integral. Our proof follows the same path. In the fifth and final section we show how I.4 and its Weyl group analogues yield the non degeneracy of the bilinear form for $m$-Quasi-Invariants.

We should again emphasize that the contents of this writing should be considered semi-expository in that many of the results we prove are well known to the expert in the subject. Our goal throughout has been to make the material accessible to beginners in the subject in the least painful manner. In fact most of our work here is simply a detailed presentation of some of the contents of a graduate topic course on the theory of $m$-quasi-invariants given at UCSD in the academic year $2003-2004$. Readers who may wish to learn more about $m$-quasi-invariants may consult the expository works in [7] and [10].

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## 1. Dunkl operators and the Mehta integral

Our point of departure here is the Mehta identity

$$\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-(x_1^2+x_2^2+\cdots+x_n^2)/2}\prod_{1\le i<j\le n}(x_i-x_j)^{2k}dx_1dx_2\cdots dx_n \;=\; \prod_{j=1}^{n}\frac{jk!}{k!}\,. \qquad 1.1$$

Zeilberger in [16] attempts a WZ evaluation of this integral by seeking for a polynomial $P(z)$ which yields 1.1 as a consequence of the simple identity

$$\sum_{i=1}^{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\partial_{x_i}\Big((\partial_{x_i}P)e^{-|x|^2/2}\Pi_n(x)^{2k}\Big)\frac{dx_1dx_2\cdots dx_n}{(2\pi)^{n/2}} \;=\; 0\,. \qquad 1.2$$

where for convenience we have set

$$|x|^2 \;=\; x_1^2+x_2^2+\cdots+x_n^2=p_2(x)\,. \qquad 1.3$$

This idea leads him to a discovery which may be best expressed by the following

### Proposition 1.1

*For any polynomial $P(x)$ we have*

$$\sum_{i=1}^{n}\partial_{x_i}\Big((\partial_{x_i}P)e^{-|x|^2/2}\Pi(x)^{2k}\Big) \;=\; \Big(\Delta_kP-\sum_{i=1}^{n}x_i\partial_{x_i}P\Big)e^{-|x|^2/2}\Pi(x)^{2k}\,. \qquad 1.4$$

*where*

$$\Delta_k \;=\; \Delta \;+\; 2k\sum_{1\le i<j\le n}\tfrac{1}{x_i-x_j}(\partial_{x_i}-\partial_{x_j}) \qquad 1.5$$

*with $\Delta=\sum_{i=1}^{n}\partial_{x_i}^2$ the ordinary Laplacian.*

### Proof

Note that for any $P(x)$ we have

$$\partial_{x_i}\Big((\partial_{x_i}P)e^{-|x|^2/2}\Pi(x)^{2k}\Big)=(\partial_{x_i}^2P)e^{-|x|^2/2}\Pi(x)^{2k}\,-\,x_i(\partial_{x_i}P)e^{-|x|^2/2}\Pi(x)^{2k}\,.\;+\;(\partial_{x_i}P)e^{-|x|^2/2}\partial_{x_i}\Pi(x)^{2k}$$

But

$$\partial_{x_i}\Pi(x)^{2k} \;=\; \Pi(x)^{2k}\partial_{x_i}\log\big(\Pi(x)^{2k}\big) \;=\; 2k\Pi(x)^{2k}\sum_{1\le r<s\le n}\frac{\partial_{x_i}(x_r-x_s)}{x_r-x_s)}$$

thus

$$\begin{aligned}\sum_{i=1}^{n}\partial_{x_i}\Big((\partial_{x_i}P)e^{-|x|^2/2}\Pi(x)^{2k}\Big) \;=\; &(\Delta P)e^{-|x|^2/2}\Pi(x)^{2k}\\ &-\Big(\sum_{i=1}^{n}x_i\partial_{x_i}P\Big)e^{-|x|^2/2}\Pi(x)^{2k} \qquad 1.6\\ &+\Big(2k\sum_{i=1}^{n}(\partial_{x_i}P)\sum_{1\le r<s\le n}\frac{\partial_{x_i}(x_r-x_s)}{x_r-x_s)}\Big)e^{-|x|^2/2}\Pi(x)^{2k}\,.\end{aligned}$$

Since

$$\sum_{i=1}^{n}(\partial_{x_i}P)\sum_{1\le r<s\le n}\frac{\partial_{x_i}(x_r-x_s)}{x_r-x_s} \;=\; \sum_{1\le r<s\le n}\frac{1}{x_r-x_s}(\partial_{x_r}-\partial_{x_s})P$$

we see that 1.6 may be simply written as 1.4.

## Proposition 1.2

*If $I_k$ denotes the Mehta integral in the left hand side of 1.1, then*

$$\frac{1}{2^d d!}\,\Delta_k^{\binom{n}{2}}\,\Pi(x)^2\,I_k \;=\; I_{k+1}. \qquad 1.7$$

## Proof

Zeilberger manages to accomplish this in one stroke by setting in 1.4

$$P(x) \;=\; \sum_{r=0}^{d-1}\frac{\Delta_k^r\,\Pi(x)^2}{2^{r+1}d(d-1)\cdots(d-r)} \qquad \left(\text{with } d=\tbinom{n}{2}\right) \qquad 1.8$$

Indeed, with this choice of $P(x)$, we derive that

$$\begin{aligned}\Delta_k P-\sum_{i=1}^{n}x_i\partial_{x_i}P \;&=\; \sum_{r=0}^{d-1}\frac{\Delta_k^{r+1}\,\Pi(x)^2}{2^{r+1}d(d-1)\cdots(d-r)}-\sum_{r=0}^{d-1}\frac{2(d-r)\Delta_k^r\,\Pi(x)^2}{2^{r+1}d(d-1)\cdots(d-r)}\\ &=\; \sum_{r=1}^{d}\frac{\Delta_k^r\,\Pi(x)^2}{2^r d(d-1)\cdots(d-r+1)}-\sum_{r=0}^{d-1}\frac{\Delta_k^r\,\Pi(x)^2}{2^r d(d-1)\cdots(d-r+1)}\\ &=\; \frac{1}{2^d d!}\,\Delta_k^d\,\Pi(x)^2\;-\;\Pi(x)^2\end{aligned} \qquad 1.9$$

Since the operator $\Delta_k$ decreases degrees by 2 we see that $\Delta_k^d\,\Pi(x)^2$ is none other than a scalar, keeping in mind this fact, the identities in 1.9 and 1.4 combined with 1.2 give

$$\begin{aligned}0 \;&=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Big(\frac{1}{2^d d!}\,\Delta_k^d\,\Pi(x)^2\;-\;\Pi(x)^2\Big)e^{-|x|^2/2}\Pi(x)^{2k}\Big)\frac{dx_1dx_2\cdots dx_n}{(2\pi)^{n/2}}\\ &=\; \frac{1}{2^d d!}\,\Delta_k^d\,\Pi(x)^2\,I_k\;-\;I_{k+1}\end{aligned}$$

proving 1.7.

Assuming the Mehta identity we immediately derive that

## Theorem 1.1

$$\frac{1}{2^d d!}\,\Delta_k^d\,\Pi(x)^2 \;=\; n!\prod_{1\le i<j\le n}(kj+i) \qquad 1.10$$

## Proof

From 1.7 and 1.1 we get

$$
\begin{aligned}
\frac{1}{2^d d!}\,\Delta_k^d\,\Pi(x)^2 &= \prod_{j=1}^{n}\frac{(kj+j)!k!}{kj!(k+1)!} \\
&= \frac{1}{(k+1)^n}\prod_{j=1}^{n}(kj+j)(kj+j-1)\cdots(kj+1) \\
&= \Big(\prod_{j=1}^{n}\frac{(kj+j)}{(k+1)}\Big)\prod_{j=1}^{n}(kj+j-1)\cdots(kj+1) \\
&= n!\prod_{1\le i<j\le n}(kj+i)
\end{aligned}
\qquad 1.11
$$

proving 1.10.

To translate 1.10 into a Dunkl operator identity we only need the following revealing fact.

## Proposition 1.3

*The actions of the operators*

$$
\Delta_k \;=\; \Delta + 2k\sum_{1\le i<j\le n}\tfrac{1}{x_i-x_j}(\partial_{x_i}-\partial_{x_j}) \qquad \textit{and} \qquad p_2(\nabla(-k)) = \nabla_1(-k)^2+\nabla_2(-k)^2+\cdots+\nabla_n(-k)^2
$$

*on symmetric polynomials are identical.*

## Proof

Note that if $f(x_1,x_2,\ldots,x_n)$ is symmetric then for any $1\le i\le n$ from I.1 we get

$$
\nabla_i(-k)f(x) \;=\; \partial_{x_i}f(x)
$$

This gives that

$$
\begin{aligned}
\nabla_i(-k)^2 f(x) &= \partial_{x_i}^2 f(x) + k\sum_{j=1,j\ne i}^{n}\tfrac{1}{x_i-x_j}(1-s_{ij})\partial_{x_i}f(x) \\
&= \partial_{x_i}^2 f(x) + k\sum_{j=1,j\ne i}^{n}\tfrac{1}{x_i-x_j}\big(\partial_{x_i}f(x)-\partial_{x_j}s_{ij}f(x)\big) \\
&= \partial_{x_i}^2 f(x) + k\sum_{j=1,j\ne i}^{n}\tfrac{1}{x_i-x_j}\big(\partial_{x_i}-\partial_{x_j}\big)f(x)
\end{aligned}
$$

Thus summing over $i$ gives

$$
p_2(\nabla(-k))f(x) \;=\; \Delta f(x) + k\sum_{i=1}^{n}\sum_{j=1,j\ne i}^{n}\tfrac{1}{x_i-x_j}\big(\partial_{x_i}-\partial_{x_j}\big)f(x) \;=\; \Delta_k f(x)
$$

proving our assertion.

We can thus derive

**Theorem 1.2**

$$\frac{1}{2^d d!}\, p_2\big(\nabla(m)\big)^d\, \Pi(x)^2 \;=\; n!(-1)^{\binom{n}{2}} \prod_{1\le i<j\le n} (mj-i) \qquad 1.12$$

## Proof

Since $\Pi(x)^{2k}$ is a symmetric polynomial we can use Proposition 1.3 and derive from 1.10 that

$$\frac{1}{2^d d!}\, p_2\big(\nabla(-k)\big)^d\, \Pi(x)^2 \;=\; n! \prod_{1\le i<j\le n} (kj+i)\,. \qquad 1.13$$

Since both sides of this identity are polynomials in $k$, it follows that the equality in 1.13 for all integers $k$ implies that these two polynomials are one and the same. This allows us to make the replacecement $k\to -m$ in 1.13 and get 1.12 precisely as asserted

To convert 1.12 into our desired identity we need but only one more Dunkl operator identity. Namely, the following remarkable fact

**Proposition 1.4**

*For any homogeneous polynomial $P(x_1,x_2,\ldots,x_n)$ we have the operator identity*

$$P\big(\nabla(m)\big) \;=\; \frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r p_2(\nabla(m))^r P(\underline{x}) p_2(\nabla(m))^{d-r} \qquad 1.14$$

*where $d=degree(P)$ and "$P(\underline{x})$" denotes the operator "multiplication by $P$".*

The proof of 1.14 is given in the next section where it will established for all Weyl groups. This given we are in a position to obtain

**Theorem 1.3**

$$\Pi_n\big(\nabla(m)\big)\Pi_n(x) \;=\; n!(-1)^{\binom{n}{2}} \prod_{1\le i<j\le n} (mj-i) \qquad 1.15$$

## Proof

Using 1.14 with $P(x)=\Pi_n(x)$ gives, for $d=\binom{n}{2}$, the operator identity

$$\Pi_n\big(\nabla(m)\big) \;=\; \frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r p_2(\nabla(m))^r \Pi_n(\underline{x}) p_2(\nabla(m))^{d-r}\,. \qquad 1.16$$

Now note that applying both sides of this identity to $\Pi_n(x)$ gives

$$\Pi_n\big(\nabla(m)\big)\Pi_n(x) \;=\; \frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r p_2(\nabla(m))^r \Pi_n(\underline{x}) p_2(\nabla(m))^{d-r}\Pi_n(x). \qquad 1.17$$

However, we see that for $r<d$ the term

$$p_2(\nabla(m))^{d-r}\Pi_n(x)$$

must identically vanish since it is an alternating polynomial of degree $< \binom{n}{2}$. It follows that 1.17 reduces to none other than

$$\Pi_n\big(\nabla(m)\big)\Pi_n(x) \;=\; \frac{1}{2^d d!}p_2(\nabla(m))^d\Pi_n(x)^2$$

and thus 1.15 follows from Theorem 1.2.

## Remark 1.1

We should note that in [16] Zeilberger asks for a direct elementary proof of 1.12 to complete his WZ derivation of the Mehta identity. Such a derivation is in fact contained in the Dunkl-Hanlon paper. Thus a combination of the results in [16] and [5] may be said to provide a completely elementary proof of the Mehta identity. However, one may wish for a simpler argument than the one provided in [5]. Moreover such an argument should be carried out in the general setting of Weyl groups and thereby also obtain an elementary proof of the general form of 1.15

## 2. Basics on Dunkl operators

Let $\Phi$ be a root system contained in $\mathbb{R}^n$ and let $\Phi^+$ a system of positive roots in $\Phi$. For

$$\alpha = (\alpha_1, \alpha_2, \dots \alpha_n) \qquad \text{and} \qquad x = (x_1, x_2, \dots x_n)$$

we set

$$(\alpha, x) \;=\; \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

As customary we shall denote by "$s_\alpha$" the the reflection across the hyperplane $(\alpha, x) = 0$. That is for any $v \in \mathbb{R}^n$ we set

$$s_\alpha v \;=\; v \,-\, 2\tfrac{(\alpha,v)}{(\alpha,\alpha)}\,\alpha\,, \tag{2.1}$$

and denote by $W$ the Weyl group generated by the $\{s_\alpha\}_{\alpha\in\Phi}$. For an element $\sigma \in W$ we shall denote by $A_\sigma$ the matrix yielding the action of $\sigma$ on the basis $x_1, x_2, \dots, x_n$. This given, for any polynomial $P(x) = P(x_1, x_2, \dots, x_n)$ we set

$$\sigma P(x) \;=\; P(xA_\sigma) \tag{2.2}$$

where $xA_\sigma$ denotes ordinary multiplication of a row $n$-vector by an $n \times n$ matrix.

The Dunkl operators are simply defined by setting for any $v \in \mathbb{R}^n$

$$\nabla_v(m) \;=\; \partial_v \,-\, m\sum_{\alpha\in\Phi^+}\tfrac{(\alpha,v)}{(\alpha,x)}(1 - s_\alpha) \tag{2.3}$$

where $\partial_v = \sum_{i=1}^n v_i \partial_{x_i}$ is the directional derivative corresponding to $v$. Since it is well known and easy to show that for any polynomial $P(x)$ the polynomial $(1 - s_\alpha)P(x)$ is divisible by $(\alpha, x)$ we see from 2.3 that $\nabla_v(m)$ is a well defined polynomial operator. Our goal in this section is to provide a self contained derivation of some basic properties of Dunkl operators. The readers familiar with this material may skip to the next section.

Our first task is to establish the commutativity relations

$$\nabla_u(m)\nabla_v(m) \;=\; \nabla_v(m)\nabla_v(m) \qquad \text{(for all } u, v \in \mathbb{R}^n). \tag{2.4}$$

To this end it is convenient to set

$$a)\qquad \theta_v \;=\; \sum_{\alpha\in\Phi^+} \tfrac{(\alpha,v)}{(\alpha,x)}(1-s_\alpha), \qquad\qquad b)\qquad T_v \;=\; \sum_{\alpha\in\Phi^+} \tfrac{(\alpha,v)}{(\alpha,x)} s_\alpha \tag{2.5}$$

and rewrite 2.3 as

$$\nabla_v(m) \;=\; \partial_v \,-\, m\,\theta_v\,. \tag{2.6}$$

It will also be good to keep in mind that

## Proposition 2.1

*For any $v\in\mathbb{R}^n$, $\alpha\in\Phi$ and $\sigma\in W$ we have*

$$a)\quad \sigma\partial_v\sigma^{-1}=\partial_{\sigma v} \qquad b)\quad \sigma s_\alpha\sigma^{-1} = s_{\sigma\alpha} \qquad b)\quad \sigma\theta_v\sigma^{-1} = \theta_{\sigma v} \qquad d)\quad \sigma T_v\sigma^{-1} = T_{\sigma v} \tag{2.7}$$

*in particular 2.6 gives*

$$\sigma\nabla_v(m)\sigma^{-1} \;=\; \nabla_{\sigma v}(m) \tag{2.8}$$

## Proof

Note first that for any polynomal $P(x)$ from 2.1 we derive that

$$\begin{aligned} s_\alpha\partial_{x_i}s_\alpha P(x) \;&=\; s_\alpha\Big(\partial_{x_i}P\big(x-\tfrac{2(\alpha,x)}{(\alpha,\alpha)}\alpha\big)\Big)\\ &=\; s_\alpha\Big(P_{x_i}(s_\alpha x)\,-\,2\sum_{j=1}^n P_{x_j}(s_\alpha x)\tfrac{\alpha_i\alpha_j}{(\alpha,\alpha)}\Big)\\ &=\; \Big(\partial_{e_i}P(x)\,-\,2\tfrac{(\alpha,e_i)}{(\alpha,\alpha)}\partial_\alpha P(x)\Big) \;=\; \partial_{s_\alpha e_i}P(x) \end{aligned}$$

where $e_i$ denotes the $i^{th}$ coordinate vector. Thus by linearity it follows that for any $v\in\mathbb{R}^n$ we have

$$s_\alpha\partial_v s_\alpha \;=\; \partial_{s_\alpha v}$$

and then we must also have 2.7 a) for all $\sigma\in W$. Note next that, again from 2.1, it follows that

$$\begin{aligned} \sigma s_\alpha\sigma^{-1}v \;&=\; \sigma\big(\sigma^{-1}v\,-\,2\tfrac{(\alpha,\sigma^{-1}v)}{(\alpha,\alpha)}\,\alpha\big)\\ &=\; v\,-\,2\tfrac{(\sigma\alpha,v)}{(\sigma\alpha,\sigma\alpha)}\,\sigma\alpha \;=\; s_{\sigma\alpha}v \end{aligned}$$

this proves 2.7 b). This given, we have

$$\begin{aligned} \sigma\theta_v\sigma^{-1} \;&=\; \sigma\big(\sum_{\alpha\in\Phi^+}\tfrac{(\alpha,v)}{(\alpha,x)}(1-s_\alpha)\big)\sigma^{-1} \;=\; \sum_{\alpha\in\Phi^+}\tfrac{(\alpha,v)}{(\alpha,\sigma^{-1}x)}(1-s_{\sigma\alpha})\\ &=\; \sum_{\alpha\in\Phi^+}\tfrac{(\sigma\alpha,\sigma v)}{(\sigma\alpha,x)}(1-s_{\sigma\alpha)} \;=\; \sum_{\alpha\in\Phi^+}\tfrac{(\alpha,\sigma v)}{(\alpha,x)}(1-s_{\alpha)} \;=\; \theta_{\sigma v} \end{aligned}$$

Here we have used the fact that

$$\frac{(-\alpha.u)}{(-\alpha,x)}\,(1-s_{-\alpha}) \;=\; \frac{(\alpha.u)}{(\alpha,x)}\,(1-s_\alpha)$$

This proves 2.7 c). Similarly we see that

$$\sigma T_v \sigma^{-1} \;=\; \sum_{\alpha\in\Phi^+} \tfrac{(\sigma\alpha,\sigma v)}{(\sigma\alpha,x)} s_{\sigma\alpha} \;=\; T_{\sigma v}$$

This completes our proof of the proposition.

Now note that we can write

$$\nabla_u(m)\nabla_v(m) \;=\; (\partial_u - m\,\theta_u)(\partial_v - m\,\theta_v) \;=\; \partial_u\partial_v - m\,\partial_u\theta_v - m\,\theta_u\partial_v + m^2\theta_u\theta_v$$

and similarly we get

$$\nabla_v(m)\nabla_u(m) \;=\; (\partial_v - m\,\theta_v)(\partial_u - m\,\theta_u) \;=\; \partial_v\partial_u - m\,\partial_v\theta_u - m\,\theta_v\partial_u + m^2\theta_v\theta_u$$

Thus we see that in order for 2.4 to be valid for all $m$ it is necessary and sufficient that we have

## Proposition 2.2

*For any* $u,v\in\mathbb{R}^n$

$$a)\quad \partial_u\theta_v \,-\, \theta_v\partial_u \;=\; \partial_v\theta_u \,-\, \theta_u\partial_v \qquad \text{and} \qquad c)\quad \theta_u\theta_v \;=\; \theta_v\theta_u \qquad 2.9$$

## Proof

To begin note that 2.5 gives

$$\begin{aligned}\partial_u\theta_v \,-\, \theta_v\partial_u \;&=\; \partial_u\Big(\sum_{\alpha\in\Phi^+}\tfrac{(\alpha,v)}{(\alpha,x)}(1-s_\alpha)\Big) - \sum_{\alpha\in\Phi^+}\tfrac{(\alpha,v)}{(\alpha,x)}(1-s_\alpha)\partial_u\\ &=\; -\sum_{\alpha\in\Phi^+}\tfrac{(\alpha,v)(\alpha,u)}{(\alpha,x)^2}(1-s_\alpha) \;+\; \sum_{\alpha\in\Phi^+}\tfrac{(\alpha,v)}{(\alpha,x)}(s_\alpha\partial_u - \partial_u s_\alpha)\end{aligned}$$

However, since 2.7 a) gives

$$s_\alpha\partial_u s_\alpha \;=\; \partial_{s_\alpha u} \;=\; \partial_u \,-\, 2\tfrac{(\alpha,u)}{(\alpha,\alpha)}\partial_\alpha$$

we see that

$$s_\alpha\partial_u - \partial_u s_\alpha \;=\; -2\tfrac{(\alpha,u)}{(\alpha,\alpha)}\partial_\alpha s_\alpha.$$

Using this we get

$$\partial_u\theta_v \,-\, \theta_v\partial_u \;=\; -\sum_{\alpha\in\Phi^+}\tfrac{(\alpha,v)(\alpha,u)}{(\alpha,x)^2}(1-s_\alpha) \,-\, 2\sum_{\alpha\in\Phi^+}\tfrac{(\alpha,v)(\alpha,u)}{(\alpha,x)(\alpha,\alpha)}\partial_\alpha s_\alpha$$

which an expression entirely symmetric in $u,v$. This proves 2.9 a).

Next note that from 2.5 a) and b) we get

$$\begin{aligned}\theta_u\theta_v \;&=\; \Big(\sum_{\alpha\in\Phi^+}\tfrac{(\alpha,u)}{(\alpha,x)}(1-s_\alpha)\Big)\Big(\sum_{\beta\in\Phi^+}\tfrac{(\beta,v)}{(\beta,x)}(1-s_\beta)\Big)\\ &=\; \sum_{\alpha,\beta\in\Phi^+}\tfrac{(\alpha,u)(\beta,v)}{(\alpha,x)(\beta,x)} - \sum_{\alpha,\beta\in\Phi^+}\tfrac{(\alpha,u)}{(\alpha,x)}\, s_\alpha\, \tfrac{(\beta,v)}{(\beta,x)} - \sum_{\alpha\in\Phi^+}\tfrac{(\alpha,u)}{(\alpha,x)}\, T_v \;+\; T_u\, T_v\\ &=\; \sum_{\alpha,\beta\in\Phi^+}\tfrac{(\alpha,u)(\beta,v)}{(\alpha,x)(\beta,x)} - \sum_{\alpha,\beta\in\Phi^+}\tfrac{(\alpha,u)}{(\alpha,x)}\, \tfrac{(\beta,s_\alpha v)}{(\beta,x)}\, s_\alpha - \sum_{\alpha\in\Phi^+}\tfrac{(\alpha,u)}{(\alpha,x)}\, T_v \,+ T_u\, T_v\\ &=\; \sum_{\alpha,\beta\in\Phi^+}\tfrac{(\alpha,u)(\beta,v)}{(\alpha,x)(\beta,x)} - \sum_{\beta\in\Phi^+}\tfrac{(\beta,v)}{(\beta,x)}\, T_u + 2\sum_{\alpha,\beta\in\Phi^+}\tfrac{(\alpha,u)(\alpha,v)(\alpha,\beta)}{(\alpha,x)(\beta,x)(\alpha,\alpha)} - \sum_{\alpha\in\Phi^+}\tfrac{(\alpha,u)}{(\alpha,x)}\, T_v \,+ T_u\, T_v \;=\; \Gamma(u,v) \;+\; T_uT_v\end{aligned}$$

Since $\Gamma(u,v)$ is symmetric in $u, v$ we see that to show 2.9 c) we need only verify that

$$T_u T_v \;=\; T_v T_u \tag{2.10}$$

To this end note that 2.7 d) gives

$$\begin{aligned} T_u T_v \;&=\; \sum_{\alpha\in\Phi^+} \tfrac{(\alpha,u)}{(\alpha,x)} s_\alpha T_v \;=\; \sum_{\alpha\in\Phi^+} \tfrac{(\alpha,u)}{(\alpha,x)}\, T_{s_\alpha v} s_\alpha \\ &=\; \sum_{\alpha\in\Phi^+} \tfrac{(\alpha,u)}{(\alpha,x)} \sum_{\beta\in\Phi^+} \tfrac{(\beta,s_\alpha v)}{(\beta,x)}\, s_\beta s_\alpha \end{aligned}$$

and since $(\beta, s_\alpha v) = (\beta, v) - 2\frac{(\alpha,v)}{(\alpha,\alpha)}(\beta,\alpha)$ we get

$$\begin{aligned} T_u T_v \;&=\; \sum_{\alpha,\beta\in\Phi^+} \frac{(\alpha,u)\Big((\beta,v)-2\frac{(\alpha,v)}{(\alpha,\alpha)}(\beta,\alpha)\Big)}{(\alpha,x)(\beta,x)}\; s_\beta\, s_\alpha \\ &=\; \sum_{\alpha,\beta\in\Phi^+} \tfrac{(\alpha,u)(\beta,v)}{(\alpha,x)(\beta,x)}\; s_\beta\, s_\alpha \;-\; 2 \sum_{\alpha,\beta\in\Phi^+} \tfrac{(\alpha,u)(\alpha,v)(\beta,\alpha)\big)}{(\alpha,x)(\beta,x)(\alpha,\alpha)}\; s_\beta\, s_\alpha \end{aligned}$$

Since the last term is symmetric in $u$ and $v$, we have thus reduced 2.9 c) to proving the following identity

$$\sum_{\alpha,\beta\in\Phi^+} \tfrac{(\alpha,u)(\alpha,v)}{(\alpha,x)(\beta,x)}\; \big(s_\beta\, s_\alpha \;-\; s_\alpha\, s_\beta\big) \;=\; 0\,, \qquad\qquad \text{(for all } u,v\ )$$

as an operator on the rational functions in x.

This follows from the following identity

$$\sum_{\alpha,\beta\in\Phi^+} \tfrac{(\alpha,u)(\alpha,v)}{(\alpha,z)(\beta,z)}\; \big(s_\beta\, s_\alpha \;-\; s_\alpha\, s_\beta\big) \;=\; 0\,, \tag{2.11}$$

as a rational function in the variables $u, v, z$ with values in the group algebra of $W$.

The verification of this identity requires auxiliary material. We will use as reference J. Humphreys, "Reflection groups and Coxeter groups" [13].

Let $W$ be a finite reflection group acting on $\mathbb{R}^n$. Let $\Phi$ be a choice of a set of roots for $W$ and $\Phi^+$ a choice of positive roots (sections 1.2,1.3 of [13]). If $\alpha, \beta \in \Phi^+$ are such that $(\alpha,\beta) < 0$ we set

$$\Phi_{\alpha,\beta} = (\mathbb{R}\alpha + \mathbb{R}\beta) \cap \Phi.$$

Obviously $\Phi_{\alpha,\beta} = \Phi_{\beta,\alpha}$. We will therefore write $\Phi_{\{\alpha,\beta\}} = \Phi_{\alpha,\beta}$. Let $\Sigma$ denote the set of $\{\alpha,\beta\}$ with $(\alpha,\beta) < 0$ such that $\{\alpha,\beta\}$ is a system of simple roots (section 1.3 of [13]) for $\Phi^+_{\{\alpha.\beta\}} = \Phi_{\{\alpha,\beta\}} \cap \Phi^+$.

## Lemma 2.1.

*If $\gamma, \delta \in \Phi^+$ are distinct and $(\gamma,\delta) \neq 0$ then there is a unique element $(\alpha,\beta) \in \Sigma$ such that $\gamma, \delta \in \Phi^+_{\{\alpha,\beta\}}$.*

**Proof**.

Set $\Psi = (\mathbb{R}\gamma + \mathbb{R}\delta) \cap \Phi$. Then the group generated by the reflections corresponding to the elements of $\Psi$ is a subgroup of $W$ hence finite and $\Psi$ is a corresponding set of roots. Let $\{\alpha,\beta\}$ be a simple system for $\Psi\cap\Phi^+$ (Theorem 1.3 (b) of [13]). Then $\Phi_{\{\alpha,\beta\}}=\Psi$. Thus $\{\alpha,\beta\}\in\Sigma$. This proves existence, to prove uniqueness note that if $\{\mu,\nu\}\in\Sigma$ and $\gamma,\delta\in\Phi_{\{\mu,\nu\}}$ then $\gamma,\delta$ is a basis of $\mathbb{R}\mu+\mathbb{R}\nu$. This implies that $\Phi^+_{\{\mu,\nu\}}=\Psi\cap\Phi^+=\Phi^+_{\{\alpha,\beta\}}$ . But since there is exactly one simple system for $\Psi\cap\Phi^+$ (Theorem 1.3 (b) of [13]) we must have $\{\mu,\nu\}=\{\alpha,\beta\}$. This completes the proof of the Lemma..

## Lemma 2.2.

*For all $u,v$ and all $x$ such that $(\alpha,z)\neq 0$ when $\alpha\in\Phi^+$ we have the identity*

$$\sum_{\alpha,\beta\in\Phi^+}\frac{(\alpha,u)(\beta,v)}{(\alpha,z)(\beta,z)}(s_\alpha s_\beta-s_\beta s_\alpha)=\sum_{\{\alpha,\beta\}\in\Sigma}\ \sum_{\gamma,\delta\in\Phi^+_{\{\alpha,\beta\}}}\frac{(\gamma,u)(\delta,v)}{(\gamma,z(\delta,z)}(s_\gamma s_\delta-s_\delta s_\gamma)$$

*in the group algebra of $W$.*

## Proof

We note that if $(\alpha,\beta)=0$ or $\alpha=\beta$ then $s_\alpha s_\beta=s_\beta s_\alpha$. We may therefore assume that the sum on the left is over $\alpha,\beta$ with $\alpha\neq\beta$ and $(\alpha,\beta)\neq 0$. This given, Lemma 2.1 assures us that the collections $\left\{\Phi^+_{\{\alpha,\beta\}}\right\}_{\{\alpha,\beta\}\in\Sigma}$ give a decomposition of the set of pairs $\alpha,\beta\in\Phi^+$ into disjoint subsets. The stated identity is thus an immediate consequence of this fact.

Lemma 2.2 reduces the proof of 2.11 for finite reflection groups to the case of rank 2 finite reflection groups. To complete our proof of 2.11 we will show that for any rank 2 finite reflection group $W$, the expression

$$Q(u,v,z)\ \ =\ \ \sum_{\alpha.\beta\in\Phi^+}\frac{(\alpha,u)}{(\alpha,z)}\frac{(\alpha,v)}{(\alpha,z)}\left(s_\alpha s_\beta-s_\beta s_\alpha\right)$$

vanishes identically, for all vectors $u=(u_1,u_2)$, $v=(v_1,v_2)$, $z=(z_1,z_2)$, as an element of the group algebra of $W$. Note that we have been using $z$ (instead of $x$) here to emphasize that, in our arguments, the reflections $s_\alpha$ and $s_\beta$ will not act on the denominators $(\alpha,z)$ and $(\beta,z)$. With this proviso, setting

$$\mathcal{T}_u=\sum_{\alpha\in\Phi^+}\frac{(\alpha,u)}{(\alpha,z)}s_\alpha\ ,\qquad \mathcal{T}_v=\sum_{\beta\in\Phi^+}\frac{(\beta,v)}{(\beta,z)}s_\beta$$

we may write

$$Q(u,v,z)\ \ =\ \ \mathcal{T}_u\mathcal{T}_v\ -\ \mathcal{T}_v\mathcal{T}_u$$

Now we clearly have

$$Q(u,v,z)\ \ =\ \ u_1Q_1(v,z)+u_2Q_2(v,z)$$

with

$$Q_1(v,z)\ \ =\ \ v_1Q_{11}(z)+v_2Q_{12}(z)$$

and

$$Q_2(v,z)\ \ =\ \ v_1Q_{21}(z)+v_2Q_{22}(z)$$

Note next that setting $u=cz$ for some scalar $c\neq 0$ we get

$$\mathcal{T}_{cz}\ \ =\ \ c\sum_{\alpha\in\Phi^+}s_\alpha$$

Since the latter is a central element of the group algebra of $W$, it follows that

$$Q(cz,v,z) \;=\; \mathcal{T}_{cz}\mathcal{T}_v - \mathcal{T}_v\mathcal{T}_{cz} \;=\; 0$$

This gives

$$c\big(z_1Q_1(v,z)+z_2Q_2(v,z)\big) \;=\; 0$$

forcing

$$Q_2(v,z) \;=\; -\tfrac{z_1}{z_2}Q_1(v,z)$$

Thus it follows that

$$v_1Q_{21}(z)+v_2Q_{22}(z) \;=\; -\tfrac{z_1}{z_2}\big(v_1Q_{11}(z)+v_2Q_{12}(z)\big)$$

and this gives

$$Q_{21}(z)=-\tfrac{z_1}{z_2}Q_{11}(z) \qquad \text{and} \qquad Q_{22}(z)=-\tfrac{z_1}{z_2}Q_{12}(z)$$

However we clearly also have

$$Q(u,v,z) \;=\; -Q(v,u,z)$$

that is

$$u_1Q_1(v,z)+u_2Q_2(v,z) \;=\; -\big(v_1Q_1(u,z)+v_2Q_2(u,z)\big)$$

and this expands to

$$\begin{aligned} u_1\big(v_1Q_{11}(z)+v_2Q_{12}(z)\big) \;-\; u_2\tfrac{z_1}{z_2}\big(v_1Q_{11}(z)+v_2Q_{12}(z)\big) \;=&\\ -\; v_1\big(u_1Q_{11}(z)+u_2Q_{12}(z)\big) \;-\; v_2\tfrac{z_1}{z_2}\big(u_1Q_{11}(z)+u_2Q_{12}(z)\big)& \end{aligned}$$

yielding the equalities

$$Q_{11}(z) \;=\; -Q_{11}(z) \qquad \text{and} \qquad -Q_{12}(z) \;=\; Q_{12}(z)$$

which force the desired vanishing of $Q(u,v,z)$.

This establishes 2.11 and completes the proof of Proposition 2.9.

Our next goal is the establishment of Proposition 1.4 in the general Weyl group setting. To carry this out we need to establish a few auxiliary results.

## Proposition 2.3

*For any two vectors $u,v\in\mathbb{R}^n$ we have the operator identity*

$$\nabla_u(m)(\underline{x},v) \;-\; (\underline{x},v)\nabla_u(m) \;=\; (u,v)I \;-\; 2m\sum_{\alpha\in\Phi^+}\tfrac{(\alpha,u)(\alpha,v)}{(\alpha,\alpha)}\,s_\alpha \tag{2.12}$$

*where "$(\underline{x},v)$" denotes the operator "multiplication by $(x,v)$".*

## Proof

Note first that for any polynomial $P$ we have

$$\begin{aligned}
\nabla_u(m)(\underline{x},v)P &= \Big(\partial_u - m\sum_{\alpha\in\Phi^+}\tfrac{(\alpha,u)}{(\alpha,x)}(1-s_\alpha)\Big)(x,v)P \\
&= (u,v)P+(x,v)\partial_u P - m(x,v)\sum_{\alpha\in\Phi^+}\tfrac{(\alpha,u)}{(\alpha,x)}\,P \;+\; m\sum_{\alpha\in\Phi^+}\tfrac{(\alpha,u)}{(\alpha,x)}\,(s_\alpha x,v)s_\alpha P \\
&= (u,v)P+(x,v)\partial_u P - m(x,v)\sum_{\alpha\in\Phi^+}\tfrac{(\alpha,u)}{(\alpha,x)}\,P \;+\; m\sum_{\alpha\in\Phi^+}\tfrac{(\alpha,u)}{(\alpha,x)}\,(x-2\tfrac{(\alpha,x)}{(\alpha,\alpha)}\alpha,v)s_\alpha P \\
&= (u,v)P+(x,v)\nabla_u P \;-\; 2m\sum_{\alpha\in\Phi^+}\tfrac{(\alpha,u)(\alpha,v)}{(\alpha,\alpha)}\,s_\alpha P
\end{aligned}$$

This proves 2.12.

As in the $S_n$ case we shall set

$$\nabla_i(m) \;=\; \nabla_{e_i}(m) \;=\; \partial_{x_i} \;-\; m\sum_{\alpha\in\Phi^+}\tfrac{(\alpha,e_i)}{(\alpha,\alpha)}\,(1-s_\alpha) \tag{2.13}$$

where $e_i$ is the $i^{th}$ coordinate vector. This given, note that 2.12 specialized at $u=e_j$ and $v=e_i$ gives

$$\nabla_j(m)\underline{x}_i \;-\; \underline{x}_i\nabla_j(m) \;=\; -2m\sum_{\alpha\in\Phi^+}\tfrac{\alpha_j\alpha_i}{(\alpha,\alpha)}\,s_\alpha \qquad \text{(for all } i\neq j) \tag{2.14}$$

On the other hand for $u=e_j$ and $v=e_j$ 2.12 gives

$$\nabla_i(m)\underline{x}_i \;-\; \underline{x}_i\nabla_i(m) \;=\; I-2m\sum_{\alpha\in\Phi^+}\tfrac{\alpha_i\alpha_i}{(\alpha,\alpha)}\,s_\alpha \qquad \text{(for all } j) \tag{2.15}$$

These two identities yield the following beautiful commutation relation.

## Proposition 2.4

$$p_2\big(\nabla(m)\big)\,\underline{x}_i \;-\; \underline{x}_i\,p_2\big(\nabla(m)\big) \;=\; 2\nabla_i(m) \qquad \text{(for all } i) \tag{2.16}$$

## Proof

Note first that for $i\neq j$ we have

$$\begin{aligned}
\nabla_j(m)^2\underline{x}_i \;-\; \underline{x}_i\nabla_j(m)^2 &= \nabla_j(m)\Big(\nabla_j(m)\underline{x}_i \;-\; \underline{x}_i\nabla_j(m)\Big) \;+\; \Big(\nabla_j(m)\underline{x}_i \;-\; \underline{x}_i\nabla_j(m)\Big)\nabla_j(m) \\
\text{(using 2.14)} &= \nabla_j(m)\Big(-2m\sum_{\alpha\in\Phi^+}\tfrac{\alpha_j\alpha_i}{(\alpha,\alpha)}\,s_\alpha\Big) \;+\; \Big(-2m\sum_{\alpha\in\Phi^+}\tfrac{\alpha_j\alpha_i}{(\alpha,\alpha)}\,s_\alpha\Big)\nabla_j(m) \\
&= -2m\sum_{\alpha\in\Phi^+}\tfrac{\alpha_j\alpha_i}{(\alpha,\alpha)}\,\big(\nabla_j(m)s_\alpha \;+\; s_\alpha\nabla_j(m)\big)
\end{aligned} \tag{2.17}$$

Similarly for $j=i$ we get

$$\begin{aligned}
\nabla_i(m)^2\underline{x}_i \;-\; \underline{x}_i\nabla_i(m)^2 &= \nabla_i(m)\Big(\nabla_i(m)\underline{x}_i \;-\; \underline{x}_i\nabla_i(m)\Big) \;+\; \Big(\nabla_i(m)\underline{x}_i \;-\; \underline{x}_i\nabla_i(m)\Big)\nabla_i(m) \\
\text{(using 2.15)} &= \nabla_i(m)\Big(I-2m\sum_{\alpha\in\Phi^+}\tfrac{\alpha_i\alpha_i}{(\alpha,\alpha)}\,s_\alpha\Big) \;+\; \Big(I-2m\sum_{\alpha\in\Phi^+}\tfrac{\alpha_i\alpha_i}{(\alpha,\alpha)}\,s_\alpha\Big)\nabla_i(m) \\
&= 2\nabla_i(m)-2m\sum_{\alpha\in\Phi^+}\tfrac{\alpha_i\alpha_i}{(\alpha,\alpha)}\,\big(\nabla_i(m)s_\alpha \;+\; s_\alpha\nabla_i(m)\big)
\end{aligned} \tag{2.18}$$

Now, for a fixed $i$ summing 2.17 for all $j \neq i$ and adding 2.18 gives

$$\begin{aligned} p_2\big(\nabla(m)\big)\underline{x}_i \;-\; \underline{x}_i p_2\big(\nabla(m)\big) \;&=\; 2\nabla_i(m) \;-\; 2m \sum_{\alpha\in\Phi^+} \tfrac{\alpha_i}{(\alpha,\alpha)} \sum_{j=1}^{n} \alpha_j \left(\nabla_j(m)s_\alpha \;+\; s_\alpha \nabla_j(m)\right) \\ &=\; 2\nabla_i(m) \;-\; 2m \sum_{\alpha\in\Phi^+} \tfrac{\alpha_i}{(\alpha,\alpha)} \big(\nabla_\alpha(m)s_\alpha \;+\; s_\alpha\nabla_\alpha(m)\big) \end{aligned} \qquad 2.19$$

But 2.8 gives

$$s_\alpha \nabla_\alpha(m) \;=\; \nabla_{-\alpha}(m)s_\alpha \;=\; -\nabla_\alpha(m)s_\alpha$$

Using this in 2.19 reduces it to 2.16 completing our proof.

To derive our next identities we need some notation. To begin, given two operators $A, B$ we shall set

$$D_A B \;=\; [A,B] \;=\; AB - BA$$

It is easily seen that for any operators $A, B_1, B_2$ we have

$$D_A B_1 B_2 \;=\; (D_A B_1)B_2 \;+\; B_1 D_A B_2$$

Thus "$D_A$" acts as differentiation on the algebra of operators. More generally we have the Leibnitz identity

$$D_A^d B_1 B_2 \cdots B_r \;=\; \sum_{a_1+a_2+\cdots+a_r=d} \frac{d!}{a_1!a_2!\cdots a_r!}(D_A^{a_1}B_1)(D_A^{a_2}B_2)\cdots(D_A^{a_r}B_r) \qquad 2.20$$

To simplify our notation let us set

$$E \;=\; p_2\big(\nabla(m)\big)/2 \qquad \text{and} \qquad T_i \;=\; \nabla_i(m)$$

so that 2.16 may be simply rewritten as

$$D_E\, \underline{x}_i \;=\; T_i\,. \qquad 2.21$$

This brings us to

## Proposition 2.5

*For all positive integers $a$ we have*

$$D_E^a x_i^a \;=\; a!\,T_i^a \qquad\qquad \text{for } i = 1, 2, \ldots, n\,. \qquad 2.22$$

## Proof

For $a = 1$ 2.22 is simply 2.21. We can thus proceed by induction on $a$. So assume 2.22 true for all integers less or equal to $a$. Now note that 2.20 gives

$$D_E^{a+1}\, \underline{x}_i^{a+1} \;=\; \sum_{a_1+a_2=a+1} \frac{(a+1)!}{a_1!a_2!}(D_E^{a_1}\, \underline{x}_i^a)(D_E^{a_2}\, \underline{x}_i) \qquad 2.23$$

Now, since $E$ commutes with all $T_i$, the inductive hypothesis immediately implies that

$$D_E^r\,\underline{x}_i^s \;=\; 0 \quad \forall \;\; s\leq a \;\;\&\;\; r>s\,. \tag{2.24}$$

But this forces all the summands in 2.23 to vanish except the one corresponding to $a_1=a$. Thus 2.23 reduces to

$$D_E^{a+1}\underline{x}_i^{a+1} \;=\; \frac{(a+1)!}{a!}(D_E^a\underline{x}_i^a)(D_E\underline{x}_i) \tag{2.25}$$

and the inductive hypothesis gives

$$D_E^{a+1}\underline{x}_i^{a+1} \;=\; \frac{(a+1)!}{a!}\;a!\,T_i^a\;\;T_i \;=\; (a+1)!T_i^{a+1}\,.$$

This completes the induction and the proof.

The identity in 2.22 has the following immediate corollary.

## Proposition 2.6

*For any exponent vector $p=(p_1,p_2,\ldots,p_n)$ we have, setting $d=p_1+p_2+\cdots+p_n$*

$$D_E^d\;\;\underline{x}_1^{p_1}\underline{x}_2^{p_2}\cdots\underline{x}_n^{p_n} \;=\; d!\;\;T_1^{p_1}T_2^{p_2}\cdots T_n^{p_n} \tag{2.26}$$

## Proof

This is another application of 2.20. Indeed using 2.20 we get

$$D_E^d\,\underline{x}_1^{p_1}\,\underline{x}_2^{p_2}\cdots\underline{x}_n^{p_n} \;=\; \sum_{a_1+a_2+\cdots+a_n=d}\frac{d!}{a_1!a_2!\cdots a_n!}D_E^{a_1}\underline{x}_1^{p_1}\;D_E^{a_2}\underline{x}_2^{p_2}\;\cdots D_E^{a_n}\underline{x}_n^{p_n}$$

But, now again since $E$ commutes with all $T_i$, the identity in 2.22 forces this sum to reduce to the single term where each $a_i=p_i$. Thus

$$D_E^d\,\underline{x}_1^{p_1}\,\underline{x}_2^{p_2}\cdots\underline{x}_n^{p_n} \;=\; \frac{d!}{p_1!p_2!\cdots p_n!}D_E^{p_1}\underline{x}_1^{p_1}\;D_E^{p_2}\underline{x}_2^{p_2}\;\cdots D_E^{p_n}\underline{x}_n^{p_n}$$

and an application of 2.22 to each of the factors on the right yields 2.26 precisely as asserted.

We are now finally in a position to prove Proposition 1.3, which we restate as

## Theorem 2.1

*For any homogeneous polynomial $P(x_1,x_2,\ldots,x_n)$ we have the operator identity*

$$\frac{1}{2^d}\sum_{r=0}^{d}\binom{d}{r}(-1)^r\big(\textstyle\sum_{i=1}^n T_i^2\big)^{d-r}P(\underline{x})\big(\textstyle\sum_{i=1}^n T_i^2\big)^r \;=\; d!\;P\big(T_1,T_2,\ldots,T_n\big) \tag{2.27}$$

*where $d=degree(P)$ and "$P(\underline{x})$" denotes the operator "multiplication by $P$".*

## Proof

If our polynomial $P(x_1,x_2,\ldots,x_n)$ has the expansion

$$P(x_1,x_2,\ldots,x_n) \;=\; \sum_p c_p x_1^{p_1} x_2^{p_2}\cdots x_n^{p_n}$$

where the sum is over all monomials of degree $d$ then by linearity from 2.26 we derive that

$$D_E^d P(\underline{x}) \;=\; d!\, P\big(T_1,T_2,\ldots,T_n\big) \qquad 2.28$$

on the other hand a straightforward induction argument yields that for any two operators $A,B$ we have

$$D_A^d B \;=\; \sum_{r=0}^{d}\binom{d}{r}(-1)^r A^{d-r}BA^r. \qquad 2.29$$

Since

$$E \;=\; \tfrac{1}{2}\textstyle\sum_{i=1}^n T_i^2$$

we see that 2.29 with $A=E$ and $B=P(\underline{x})$ reduces to the left hand side of 2.27. This given 2.27 is an immediate consequence of 2.28. This completes our proof of 2.27.

We terminate this section by showing that what we did in for $S_n$ section 1 can be carried out almost verbatim for all reflection groups as long as we are in possession of the corresponding analogue of the Mehta integral. In fact it was conjectured by Macdonald in [14] that for a Coxeter group $W$ of isometries of $\mathbb{R}^n$ we have

$$\tfrac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-|x|^2/2}\Pi_W(x)^{2k}\,dx_1dx_2\cdots dx_n \;=\; \prod_{i=1}^n\frac{(kd_i)!}{k!} \qquad 2.30$$

with $d_1,d_2,\ldots,d_n$ are the degrees of the fundamental invariants of $W$, and

$$\Pi_W(x) \;=\; \prod_{\alpha\in\Phi^+}(\alpha,x), \qquad 2.31$$

here we denote by $\Phi^+$ a complete collection of reflecting vectors of $W$ normalized by the requirement that $(\alpha,\alpha)=2\;\forall\alpha\in\Phi^+$. We should mention that 2.30 for $\mathcal{B}_n$ and $\mathcal{D}_n$ was first proved by Regev [14] who showed that also in these cases it is a consequence of the Selberg integral. Accordingly, we will prove 2.30 here only for $S_n$, $B_n$ and $D_n$. For the other Coxeter groups we shall assume it to be true and refer to the original papers for a proof.

This given we begin by noting that we have a complete analogue of the Zeilberger identity of Proposition 1.1.

## Proposition 2.7

*For any polynomial $P(x)$ we have*

$$\sum_{i=1}^n \partial_{x_i}\Big((\partial_{x_i}P)e^{-|x|^2/2}\Pi_W(x)^{2k}\Big) \;=\; \Big(\Delta_{W,k}P-\sum_{i=1}^n x_i\partial_{x_i}P\Big)e^{-|x|^2/2}\Pi_W(x)^{2k}\,. \qquad 2.32$$

*where*

$$\Delta_{W,k} \;=\; \Delta \,+\, 2k \sum_{\alpha\in\Phi^+} \tfrac{1}{(\alpha,x)}\,\partial_\alpha \tag{2.33}$$

*with* $\Delta=\sum_{i=1}^n \partial_{x_i}^2$ *the ordinary Laplacian.*

## Proof

Note that for any $P(x)$ we have

$$\partial_{x_i}\Big((\partial_{x_i}P)e^{-|x|^2/2}\Pi_W(x)^{2k}\Big) = (\partial_{x_i}^2 P)e^{-|x|^2/2}\Pi_W(x)^{2k} - x_i(\partial_{x_i}P)e^{-|x|^2/2}\Pi_W(x)^{2k} \,+\, (\partial_{x_i}P)e^{-|x|^2/2}\partial_{x_i}\Pi_W(x)^{2k}$$

But

$$\partial_{x_i}\Pi_W(x)^{2k} \;=\; \Pi_W(x)^{2k}\partial_{x_i}\log\big(\Pi_W(x)^{2k}\big) \;=\; 2k\Pi_W(x)^{2k}\sum_{\alpha\in\Phi^+}\tfrac{\partial_{x_i}(\alpha,x)}{(\alpha,x)}$$

thus

$$\begin{aligned}\sum_{i=1}^n \partial_{x_i}\Big((\partial_{x_i}P)e^{-|x|^2/2}\Pi_W(x)^{2k}\Big) \;=&\; (\Delta P)e^{-|x|^2/2}\Pi_W(x)^{2k}\\ &-\Big(\sum_{i=1}^n x_i\partial_{x_i}P\Big)e^{-|x|^2/2}\Pi_W(x)^{2k}\\ &+\Big(2k\sum_{i=1}^n(\partial_{x_i}P)\sum_{\alpha\in\Phi^+}\tfrac{\alpha_i}{(\alpha,x)}\Big)e^{-|x|^2/2}\Pi_W(x)^{2k}\end{aligned} \tag{2.34}$$

and since

$$2k\sum_{i=1}^n(\partial_{x_i}P)\sum_{\alpha\in\Phi^+}\tfrac{\alpha_i}{(\alpha,x)} \;=\; 2k\sum_{\alpha\in\Phi^+}\tfrac{1}{(\alpha,x)}\partial_\alpha P$$

we see that 2.34 may be simply written as 2.32.

As in section 1 it follows from this that

## Proposition 2.8

*If $I_k$ denotes the Macdonald-Mehta integral in the left hand side of 2.30, then with $d=\sum_{i=1}^n(d_i-1)$* we have

$$\frac{1}{2^d d!}\,\Delta_{W,k}^d\,\Pi(x)^2\,I_k \;=\; I_{k+1}. \tag{2.35}$$

## Proof

Here again we simply set

$$P(x) \;=\; \sum_{r=0}^{d-1}\frac{\Delta_{W,k}^r\,\Pi_W(x)^2}{2^{r+1}d(d-1)\cdots(d-r)} \tag{2.36}$$

and derive 2.32 by the same identical steps we carried out in the proof of Proposition 1.2.

Assuming the Macdonald-Mehta identity, as before, we immediately derive that

## Theorem 2.2

$$\frac{1}{2^d d!}\,\Delta_{W,k}^d\,\Pi_W(x)^2 \;=\; d_1 d_2\cdots d_n \prod_{j=1}^n \prod_{1\le i<d_j}(kd_j+i) \qquad 2.37$$

## Proof

From 2.30 and 2.35 we get

$$\begin{aligned}\frac{1}{2^d d!}\,\Delta_{W,k}^d\,\Pi_W(x)^2 &= \prod_{j=1}^n \frac{(kd_j+d_j)!k!}{kd_j!(k+1)!}\\ &= \frac{1}{(k+1)^n}\prod_{j=1}^n (kd_j+d_j)(kd_j+d_j-1)\cdots(kd_j+1)\\ &= \Big(\prod_{j=1}^n \frac{(kd_j+d_j)}{(k+1)}\Big)\prod_{j=1}^n (kd_j+d_j-1)\cdots(kd_j+1)\\ &= \prod_{j=1}^n d_j \prod_{j=1}^n \prod_{1\le i<d_j}(kd_j+i)\end{aligned} \qquad 2.38$$

proving 2.37

Again we can translate 2.37 into a Dunkl operator identity using the following analogue of Proposition 1.3.

## Proposition 2.9

*The actions of the operators*

$$\Delta_{W,k} \;=\; \Delta + 2k\sum_{\alpha\in\Phi^+}\tfrac{1}{(\alpha,x)}\partial_\alpha \qquad \textit{and} \qquad p_2(\nabla(-k)) = \nabla_1(-k)^2+\nabla_2(-k)^2+\cdots+\nabla_n(-k)^2$$

*on $W$-invariant polynomials are identical.*

## Proof

Note that if $f(x)$ is $W$-invariant then for any $1\le i\le n$ from 2.13 for $m=-k$ we get

$$\nabla_i(-k)f(x) \;=\; \partial_{x_i}f(x)$$

Thus

$$\begin{aligned}\nabla_i(-k)^2 f(x) &= \partial_{x_i}^2 f(x) \;+\; k\sum_{\alpha\in\Phi^+}\tfrac{(\alpha,e_i)}{(\alpha,x)}(1-s_\alpha)\partial_{x_i}f(x)\\ &= \partial_{x_i}^2 f(x) \;+\; k\sum_{\alpha\in\Phi^+}\tfrac{\alpha_i}{(\alpha,x)}\big(\partial_{x_i}-s_\alpha\partial_{x_i}s_\alpha\big)f(x)\end{aligned} \qquad 2.39$$

But we can write

$$\partial_{x_i}-s_\alpha\partial_{x_i}s_\alpha \;=\; \partial_{e_i}-s_\alpha\partial_{e_i}s_\alpha \qquad 2.40$$

and from 2.7 a) it follows that

$$\partial_{e_i}-s_\alpha\partial_{e_i}s_\alpha \;=\; \partial_{e_i}\;-\;\Big(\partial_{e_i}-\tfrac{2(\alpha,e_i)}{(\alpha,\alpha)}\,\partial_\alpha\Big) \;=\; \tfrac{2(\alpha,e_i)}{(\alpha,\alpha)}\,\partial_\alpha$$

Combining this with 2.40 and 2.39 gives

$$\nabla_i(-k)^2 f(x) \;=\; \partial_{x_i}^2 f(x) \;+\; k \sum_{\alpha\in\Phi^+} \tfrac{1}{(\alpha,x)}\tfrac{2\alpha_i^2}{(\alpha,\alpha)}\,\partial_\alpha f(x)$$

Thus summing over $i$ we get

$$\sum_{i=1}^n \nabla_i(-k)^2 f(x) \;=\; \Delta f(x) \;+\; 2k \sum_{\alpha\in\Phi^+} \tfrac{1}{(\alpha,x)}\,\partial_\alpha f(x)$$

This completes our proof.

This given we have

## Theorem 2.3

$$\Pi_W\big(\nabla(m)\big)\Pi_W(x) \;=\; d_1 d_2\cdots d_n(-1)^{\sum_{j=1}(d_j-1)} \prod_{j=1}^n \prod_{1\le i<d_j} (md_j - i) \qquad 2.41$$

## Proof

Again setting $T_i=\nabla_i(-k)$ from Theorem 2.1 it follows that for $d=\sum_{j=1}^n(d_j-1)$ we have

$$\frac{1}{2^d d!}\sum_{r=0}^d \binom{d}{r}(-1)^r\big(\textstyle\sum_{i=1}^n T_i^2\big)^{d-r}\Pi_W(\underline{x})\big(\textstyle\sum_{i=1}^n T_i^2\big)^r\Pi_W(x) \;=\; \Pi_W\big(T_1,T_2,\ldots,T_n\big)\Pi_W(x) \qquad 2.42$$

Since $\sum_{i=1}^n T_i^2$is a $W$-invariant operator, the polynomial $\big(\sum_{i=1}^n T_i^2\big)^r\Pi_W(x)$ is a $W$-alternant, and thus it must necessarily identically vanish for any $r>0$. Thus 2.42 reduces to

$$\frac{1}{2^d d!}\big(\sum_{i=1}^n T_i^2\big)^d\Pi_W(x)^2 \;=\; \Pi_W\big(T_1,T_2,\ldots,T_n\big)\Pi_W(x) \qquad 2.43$$

and since $\Pi_W(x)^2$ is a $W$-invariant from Proposition 2.9 it follows that

$$\frac{1}{2^d d!}\big(\sum_{i=1}^n T_i^2\big)^d\Pi_W(x)^2 \;=\; \frac{1}{2^d d!}\Delta_{W,k}^d\Pi_W(x)^2$$

and 2.43 becomes

$$\Pi_W\big(T_1,T_2,\ldots,T_n\big)\Pi_W(x) \;=\; \frac{1}{2^d d!}\Delta_{W,k}^d\Pi_W(x)^2.$$

Thus 2.37 may be rewritten as

$$\Pi_W\big(\nabla_i(-k),\nabla_i(-k),\ldots,\nabla_i(-k)\big)\Pi_W(x) \;=\; d_1 d_2\cdots d_n \prod_{j=1}^n\prod_{1\le i<d_j}(kd_j+i)$$

Since both sides are polynomials in $k$, the validity of this identity for all positive integers $k$ forces the equality of the two polynomials. This allows us to replace $k$ by $-m$ and obtain 2.41 precisely as stated.

## 3. The Selberg integral

Our task in this section is to present the evaluation of the following multiple integral

$$J_n(x,y,k) \;=\; \int_0^1\int_0^1\cdots\int_0^1 \Big(\prod_{i=1}^n t_i^{x-1}(1-t_i)^{y-1}\Big)\prod_{1\le i<j\le n}(t_i-t_j)^{2k}dt_1dt_2\cdots dt_n \qquad 3.1$$

We shall establish the following fundamental identity due to Selberg

### Theorem 3.1

$$J_n(x,y,k) \;=\; \prod_{j=1}^n \frac{\Gamma\big(x+(j-1)k\big)\Gamma\big(y+(j-1)k\big)}{\Gamma\big(x+y+(n+j-2)k\big)}\prod_{j=1}^n\frac{\Gamma(1+jk)}{\Gamma(1+k)}\,.$$

Our presentation follows very closely Selberg' s original argument as presented by Andrews in [1]. The ideas are very simple, in principle, yet as we shall see, when all the (usually skipped) details are included, it does end up taking quite a few pages.

To begin we must recall the following well known identities satisfied by the Gamma function.

$$a)\quad \Gamma(x+1)=x\Gamma(x)\,,\qquad b)\quad \Gamma(x)\Gamma(1-x)=\frac{\pi}{sin\,\pi x}\,,\qquad c)\quad \Gamma(1)=1 \qquad 3.2$$

and

$$\int_0^1 t^{x-1}(1-t)^{y-1}dt \;=\; \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,. \qquad 3.3$$

Note that it follows from 3.2 a) that for any integer $n$ we have

$$\Gamma(x+n) \;=\; (x)\uparrow^n\Gamma(x) \qquad 3.4$$

where for convenience we set

$$(x)\uparrow^n \;=\; (x+n-1)(x+n-2)\cdots(x+1)$$

The crucial first step is a remarkably simple observation about the expansion of even powers of the Vandermonde determinant

$$\Pi(t_1,t_2,\ldots,t_n) \;=\; \prod_{1\le i<j\le n}(t_i-t_j)\,.$$

## Proposition 3.1

*Let*

$$\Pi(t_1,t_2,\ldots,t_n)^{2k} \;=\; \sum_\alpha c(\alpha)\; t_1^{\alpha_1}t_2^{\alpha_2}\cdots t_n^{\alpha_n} \tag{3.5}$$

*then when* $\alpha_1\le\alpha_2\le\cdots\le\alpha_n$ *we have*

$$(j-1)k \;\le\; \alpha_j \;\le\; (n-2)k+jk \tag{3.6}$$

## Proof

Note that for any $1\le j\le n$ we have

$$\Pi(t_1,t_2,\ldots,t_n)^{2k} \;=\; \Pi(t_1,t_2,\ldots,t_j)^{2k}\Big(\prod_{1\le r\le j}\prod_{j<s\le n}(t_r-t_s)^{2k}\Big)\Pi(t_{j+1},\ldots,t_n)^{2k} \tag{3.7}$$

Note further that for any term of the expansion

$$\Pi(t_1,t_2,\ldots,t_j)^{2k} \;=\; \sum_\alpha c(\alpha)\; t_1^{\alpha_1}t_2^{\alpha_2}\cdots t_j^{\alpha_j} \tag{3.8}$$

we shall necessarily have

$$j\max_{1\le i\le j}\alpha_i \ge \alpha_1+\alpha_2+\cdots+\alpha_j \;=\; 2\,k\binom{j}{2} \;=\; k\,j(j-1)$$

or better

$$\max_{1\le i\le j}\alpha_i \ge k\,(j-1)$$

and the left hand side of 3.6 immediately follows from the factorization in 3.7. To get the other side we use the identity

$$\Pi(t_1,t_2,\ldots,t_n)^{2k} \;=\; (-1)^{\binom{n}{2}}\Pi(t_1^{-1},t_2^{-1},\ldots,t_n^{-1})(t_1t_2\cdots t_n)^{n-1}$$

and derive from 3.5 that

$$\Pi(t_1,t_2,\ldots,t_n)^{2k} \;=\; (-1)^{\binom{n}{2}}\sum_\alpha c(\alpha)\; t_1^{2k(n-1)-\alpha_1}t_2^{2k(n-1)-\alpha_2}\cdots t_n^{2k(n-1)-\alpha_n}$$

so for $\alpha_1\le\alpha_2\le\cdots\le\alpha_n$ we derive that

$$2k(n-1)-\alpha_1\ge 2k(n-1)-\alpha_2\ge\cdots\ge 2k(n-1)-\alpha_n$$

and the left hand side of 3.6 gives

$$2k(n-1)-\alpha_{n+1-j}\ge (j-1)k$$

or better

$$k(2n-j-1)\ge \alpha_{n+1-j}$$

and this gives

$$k(2n-(n+1-j)-1)\geq \alpha_j$$

which is another way of writing the right hand side of (6).

Now substituting the expansion in 3.5 in the definition of the Selbert integral we get using 3.3

$$J_n(x,y;k) \;=\; \sum_{\alpha} c(\alpha)\prod_{i=1}^{n}\frac{\Gamma(x+\alpha_i)\,\Gamma(y)}{\Gamma(x+y+\alpha_i)} \tag{3.9}$$

Now when $\alpha_1\leq\alpha_2\leq\cdots\leq\alpha_n$ from 3.6 and 3.4 we derive that

$$\Gamma(x+\alpha_j) \;=\; \Gamma(x+(j-1)k)\big(x+(j-1)k\big)\uparrow^{\alpha_j-(j-1)k} \tag{3.10}$$

similarly we also derive from 3.6 and 3.4

$$\Gamma(x+y+(n+j-2)k) \;=\; \Gamma(x+y+\alpha_j)\big(x+y+\alpha_j\big)\uparrow^{(n+j-2)k-\alpha_j} \tag{3.11}$$

In summary we can write, using 3.10 and 3.11

$$\prod_{j=1}^{n}\frac{\Gamma(x+\alpha_j)\,\Gamma(y)}{\Gamma(x+y+\alpha_j)} \;=\; \Big(\prod_{j=1}^{n}\frac{\Gamma\big(x+(j-1)k)\Gamma(y)\big)}{\Gamma\big(x+y+(n+j-2)k\big)}\Big)P_\alpha(x,y;k) \tag{3.12}$$

where for convenience we have set

$$P_\alpha(x,y;k) \;=\; \prod_{j=1}^{n}\big(x+(j-1)k\big)\uparrow^{\alpha_j-(j-1)k}\,\big(x+y+\alpha_j\big)\uparrow^{(n+j-2)k-\alpha_j}\ . \tag{3.13}$$

Now in view of the symmetry of this expression in $\alpha_1,\alpha_2,\ldots,\alpha_n$ and the invariance of the coefficients $c(\alpha)$ under permutations of $\alpha_1,\alpha_2,\ldots,\alpha_n$ we can use 3.12 in every term of 3.9 and obtain

$$J_n(x,y;k) \;=\; \Big(\prod_{j=1}^{n}\frac{\Gamma\big(x+(j-1)k)\Gamma(y)\big)}{\Gamma\big(x+y+(n+j-2)k\big)}\Big)\sum_{\alpha}c(\alpha)P_\alpha(x,y;k)\,. \tag{3.14}$$

Our next task is to determine the polynomial

$$P(x,y;k) \;=\; \sum_{\alpha}c(\alpha)P_\alpha(x,y;k)\,. \tag{3.15}$$

To this end it is convenient to use the identity

$$\Gamma(y) \;=\; \frac{\Gamma\big(y+(j-1)k\big)}{(y)\uparrow^{(j-1)k}}$$

and rewrite 3.14 in the form

$$J_n(x,y;k) \;=\; \Big(\prod_{j=1}^{n}\frac{\Gamma\big(x+(j-1)k)\Gamma\big(y+(j-1)k)\big)}{\Gamma\big(x+y+(n+j-2)k\big)}\Big)\frac{P(x,y;k)}{\prod_{j=1}^{n}(y)\uparrow^{(j-1)k}}\,. \tag{3.16}$$

However note that the change of variables $t_i \to 1 - t_i$ in 3.1 immediately proves that

$$J_n(x, y; k) \;=\; J_n(y, x; k)$$

Combining this with 3.16 shows that we must have

$$\frac{P(x, y; k)}{\prod_{j=1}^{n} (y) \uparrow^{(j-1)k}} \;=\; \frac{P(y, x; k)}{\prod_{j=1}^{n} (x) \uparrow^{(j-1)k}}$$

or better

$$P(x, y; k) \prod_{j=1}^{n} (x) \uparrow^{(j-1)k} \;=\; P(y, x; k) \prod_{j=1}^{n} (y) \uparrow^{(j-1)k} \qquad 3.17$$

Now it is easily seen from 3.13 and 3.5 that $P(x, y; k)$ is a polynomial in $y$ of degree at most

$$\sum_{j=1} \big((n + j - 2)k - \alpha_j\big) \;=\; n(n-1)k + n(n-1)k/2 - 2\tbinom{n}{2}k \;=\; \tbinom{n}{2}k$$

Since the degree in $y$ of $\prod_{j=1}^{n} (y) \uparrow^{(j-1)k}$ is also $\binom{n}{2}k$ we immediately derive from 3.17 that for some polynomial $R(x, k)$ we must have

$$P(x, y; k) \;=\; R(x, k) \prod_{j=1}^{n} (y) \uparrow^{(j-1)k} \;. \qquad 3.18$$

But we can interchange $x, y$ in this relation and obtain that we must also have

$$P(y, x; k) \;=\; R(y, k) \prod_{j=1}^{n} (x) \uparrow^{(j-1)k} \;. \qquad 3.19$$

Using 3.18 and 3.19 in 3.17 reduces it to

$$R(x, k) \prod_{j=1}^{n} (y) \uparrow^{(j-1)k} \prod_{j=1}^{n} (x) \uparrow^{(j-1)k} \;=\; R(y, k) \prod_{j=1}^{n} (x) \uparrow^{(j-1)k} \prod_{j=1}^{n} (y) \uparrow^{(j-1)k}$$

Cancelling the common factors yields

$$R(x, k) \;=\; R(y, k)$$

and this can only hold true when $R(x; k)$ does not depend on $x$. In other words we can now conclude from 3.16 and 3.18 that for some polynomial $R_n(k)$ we must have

$$J_n(x, y; k) \;=\; R_n(k) \prod_{j=1}^{n} \frac{\Gamma\big(x + (j-1)k\big)\Gamma\big(y + (j-1)k\big)}{\Gamma\big(x + y + (n + j - 2)k\big)} \,. \qquad 3.20$$

The next step is to obtain a recursion for $R_n(k)$.

To this end we begin by noting that the integrand in 3.1 is a symmetric function of $t_1, t_2, \ldots, t_n$. This permits us to break up the integral into a sum of identical terms obtained by separately integrating over the images of the simplex $0 \leq t_n \leq t_{n-1} \leq \cdots \leq t_1 \leq 1$ under the action of the symmetric group $S_n$. Since these images decompose the unit $n$-dimensional cube into $n!$ simplices we may write

$$J_n(x,y;k) \;=\; n! \int_0^1 \int_{t_n}^1 \int_{t_{n-1}}^1 \cdots \int_{t_2}^1 \prod_{i=1}^n t_i^{x-1}(1-t_i)^{y-1} \prod_{1\leq i<j\leq n} (t_i-t_j)^{2k} dt_1 dt_2 \cdots dt_{n-1} dt_n \tag{3.21}$$

Now let us set

$$\begin{aligned} f(t_n,x) \;&=\; (1-t_n)^{y-1}\times \\ &\int_{t_n}^1 \int_{t_{n-1}}^1 \cdots \int_{t_2}^1 \prod_{i=1}^{n-1} t_i^{x-1}(1-t_i)^{y-1} \prod_{i=1}^{n-1}(t_i-t_n)^{2k} \prod_{1\leq i<j\leq n-1} (t_i-t_j)^{2k} dt_1 dt_2 \cdots dt_{n-1} \end{aligned} \tag{3.22}$$

so 3.21 becomes

$$J_n(x,y;k) \;=\; n! \int_0^1 t_n^{x-1} f(t_n,x)\, dt_n\,.$$

An integration by parts then gives (for $x, y > 0$ )

$$\begin{aligned} xJ_n(x,y;k) \;&=\; n!\Big( t_n^x f(t_n,x)\big|_0^1 \;-\; \int_0^1 t_n^x \partial_{t_n} f(t_n,x)\, dt_n\Big) \\ &=\; -n! \int_0^1 t_n^x \partial_{t_n} f(t_n,x)\, dt_n \\ &=\; n! \int_0^1 (1-t_n^x)\partial_{t_n} f(t_n,x)\, dt_n \;-\; n! \int_0^1 \partial_{t_n} f(t_n,x)\, dt_n \\ &=\; n! \int_0^1 (1-t_n^x)\partial_{t_n} f(t_n,x)\, dt_n \;-\; n!\Big(f(1,x) \;-\; f(0,x)\Big)\,. \end{aligned}$$

Since the definition in 3.22 immediately gives that $f(1,x) = 0$ this reduces to

$$xJ_n(x,y;k) \;=\; n! \int_0^1 (1-t_n^x)\partial_{t_n} f(t_n,x)\, dt_n \;\;+\;\; n!\, f(0,x)\,. \tag{3.23}$$

We aim to take the limit in 3.23 as $x \to 0$. To begin note that 3.22 gives (for $x > 0$)

$$f(0,x) \;=\; \int_0^1 \int_{t_{n-1}}^1 \cdots \int_{t_2}^1 \prod_{i=1}^{n-1} t_i^{x+2k-1}(1-t_i)^{y-1} \prod_{1\leq i<j\leq n-1} (t_i-t_j)^{2k} dt_1 dt_2 \cdots dt_{n-1}$$

and so for $k \geq 1$ we have

$$\begin{aligned} \lim_{x\to 0} f(0,x) \;&=\; \int_0^1 \int_{t_{n-1}}^1 \cdots \int_{t_2}^1 \prod_{i=1}^{n-1} t_i^{2k-1}(1-t_i)^{y-1} \prod_{1\leq i<j\leq n-1} (t_i-t_j)^{2k} dt_1 dt_2 \cdots dt_{n-1} \\ &=\; \tfrac{1}{(n-1)!} J_{n-1}(2k,y,k) \;=\; \tfrac{1}{(n-1)!} R_{n-1}(k) \prod_{j=1}^{n-1} \frac{\Gamma\big(2k+(j-1)k\big)\Gamma\big(y+(j-1)k\big)}{\Gamma\big(2k+y+(n-1+j-2)k\big)}\,. \end{aligned} \tag{3.24}$$

Note next that 3.2 b) and c) give

$$\lim_{x\to 0} x\Gamma(x) \;=\; \lim_{x\to 0} \frac{x\pi}{sin\,\pi x} \;=\; 1\,.$$

and so from 3.20 we derive that

$$\begin{aligned}\lim_{x\to 0} xJ_n(x,y;k) \;&=\; R_n(k)\lim_{x\to 0} x\Gamma(x)\Gamma(y)\,\frac{\prod_{j=2}^{n}\Gamma\big(x+(j-1)k\big)\Gamma\big(y+(j-1)k\big)\big)}{\prod_{j=1}^{n}\Gamma\big(x+y+(n+j-2)k\big)}\\ &=\; R_n(k)\,\Gamma(y)\,\frac{\prod_{j=2}^{n}\Gamma\big((j-1)k\big)\Gamma\big(y+(j-1)k\big)\big)}{\prod_{j=1}^{n}\Gamma\big(y+(n+j-2)k\big)}\,.\end{aligned} \qquad 3.25$$

As may be suspected, the first term in the right hand side of 3.23 will bear no contribution, since it turns out that we do have

$$\lim_{x\to 0}\int_0^1 (1-t_n^x)\partial_{t_n} f(t_n,x)\,dt_n \;=\; 0\,. \qquad 3.26$$

This passage to the limit under the integral sign is somewhat delicate and it is usually skipped in most presentations. For sake of completeness we shall carry it out here in full detail. But before we do that it will be good to see what 3.23 yields us. To this end note that combining 3.23, 3.24, 3.25 and 3.26 we derive that

$$R_n(k)\,\Gamma(y)\,\frac{\prod_{j=2}^{n}\Gamma\big((j-1)k\big)\Gamma\big(y+(j-1)k\big)\big)}{\prod_{j=1}^{n}\Gamma\big(y+(n+j-2)k\big)} \;=\; \tfrac{n!}{(n-1)!}R_{n-1}(k)\prod_{j=1}^{n-1}\frac{\Gamma\big(2k+(j-1)k\big)\Gamma\big(y+(j-1)k\big)\big)}{\Gamma\big(2k+y+(n-1+j-2)k\big)}\,.$$

Cancelling the obvious common factors this reduces to

$$R_n(k)\;\frac{\Gamma\big(y+(n-1)k\big)\prod_{j=2}^{n}\Gamma\big((j-1)k\big)}{\prod_{j=1}^{n}\Gamma\big(y+(n+j-2)k\big)} \;=\; n\,R_{n-1}(k)\frac{\prod_{j=1}^{n-1}\Gamma\big((j+1)k\big)}{\prod_{j=1}^{n-1}\Gamma\big(y+(n-1+j)k\big)}\,.$$

Now a simple manipulation of indices in these products gives

$$R_n(k)\;\frac{\Gamma\big(y+(n-1)k\big)\prod_{j=1}^{n-1}\Gamma\big(jk\big)}{\prod_{j=0}^{n-1}\Gamma\big(y+(n+j-1)k\big)} \;=\; n\,R_{n-1}(k)\frac{\prod_{j=2}^{n}\Gamma(jk)}{\prod_{j=1}^{n-1}\Gamma\big(y+(n-1+j)k\big)}\,,$$

and miraculously all dependence on $y$ disappears yieding the simple recursion

$$R_n(k) \;=\; n\,R_{n-1}(k)\frac{\Gamma(nk)}{\Gamma(k)} \;=\; R_{n-1}(k)\;\frac{kn\Gamma(nk)}{k\Gamma(k)} \;=\; R_{n-1}(k)\frac{\Gamma(nk+1)}{\Gamma(k+1)}\,. \qquad 3.27$$

Note that setting $n=1$ in 3.1 and using 3.3 we get

$$J_1(x,y,k) \;=\; \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

On the other hand doing the same in 3.20 gives

$$J_1(x,y,k) \;=\; \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}R_1(k)$$

Thus we must take $R_1(k)=1$. This given, successive applications of 3.27 finally yields

$$R_n(k) \;=\; \prod_{j=1}^{n} \frac{\Gamma(jk+1)}{\Gamma(k+1)}\,. \tag{3.28}$$

We clearly see then that 3.20 combined with 3.28 will complete the proof of Theorem 3.1 once we verify 3.26. To this end it is convenient to set for a moment

$$G(t_n,t_{n-1},x)=\int_{t_{n-1}}^1\cdots\int_{t_2}^1 \prod_{i=1}^{n-1} t_i^{x-1}(1-t_i)^{y-1}\prod_{i=1}^{n-1}(t_i-t_n)^{2k}\prod_{1\le i<j\le n-1}(t_i-t_j)^{2k}dt_1dt_2\cdots dt_{n-2}\,. \tag{3.29}$$

so that 3.22 may be written in the form

$$f(t_n,x) \;=\; (1-t_n)^{y-1}\int_{t_n}^1 G(t_n,t_{n-1},x)dt_{n-1} \tag{3.30}$$

with

This gives

$$\begin{aligned}\partial_{t_n} f(t_n,x) \;&=\; (y-1)(1-t_n)^{y-2}\int_{t_n}^1 G(t_n,t_{n-1},x)dt_{n-1}\\ &\qquad -(1-t_n)^{y-1}G(t_n,t_n,x)\\ &\qquad +\;(1-t_n)^{y-1}\int_{t_n}^1 \partial_{t_n} G(t_n,t_{n-1},x)dt_{n-1}\end{aligned}$$

Now we immediately see from 3.29 that $G(t_n,t_n,x)=0$ and so we may write

$$\partial_{t_n} f(t_n,x) \;=\; A(t_n,x) \;+\; B(t_n,x) \tag{3.31}$$

with

$$A(t_n,x) \;=\; (y-1)(1-t_n)^{y-2}\int_{t_n}^1\int_{t_{n-1}}^1\cdots\int_{t_2}^1 \prod_{i=1}^{n-1} t_i^{x-1}(1-t_i)^{y-1}\prod_{i=1}^{n-1}(t_i-t_n)^{2k}\prod_{1\le i<j\le n-1}(t_i-t_j)^{2k}dt_1dt_2\cdots dt_{n-1} \tag{3.32}$$

and

$$B(t_n,x) \;=\; (1-t_n)^{y-1}\int_{t_n}^1\int_{t_{n-1}}^1\cdots\int_{t_2}^1 \prod_{i=1}^{n-1} t_i^{x-1}(1-t_i)^{y-1}\Big(\partial_{t_n}\prod_{i=1}^{n-1}(t_i-t_n)^{2k}\Big)\prod_{1\le i<j\le n-1}(t_i-t_j)^{2k}dt_1dt_2\cdots dt_{n-1} \tag{3.33}$$

Since in the process of proving the recursion in 3.27 the dependence on $y$ disappeared in the end, there is no loss at this point to assume that $y>2$. This given it follows that

$$(1-t_n)^{y-2}\le 1$$

and, using $t_i-t_n\le t_i$, we get from 3.32

$$\big|A(t_n,x)\big|\le |y-1|\int_{t_n}^1\int_{t_{n-1}}^1\cdots\int_{t_2}^1 \prod_{i=1}^{n-1} t_i^{x+2k-1}(1-t_i)^{y-1}\prod_{1\le i<j\le n-1}(t_i-t_j)^{2k}dt_1dt_2\cdots dt_{n-1}$$

and for $k\geq 1$ we can let $t_n\rightarrow 0$ to obtain a final estimate which is a scalar independent of $x$:

$$\big|A(t_n,x)\leq |y-1|\int_0^1\int_{t_{n-1}}^1\cdots\int_{t_2}^1\prod_{i=1}^{n-1}t_i^{2k-1}(1-t_i)^{y-1}\prod_{1\leq i<j\leq n-1}(t_i-t_j)^{2k}dt_1dt_2\cdots dt_{n-1}\,. \qquad 3.34$$

Now note that we have

$$\begin{aligned}\partial_{t_n}\prod_{i=1}^{n-1}(t_i-t_n)^{2k} &= \prod_{i=1}^{n-1}(t_i-t_n)^{2k}\partial_{t_n}log\Big(\prod_{j=1}^{n-1}(t_j-t_n)^{2k}\Big)\\ &= 2k\prod_{i=1}^{n-1}(t_i-t_n)^{2k}\sum_{j=1}^{n-1}\partial_{t_n}log(t_j-t_n)\\ &= -2k\prod_{i=1}^{n-1}(t_i-t_n)^{2k}\sum_{j=1}^{n-1}\frac{1}{t_j-t_n}\end{aligned}$$

thus

$$|\partial_{t_n}\prod_{i=1}^{n-1}(t_i-t_n)^{2k}|\leq 2k\sum_{j=1}^{n}(t_j-t_n)^{2k-1}\prod_{i=1;i\neq j}^{n-1}(t_i-t_n)^{2k}\leq 2k\sum_{j=1}^{n}t_j^{2k-1}\prod_{i=1;i\neq j}^{n-1}t_i^{2k}\,.$$

Using this in 3.33 we get for $x,y>0$ and $k\geq 1$

$$\begin{aligned}|B(t_n,x)| &\leq 2k\sum_{j=1}^{n}\int_{t_n}^1\int_{t_{n-1}}^1\cdots\int_{t_2}^1 t_j^{x+2k-2}(1-t_j)^{y-1}\prod_{i=1;i\neq j}^{n-1}t_i^{x+2k-1}(1-t_i)^{y-1}\prod_{1\leq i<j\leq n-1}(t_i-t_j)^{2k}dt_1dt_2\cdots dt_{n-1}\,.\\ &\leq 2k\sum_{j=1}^{n}\int_0^1\int_{t_{n-1}}^1\cdots\int_{t_2}^1 t_j^{2k-2}(1-t_j)^{y-1}\prod_{i=1;i\neq j}^{n-1}t_i^{2k-1}(1-t_i)^{y-1}\prod_{1\leq i<j\leq n-1}(t_i-t_j)^{2k}dt_1dt_2\cdots dt_{n-1}\,.\end{aligned} \qquad 3.35$$

Combining 3.31, 3.34 and 3.35 we derive that $|\partial_{t_n}f(t_n,x)|$ is bounded by a scalar independent of $x$. Thus we can pass to the limit under the integral sign and conclude that

$$\lim_{x\rightarrow 0}\int_0^1(1-t_n^x)\partial_{t_n}f(t_n,x)\,dt_n \;=\; 0$$

This completes our proof of Theorem 3.1

## 4. The Mehta integrals for $\mathbf{S_n}$, $\mathbf{B_n}$ and $\mathbf{D_n}$

Our point of departure in each case is the the Selberg identity

$$\int_0^1 \cdots \int_0^1 \Big(\prod_{i=1}^n t_i^{x-1}(1-t_i)^{y-1}\Big) \prod_{1\le i<j\le n}(t_i-t_j)^{2k} dt_1 dt_2 \cdots dt_n = \prod_{j=1}^n \frac{\Gamma(x+(j-1)k)\Gamma(y+(j-1)k)\Gamma(1+jk)}{\Gamma(x+y+(n+j-2)k)\Gamma(1+k)} \qquad 4.1$$

For $S_n$ we make the substitutions

$$t_i = \tfrac{1}{2}(1-x_i/N), \qquad x = 1+\tfrac{N^2}{2} \qquad y = 1+\tfrac{N^2}{2}+\tfrac{1}{2}$$

and obtain

$$\int_{-N}^{N} \cdots \int_{-N}^{N} \prod_{i=1}^n \Big(\frac{1}{4} - \frac{x_i^2}{4N^2}\Big)^{\frac{N^2}{2}} \Big(\frac{1}{2}+\frac{x_i}{2N}\Big)^{\frac{1}{2}} \prod_{1\le i<j\le n}\Big(\frac{x_i-x_j}{2N}\Big)^{2k} \frac{dx_1 dx_2\cdots dx_n}{(2N)^n} =$$

$$= \prod_{j=1}^n \frac{\Gamma(x+(j-1)k)\Gamma(x+(j-1)k+\frac{1}{2})}{\Gamma(2+N^2+(n+j-2)k+\frac{1}{2})} \prod_{j=1}^n \frac{\Gamma(1+jk)}{\Gamma(1+k)}$$

Using the Legendre duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\,\Gamma(z)\,\Gamma(z+\tfrac{1}{2}) \qquad 4.2$$

with $z = x+(j-1)k$ this may be rewritten as

$$\int_{-N}^{N} \cdots \int_{-N}^{N} \prod_{i=1}^n \Big(1 - \frac{x_i^2}{N^2}\Big)^{\frac{N^2}{2}} \Big(\frac{1}{2}+\frac{x_i}{2N}\Big)^{\frac{1}{2}} \prod_{1\le i<j\le n}(x_i-x_j)^{2k} dx_1 dx_2\cdots dx_n =$$

$$= 2^{nN^2}(2N)^{k\,n(n-1)}(2N)^n \prod_{j=1}^n \frac{\sqrt{\pi}\;\;\Gamma(2+N^2+2(j-1)k)}{2^{1+N^2+2(j-1)k}\;\Gamma(2+N^2+(n+j-2)k+\frac{1}{2})} \prod_{j=1}^n \frac{\Gamma(1+jk)}{\Gamma(1+k)} \qquad 4.3$$

$$= \pi^{\frac{n}{2}} N^{kn(n-1)+n} \prod_{j=1}^n \frac{\Gamma(2+N^2+2(j-1)k)}{\Gamma(2+N^2+(n+j-2)k+\frac{1}{2})} \prod_{j=1}^n \frac{\Gamma(1+jk)}{\Gamma(1+k)}$$

Our next step is to pass to the limit as $N\to\infty$. The left hand side of 4.3 is almost immediate. In fact the simple inequality

$$1-u \le e^{-u} \qquad \text{(for all } u\ge 0\text{ )} \qquad 4.4$$

which follows from the identity

$$1-u = e^{-u-\frac{u^2}{2}-\frac{u^3}{3}\cdots} \qquad 4.5$$

gives (for $|x_i|\le N$)

$$\Big(1-\frac{x_i^2}{N^2}\Big)^{\frac{N^2}{2}}\Big(\frac{1}{2}+\frac{x_i}{2N}\Big)^{\frac{1}{2}} \le e^{\frac{-zi^2}{2}} \qquad 4.6$$

and so the Dominated Convergence Theorem gives

$$\lim_{N\to\infty}\int_{-N}^{N} \cdots \int_{-N}^{N} \prod_{i=1}^n \Big(1 - \frac{x_i^2}{N^2}\Big)^{\frac{N^2}{2}} \Big(\frac{1}{2}+\frac{x_i}{2N}\Big)^{\frac{1}{2}} \prod_{1\le i<j\le n}(x_i-x_j)^{2k} dx_1 dx_2\cdots dx_n =$$

$$= \frac{1}{2^{\frac{n}{2}}}\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-(x_1^2+x_2^2+\cdots+x_n^2)/2} \prod_{1\le i<j\le n}(x_i-x_j)^{2k} dx_1 dx_2\cdots dx_n \qquad 4.7$$

To compute the limit of the right hand side we shall resort to the Stirling formula

$$\Gamma(x+1) \approx (2\pi)^{\frac{1}{2}} x^{x+\frac{1}{2}} e^{-x} \qquad 4.8$$

Concentrating our efforts on the part of the right hand side of 4.3 that depends on $N$ from 4.4 we derive that

$$\begin{aligned}
\lim_{N\to\infty} N^{kn(n-1)+n} \prod_{j=1}^{n} \tfrac{\Gamma(2+N^2+2(j-1)k)}{\Gamma(2+N^2+(n+j-2)k+\frac{1}{2})} \quad = \\
&= \lim_{N\to\infty} N^{kn(n-1)+n} \prod_{j=1}^{n} \frac{\Big(1+N^2+2(j-1)k\Big)^{1+N^2+2(j-1)k+\frac{1}{2}} e^{-(1+N^2+2(j-1)k)}}{\Big(1+N^2+(n+j-2)k+\frac{1}{2}\Big)^{2+N^2+(n+j-2)k+1} e^{-(1+N^2+(n+j-2)k+\frac{1}{2})}} \\
&= \lim_{N\to\infty} N^{kn(n-1)+n} \prod_{j=1}^{n} \frac{\Big(1+N^2+2(j-1)k\Big)^{1+N^2}}{\Big(1+N^2+(n+j-2)k+\frac{1}{2}\Big)^{1+N^2}} \prod_{j=1}^{n} \frac{\Big(1+N^2+2(j-1)k\Big)^{2(j-1)k+\frac{1}{2}} \; e^{(n-j)k+\frac{1}{2}}}{\Big(1+N^2+(n+j-2)k+\frac{1}{2}\Big)^{(n+j-2)k+1}} \\
&= \lim_{N\to\infty} \prod_{j=1}^{n} \frac{\Big(1+\frac{2(j-1)k}{1+N^2}\Big)^{1+N^2}}{\Big(1+\frac{(n+j-2)k+\frac{1}{2}}{1+N^2}\Big)^{1+N^2}} \prod_{j=1}^{n} e^{(n-j)k+\frac{1}{2}} \;=\; \prod_{j=1}^{n} \frac{e^{2(j-1)k}}{e^{(n+j-2)k+\frac{1}{2}}} \prod_{j=1}^{n} e^{(n-j)k+\frac{1}{2}} \;=\; 1
\end{aligned}$$

Combining this with 4.3 and 4.7 gives the Metha identity

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-(z_1^2+z_2^2+\cdots+z_n^2)/2} \prod_{1\le i<j\le n} (z_i-z_j)^{2k} dz_1 dz_2 \cdots dz_n \;=\; \prod_{j=1}^{n} \frac{\Gamma(1+jk)}{\Gamma(1+k)}$$

To get the Macdonald-Mehta identities for $B_n$ and $D_n$, following Regev's idea, we start again with the Selberg identity

$$\begin{aligned}
\int_0^1 \cdots \int_0^1 \Big(\prod_{i=1}^{n} t_i^{x-1}(1-t_i)^{y-1}\Big) \prod_{1\le i<j\le n} (t_i-t_j)^{2k} dt_1 \cdots dt_n \quad = \\
= \prod_{j=1}^{n} \frac{\Gamma\Big(x+(j-1)k)\Gamma\Big(y+(j-1)k\Big)}{\Gamma\Big(x+y+(n+j-2)k\Big)} \prod_{j=1}^{n} \tfrac{\Gamma(1+jk)}{\Gamma(1+k)}
\end{aligned} \qquad 4.9$$

but now we make the substitutions

$$t_i = 1-\tfrac{x_i}{N} \qquad \text{and} \qquad x = N+1\,.$$

and get

$$\begin{aligned}
\int_0^N \cdots \int_0^N \Big(\prod_{i=1}^{n} \big(1-\tfrac{x_i}{N}\big)^N\Big) (x_1x_2\cdots x_n)^{y-1} \prod_{1\le i<j\le n} (x_i-x_j)^{2k} dx_1 \cdots dx_n \quad = \\
= \; N^{ny} N^{kn(n-1)} \prod_{j=1}^{n} \tfrac{\Gamma(N+1+(j-1)k)}{\Gamma(N+1+y+(n+j-2)k)} \prod_{j=1}^{n} \tfrac{\Gamma(y+(j-1)k)\Gamma(1+jk)}{\Gamma(1+k)}
\end{aligned} \qquad 4.10$$

Now a use of the dominated convergence theorem as we did before yields that the left hand side, as $N \to \infty$, converges to

$$\int_0^\infty \cdots \int_0^\infty e^{-x_1-x_2-\cdots-x_n}\,(x_1x_2\cdots x_n)^{y-1}\prod_{1\le i<j\le n}(x_i-x_j)^{2k}dx_1\cdots dx_n\,.$$

To calculate the limit of the right hand side we use 4.4 (Stirling's formula) and get

$$\begin{aligned}
N^{ny}N^{kn(n-1)}\prod_{j=1}^n \tfrac{\Gamma(N+1+(j-1)k)}{\Gamma(N+1+y+(n+j-2)k)} &\approx \\
\approx N^{ny}N^{kn(n-1)}\prod_{j=1}^n &\frac{\big(N+1+(j-1)k\big)^{N+1+(j-1)k+\frac12}e^{-(N+1+(j-1)k)}}{\big(N+1+y+(n+j-2)k\big)^{N+1+y+(n+j-2)k+\frac12}e^{-(N+1+y+(n+j-2)k)}}\\
= N^{ny}N^{kn(n-1)}\prod_{j=1}^n &\frac{\big(N+1+(j-1)k\big)^{N+1+(j-1)k+\frac12}\;e^{(y+(n-1)k)}}{\big(N+1+y+(n+j-2)k\big)^{N+1+y+(n+j-2)k+\frac12}}\\
= \prod_{j=1}^n &\frac{\big(1+\frac{1+(j-1)k}{N}\big)^{N+1+(j-1)k+\frac12}\;e^{(y+(n-1)k)}}{\big(1+\frac{1+y+(n+j-2)k}{N}\big)^{N+1+y+(n+j-2)k+\frac12}} \longrightarrow \prod_{j=1}^n \frac{e^{(1+(j-1)k)}\;e^{(y+(n-1)k)}}{e^{(1+y+(n+j-2)k)}} = 1\,.
\end{aligned}$$

Thus passing to the limit as $N \to \infty$ in 4.10 we obtain the identity

$$\int_0^\infty \cdots \int_0^\infty e^{-x_1-x_2-\cdots-x_n}\,(x_1x_2\cdots x_n)^{y-1}\prod_{1\le i<j\le n}(x_i-x_j)^{2k}dx_1\cdots dx_n = \prod_{j=1}^n \frac{\Gamma(y+(j-1)k)\Gamma(1+jk)}{\Gamma(1+k)} \tag{4.11}$$

To get to our desired identities we need the further change of variables

$$x_i = \tfrac{z_i^2}{2}$$

and since $dx_i = z_i dz_i$ , 4.11 becomes

$$\begin{aligned}
\tfrac{1}{2^{n(n-1)k+n(y-1)}}\int_0^\infty \cdots \int_0^\infty e^{-\frac{z_1^2+z_2^2+\cdots+z_n^2}{2}}\,(z_1z_2\cdots z_n)^{2y-1}\prod_{1\le i<j\le n}(z_i^2-z_j^2)^{2k}dz_1\cdots dz_n &= \\
= \prod_{j=1}^n \frac{\Gamma(y+(j-1)k)\Gamma(1+jk)}{\Gamma(1+k)}
\end{aligned} \tag{4.12}$$

To obtain the $B_n$ identity we make the specialization

$$y = k+\tfrac12$$

and get from 4.12

$$\begin{aligned}
\int_0^\infty \cdots \int_0^\infty e^{-\frac{z_1^2+z_2^2+\cdots+z_n^2}{2}}\,(z_1z_2\cdots z_n)^{2k}\prod_{1\le i<j\le n}(z_i^2-z_j^2)^{2k}dz_1\cdots dz_n &= \\
= 2^{n^2k-\frac{n}{2}}\prod_{j=1}^n \frac{\Gamma(\frac12+jk)\Gamma(1+jk)}{\Gamma(1+k)}
\end{aligned}$$

and since the integrand is an even function we also have

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{z_1^2+z_2^2+\cdots+z_n^2}{2}}\,(z_1z_2\cdots z_n)^{2k}\prod_{1\le i<j\le n}(z_i^2-z_j^2)^{2k}dz_1\cdots dz_n \;=\; 2^{n^2k+\frac{n}{2}}\prod_{j=1}^{n}\frac{\Gamma(\frac12+jk)\Gamma(1+jk)}{\Gamma(1+k)} \qquad 4.13$$

Using the Legendre duplication formula 4.8 with $z=\frac12+jk$, the right hand side of 4.13 becomes

$$\begin{aligned}\mathrm{RHS} &= 2^{n^2k+\frac{n}{2}}\prod_{j=1}^{n}\frac{\Gamma(\frac12+jk)\Gamma(1+jk)}{\Gamma(1+k)}\\ &= 2^{n^2k+\frac{n}{2}}\prod_{j=1}^{n}\frac{\sqrt{\pi}\,\Gamma(1+2jk)}{2^{2jk}\Gamma\Big(1+k\Big)} \;=\; \pi^{\frac{n}{2}}\,\frac{2^{n^2k+\frac{n}{2}}}{2^{n(n+1)k}}\prod_{j=1}^{n}\frac{\Gamma(1+2jk)}{\Gamma(1+k)} \;=\; \frac{(2\pi)^{\frac{n}{2}}}{2^{nk}}\prod_{j=1}^{n}\frac{\Gamma(1+2jk)}{\Gamma(1+k)}\end{aligned}$$

Using this in 4.12 we finally obtain the $B_n$ Macdonald-Mehta identity

$$\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{z_1^2+z_2^2+\cdots+z_n^2}{2}}\,(z_1z_2\cdots z_n)^{2k}\prod_{1\le i<j\le n}(z_i^2-z_j^2)^{2k}dz_1\cdots dz_n \;=\; \frac{1}{2^{nk}}\prod_{j=1}^{n}\frac{\Gamma(1+2jk)}{\Gamma(1+k)}$$

To obtain the $D_n$ identity we substitute $y=\frac12$ in 4.12 and get

$$\begin{aligned}&\int_{0}^{\infty}\cdots\int_{0}^{\infty} e^{-\frac{z_1^2+z_2^2+\cdots+z_n^2}{2}}\prod_{1\le i<j\le n}(z_i^2-z_j^2)^{2k}dz_1\cdots dz_n \;=\\ &\qquad = 2^{n(n-1)k-\frac{n}{2}}\prod_{j=1}^{n}\frac{\Gamma(\frac12+(j-1)k)\Gamma(1+jk)}{\Gamma(1+k)} \;=\; \frac{2^{n(n-1)k-\frac{n}{2}}\Gamma(\frac12)\Gamma(1+nk)}{\Gamma(1+k)}\prod_{j=1}^{n-1}\frac{\Gamma(\frac12+jk)\Gamma(1+jk)}{\Gamma(1+k)}\end{aligned}$$

but again, since the integrand is even, we also have

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{z_1^2+z_2^2+\cdots+z_n^2}{2}}\prod_{1\le i<j\le n}(z_i^2-z_j^2)^{2k}dz_1\cdots dz_n \;=\; \frac{2^{n(n-1)k+\frac{n}{2}}\Gamma(\frac12)\Gamma(1+nk)}{\Gamma(1+k)}\prod_{j=1}^{n-1}\frac{\Gamma(\frac12+jk)\Gamma(1+jk)}{\Gamma(1+k)} \qquad 4.14$$

Using the duplication formula 4.8 once more with $z=\frac12+jk$ the right hand side becomes

$$RHS=\frac{2^{n(n-1)k}(2\pi)^{\frac{n}{2}}\Gamma(1+nk)}{\Gamma(1+k)}\prod_{j=1}^{n-1}\frac{\Gamma(1+2jk)}{2^{2jk}\Gamma(1+k)} \;=\; \frac{2^{n(n-1)k}(2\pi)^{\frac{n}{2}}\Gamma(1+nk)}{\Gamma(1+k)\;2^{n(n-1)k}}\prod_{j=1}^{n-1}\frac{\Gamma(1+2jk)}{\Gamma(1+k)} \;=\; \frac{(2\pi)^{\frac{n}{2}}\Gamma(1+nk)}{\Gamma(1+k)}\prod_{j=1}^{n-1}\frac{\Gamma(1+2jk)}{\Gamma(1+k)}$$

and we finally obtain from 4.14 the the $D_n$ Macdonald-Mehta identity

$$\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\frac{z_1^2+z_2^2+\cdots+z_n^2}{2}}\prod_{1\le i<j\le n}(z_i^2-z_j^2)^{2k}dz_1\cdots dz_n \;=\; \frac{\Gamma(1+nk)}{\Gamma(1+k)}\prod_{j=1}^{n-1}\frac{\Gamma(1+2jk)}{\Gamma(1+k)}$$

## 5. Shift-differential operators and m-Quasi-Invariants

Given a Coxeter group $W$ of $n \times n$ matrices a polynomial $P(x_1, x_2, \ldots, x_n) \in \mathbb{Q}[x_1, x_2, \ldots, x_n]$ is said to be $W$-$m$-quasi-invariant if and only if

$$\tfrac{1}{(\alpha,x)^{2m+1}}(1 - s_\alpha)P \in \mathbb{Q}[x_1, x_2, \ldots, x_n] \qquad (\text{for all } \alpha \in \Phi^+) \tag{5.1}$$

where $\{(\alpha, x) : \alpha \in \Phi^+\}$ as before denotes the collection of reflecting hyperplanes of $W$. It is easy to see that the polynomials satisfying 5.1 form a finitely generated graded algebra, we shall denote it here "$\mathcal{QI}_m^W$". We see that 5.1 is no restriction when $m = 0$ and for $m = \infty$ we may interpret 5.1 as requiring that $P$ is a $W$-invariant polynomial. Thus we have a strictly descending chain of algebras

$$\mathbb{Q}[x] = \mathcal{QI}_0^W \supset \mathcal{QI}_1^W \supset \mathcal{QI}_2^W \supset \cdots \supset \mathcal{QI}_m^W \supset \cdots \supset \mathcal{QI}_\infty^W = \mathbb{Q}[x]^W$$

that interpolates betwen $\mathbb{Q}[x]$ and $\mathbb{Q}[x]^W$. These algebras have been introduced by Chalykh, Feigin and A. P. Veselov [4],[9] and intensely studied in recent years (see [2],[3],[8]). They have been shown to have truly remarkable properties. In particular in the $S_n$ case they display some surprising combinatorial properties ([11],[12]). It was conjectured by Feigin and Veselov and proved by Etingof-Ginsburg that each $\mathcal{QI}_m^W$ is a free module over $\mathbb{Q}[x]^W$ of rank the order of $W$. In fact, each of these algebras affords analogues of every fundamental property of the ordinary polynomial algebra. For instance, let us recall that the polynomial ring $\mathbb{Q}[x]$ has a natural scalar product $\langle \, , \, \rangle$ obtained by setting for $P, Q \in \mathbb{Q}[x]$

$$\langle P \, , \, Q \rangle \;=\; Q(\partial_x)P(x)\big|_{x=0} \tag{5.2}$$

Now the space $H_W$ of "$W$-*Harmonics*" is defined as the orthogonal complement of the ideal $\mathcal{J}_W$ generated by the homogeneous $W$-invariants of positive degree with respect to this scalar product.

It is well known that for a Coxeter group $W$ of $n \times n$ matrices the ring of $W$-invariants $\mathbb{Q}[X]^W$ is a free polynomial ring on $n$ homogeneous generators $f_1(x), f_2(x), \ldots f_n(x)$. It follows from this that we have

$$H_W \;=\; \big\{ P \in \mathbb{Q}[X_n] \, : \, f_k(\partial_x)P(x) = 0 \;\; \forall \quad k = 1, 2, \ldots, n \big\} \tag{5.3}$$

where for a polynomial $P(x)$ we set $P(\partial_x) = P(\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n})$. It is also well known that $H_W$ is the linear span of the partial derivatives of the discriminant $\Pi_W(x) \;=\; \prod_{\alpha \in \Phi^+}(\alpha, x)$. In symbols

$$H_W \;=\; \big\{ Q(\partial_x)\Pi_W(x) \; Q \in \mathbb{Q}[X_n] \, \big\}. \tag{5.4}$$

Now Feigin and Veselov conjectured [8] and and Etingof and Ginsburg proved [6] an entirely analogous result for each $m$-Quasi-Invariant algebra. To state this result we need to recall that in [4] Chalykh and Veselov show that to each homogeneous $m$-Quasi-Invariant $Q(x)$ of degree $d$ there corresponds a unique homogeneous differential operator, acting on $\mathcal{QI}_m$, of the form

$$\gamma_Q(x, \partial_x) \;=\; Q(\partial_x) + \sum_{|q|<d} c_q(x)\partial_x^q \tag{5.5}$$

where $\partial_x^q = \partial_{x_1}^{q_1}\partial_{x_2}^{q_2}\cdots\partial_{x_n}^{q_n}$ and $|q| = q_1 + q_2 + \cdots + q_n$. With $c_q(x)$ a rational function in $x_1, x_2, \ldots, x_n$ with a denominator which factors into a product of the linear forms $(x, \alpha)$. In fact, there is even an explicit formula for $\gamma_q(x, \partial_x)$ which is due to Berest [2]. This is

$$\gamma_Q(x, \partial_x) \;=\; \frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^{d-r}L_m(W)^{d-r}Q(\underline{x})L_m(W)^r \qquad 5.6$$

where $Q(\underline{x})$ denotes the operator "*multiplication by* $Q(x)$", and $L_m(W)$ is our now familiar operator $\Delta_{W,-m}$ which (in the $S_n$ case) Zeilberger rediscovered in his attempt to give a WZ proof of the Mehta identity. That is

$$L_m(W) \;=\; \Delta_2 \,-\, 2m\sum_{\alpha\in\Phi^+}\frac{1}{(x, \alpha)}\partial_{\alpha_s}. \qquad 5.7$$

In fact, the $m$-Quasi-Invariant algebras naturally arise in seeking for operators that commute with $L_m(W)$. More precisely it follows from the quoted work of Chalykh, Feigin and Veselov that the linear extension of the map $Q \to \gamma_Q(x, \partial_x)$ defined by 5.6 yields an algebra isomorphism of $\mathcal{QI}_m$ onto the algebra of operators of the form 5.5 that commute with $L_m(W)$. In particular for all $P, Q \in \mathcal{QI}_m$ we have

$$\gamma_{PQ}(x, \partial_x) \;=\; \gamma_P(x, \partial_x)\gamma_Q(x, \partial_x). \qquad 5.8$$

This given, we can see that by setting, for $P, Q \in \mathcal{QI}_m$

$$\big\langle P \,,\, Q\big\rangle_m \;=\; \gamma_P(s, \partial_x)Q(x)\big|_{x=0} \qquad 5.9$$

we obtain what should be the natural $m$-quasi-invariant analogue of the customary bilinear form in 5.2. Now all the quoted results on $m$-quasi-invariants hinge on non-degeneracy of the form $\big\langle\ ,\ \big\rangle_m$ on $\mathcal{QI}_m \times \mathcal{QI}_m$. In particular this allowed Feigin and Veselov to define the space $\mathbf{H}_W(m)$ of $m$-Harmonics as the orthogonal complement, with respect to $\big\langle\ ,\ \big\rangle_m$, of the ideal $\mathcal{J}_W(m)$ generated in $\mathcal{QI}_m$ by the homogeneous $G$ invariants. This gives

$$\mathbf{H}_W(m) \;=\; \big\{P \in \mathbb{Q}[X_n] \,:\, \gamma_{f_k}(x, \partial_x)P(x) = 0 \;\;\forall\;\;\; k = 1, 2, \ldots, n\big\} \qquad 5.10$$

It should be mentioned that it follows from this that $\mathbf{H}_W(m) \subseteq \mathcal{QI}_m$. This is a immediate consequence of the remarkable property of the operator $L_m(W)$ to the effect that for any two polynomials $P, Q$ we have $L_m(W)P = Q$ with $Q \in \mathcal{QI}_m$ if and only if $P \in \mathcal{QI}_m$. In particular any polynomial in the kernel of $L_m(W)$ is necessarily in $\mathcal{QI}_m$. This given, the $m$-analogue of 5.4 conjectured by Feigin-Veselov and proved by Etingof-Ginsburg may be stated as follows

**Theorem 5.1** (Theorem 6.20 of [6])

$$\mathbf{H}_W(m) \;=\; \big\{\gamma_Q(x, \partial_x)\Pi_W^{2m+1}(x) \,:\, Q \in \mathcal{QI}_m\,\big\}. \qquad 5.11$$

*In fact, if $\mathcal{B} \subset \mathcal{QI}_m$ is any basis for the quotient $\mathcal{QI}_m/\mathcal{J}_W(m)$, then the collection*

$$\mathcal{F} \;=\; \big\{\gamma_b(x, \partial_x)\,\Pi_W^{2m+1}(x) \,:\, b \in \mathcal{B}\big\} \qquad 5.12$$

*is a basis for $\mathbf{H}_W(m)$*

We should mention that the recent new proof of this result given in [11] also hinges on the non-degeneracy of $\langle\, ,\, \rangle_m$. This non-degeneracy, in full generality follows from a deep result of Opdam [15]. The present work was prompted by the desire to find a more accessible proof of this non-degeneracy. This section is to indicate the path by which this remarkable result is derived from the identity of Theorem 2.3. To this end we need to review some definitions and facts from the theory of "*shift differential operators*". To be precise we need to deal here, for a given Coxeter group $W$, with the family $\mathcal{SD}_W$ of operators which, may be written in the form

$$A \;=\; \sum_{\sigma\in W} a_\sigma(x,\partial_x)\sigma \tag{5.13}$$

where each $a_\sigma(s,\partial_x)$ is a differential operator of the form

$$a(x,\partial_x) \;=\; \sum_p a_p(x)\partial_{x_1}^{p_1}\partial_{x_2}^{p_2}\cdots\partial_{x_n}^{p_n} \tag{5.14}$$

with $a_p(x)$ in the ring of rational functions in the algebra generated by

$$\{x_1,x_2,\ldots,x_n \quad \text{and} \quad \tfrac{1}{(\alpha,x)} \text{ with } \alpha\in\Phi^+\,\}.$$

Since the algebra of operators given by 5.14 is invariant under conjugation by elements of $W$ it follows that the operators in $\mathcal{SD}_W$ form an algebra. Indeed, we can see that if $A$ is given by 5.13 and

$$B \;=\; \sum_{\tau\in W} b_\tau(x,\partial_x)\tau \tag{5.15}$$

Then

$$\begin{aligned} AB \;&=\; \sum_{\sigma\in W} a_\sigma(x,\partial_x)\sigma \sum_{\tau\in W} b_\tau(x,\partial_x)\tau \\ &=\; \sum_{\sigma\in W} a_\sigma(x,\partial_x) \sum_{\tau\in W} \sigma b_\tau(x,\partial_x)\sigma^{-1}\sigma\tau \\ &=\; \sum_{\gamma\in W}\Big(\sum_{\sigma\tau=\gamma} a_\sigma(x,\partial_x)b_\tau^\sigma(x,\partial_x)\Big)\gamma \end{aligned} \tag{5.16}$$

where we have set

$$b_\tau^\sigma(x,\partial_x) \;=\; \sigma b_\tau(x,\partial_x)\sigma^{-1}$$

A shift-differential operator $B$ as in 5.15 is called "$W$*-invariant*" if and only if

$$\sigma B\sigma^{-1} \;=\; B \qquad\qquad (\text{for all } \sigma\in W)$$

Note that this requires that

$$\sum_{\tau\in W} {}^{\sigma}b_\tau(x,\partial_x)\sigma\tau\sigma^{-1} \;=\; \sum_{\tau\in W} a_\tau(x,\partial_x)\tau\,. \tag{5.18}$$

There is a natural map $\Gamma$ on $\mathcal{SD}_W$ we call the "*Forgetting Map*" that is simply obtained by setting

$$\Gamma B \;=\; \Gamma \sum_{\tau\in W} b_\tau(x,\partial_x)\tau \;=\; \sum_{\tau\in W} b_\tau(x,\partial_x)\,. \qquad 5.19$$

It is important to note that

## Proposition 5.1

*$\Gamma B$ is $W$-invariant if and only if*

$$\sum_{\tau\in W} {}^{\sigma}b_\tau(x,\partial_x) \;=\; \sum_{\tau\in W} b_\tau(x,\partial_x) \qquad (\textit{for all } \sigma\in W) \qquad 5.20$$

*In particular if $B$ is $W$-invariant then $\Gamma B$ is $W$-invariant*

## Proof

From 5.19 we see that

$$\sigma\,\Gamma B\,\sigma^{-1} \;=\; B$$

if and only if

$$\sum_{\tau\in W} \sigma\, b_\tau(x,\partial_x)\sigma^{-1} \;=\; \sum_{\tau\in W} b_\tau(x,\partial_x) \qquad (\text{for all } \sigma\in W)$$

and this is 5.20. Finally, if $B$ is $W$-invariant then applying $\Gamma$ to both sides of 5.18 gives 5.20 and completes our proof.

The map $\Gamma$ is clearly linear but is not multiplicative. Yet it is so in a variety of special cases, an instance in point is given by the following basic fact

## Proposition 5.2

*If $A,B\in\mathcal{SD}_W$ and $\Gamma B$ is $W$-invariant then*

$$\Gamma AB \;=\; (\Gamma A)(\Gamma B)\,. \qquad 5.21$$

*In particular 5.21 will hold true if $B$ itself is $W$-invariant*

## Proof

Assuming that $A$ and $B$ are given by 5.13 and 5.15 from 5.16 and 5.20 we derive that

$$\Gamma AB \;=\; \sum_{\sigma\in W}\sum_{\tau\in W} a_\sigma(x,\partial_x)\;{}^{\sigma}b_\tau(x,\partial_x) \;=\; \sum_{\sigma\in W} a_\sigma(x,\partial_x)\sum_{\tau\in W} b_\tau(x,\partial_x)$$

Thus the assertions are immediate consequences of Proposition 5.1 .

The following basic fact considerably simplifies our dealing with the forgetting map $\Gamma$.

## Proposition 5.3

*Two differential operators $A$ and $B$ that have identical actions on $W$-invariants are necessarily identical. In particular it follows that to test the equality*

$$\Gamma A \;=\; \Gamma B$$

*it is sufficient to verify that we have*

$$AQ(x) \;=\; BQ(x) \qquad \text{(for all } Q(x)\in\mathbb{Q}[x]^W\text{)}$$

## Proof

From the Leibnitz formula we derive that for any two polynomials $f(x), g(x) \in \mathbb{Q}[x_1, x_2, \ldots, x_n]$ we have

$$\partial_{x_1}^{p_1}\partial_{x_2}^{p_2}\cdots\partial_{x_n}^{p_n} f(x)g(x) = \sum_{r_1\ge 0}\sum_{r_2\ge 0}\cdots\sum_{r_n\ge 0}\tbinom{p_1}{r_1}\tbinom{p_2}{r_2}\cdots\tbinom{p_n}{r_n}\partial_{x_1}^{r_1}\partial_{x_2}^{r_2}\cdots\partial_{x_n}^{r_n} f(x)\;\; \partial_{x_1}^{p_1-r_1}\partial_{x_2}^{p_2-r_2}\cdots\partial_{x_n}^{p_n-r_n} g(x)$$

Viewing $f(x)$ as the "*multiplication by $f(x)$ operator*" this Leibnitz formula may be viewed as expressing the operator identity

$$\partial_{x_1}^{p_1}\partial_{x_2}^{p_2}\cdots\partial_{x_n}^{p_n} f(\underline{x}) \;=\; \sum_{r\ge 0}\frac{1}{r!}\,\partial_x^r f(\underline{x})\;(p_1)_{r_1}(p_2)_{r_2}\cdots(p_n)_{r_n}\,\partial_{x_1}^{p_1-r_1}\partial_{x_2}^{p_2-r_2}\cdots\partial_{x_n}^{p_n-r_n}\quad (*)$$

By linearity it follows that for any differential operator

$$A(x,\partial_x) \;=\; \sum_p a_p(x)\partial_x^p$$

we have

$$A(x,\partial_x) f(\underline{x}) \;=\; \sum_{r\ge 0}\frac{\partial_x^r f(\underline{x})}{r!}\,A^{(r)}(x,\partial_x) \tag{5.22}$$

with

$$A^{(r)}(x,y) \;=\; \partial_{y_1}^{r_1}\partial_{y_2}^{r_2}\cdots\partial_{y_n}^{r_n} A(x,y)$$

To prove the assertion we must show that if for some operator $A(x,\partial_x)$ we have

$$A(x,\partial_x)Q(x) \;=\; 0 \qquad \text{(for all } Q(x)\in\mathbb{Q}[x]^W\text{)}$$

then the $y$-polynomial

$$A(x,y) \;=\; \sum_p a_p(x)\, y^p$$

vanishes identically. Clearly there is nothing to prove if $A(x,y)$ is of degree 0 in $y$. So we can proceed by induction on the $y$-degree of $A(x,y)$ and suppose that this assertion holds true up to $y$-degree $\le d-1$. This

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(*) Here we set $(a)_k = a(a-1)(a-2)\cdots(a-k+1)$

given, suppose if possible that $A(x,y)$ is of $y$-degree $d$ and $A(x,\partial_x)$ kills all $W$-invariants. It then follows that for any $W$-invariant $f(x)$ the operator

$$B(x,\partial_x) \;=\; A(x,\partial_x)f(\underline{x}) - f(\underline{x})A(x,\partial_x)$$

will also kill all $W$-invariants. But from 5.22 we derive that

$$A(x,\partial_x)f(\underline{x}) - f(\underline{x})A(x,\partial_x) \;=\; \sum_{r\geq 0\ ,\ r\neq 0} \frac{\partial_x^r f(\underline{x})}{r!}\, A^{(r)}(x,\partial_x)$$

since $B(x,y)$ is of $y$-degree $\leq d-1$ the inductive hypothesis gives that $B(x,y)$ must identically vanish. But the $y$-homogeneous component of highest $y$-degree in $B(x,y)$ is simply

$$\sum_{i=1}^n \partial_{x_i} f(\underline{x})\, \partial_{y_i} A_d(x,y)$$

where $A_d(x,y)$ is the $y$-homogeneous component of $y$-degree $d$ in $A(x,y)$. It follows then that if $f_1(x), f_2(x), \ldots, f_n(x)$ are the fundamental invariants of $W$ we must necessarily have the relations

$$\sum_{i=1}^n \partial_{x_i} f_j(\underline{x})\, \partial_{y_i} A_d(x,y) \;=\; 0 \qquad (\text{for all } 1\leq j\leq n) \tag{5.23}$$

But it is well known that the Jacobian determinant $det\big\|\partial_{x_i} f_j(x)\big\|_{i,j=1}^n$ factors into the product of the linear forms $(\alpha,x)$ with $\alpha\in\Phi^+$, thus it only vanishes on the reflecting hyperplanes of $W$ thus 5.23 forces

$$\partial_{y_i} A_d(x,y) \;=\; 0 \qquad (\text{for all } 1\leq j\leq n)$$

contraddicting the hypothesis that $A(x,y)$ is of $y$-degree $d$. This contraddiction completes the induction and the proof of the Proposition.

We can immediately see that the forgetting operator $\Gamma$ may yield surprising results. For instance as a corollary of Proposition 5.3 we obtain that Proposition 2.9 (for $k=-m$) can now be restated as follows

## Proposition 5.4

*For all $m\geq 1$ we have*

$$\Gamma p_2\big(\nabla(m)\big) \;=\; L_m(W) \tag{5.24}$$

## Remark 5.1

We must note that there is a certain asymmetry in the assertion of Proposition 5.2. In fact, 2.21 may **not** be valid if only $\Gamma A$ is known to be $W$-invariant. Wrong conclusions are quickly reached if we fail to take account of this fact. For example it follows from the identity in 2.7 that for any polynomial $Q(x)$ we have

$$Q\big(\big(\nabla(m)\big) \;=\; \frac{1}{2^d d!}\sum_{r=0}^d \binom{d}{r}(-1)^r \big(p_2(\nabla(m)\big)^{d-r} Q(\underline{x})\big(p_2(\nabla(m)\big)^r$$

Now, if $Q(x)$ is $W$-invariant it follows from Proposition 5.2 and 5.22 that

$$\Gamma Q\big((\nabla(m)\big) \;=\; \frac{1}{2^d d!}\sum_{r=0}^{d}\binom{d}{r}(-1)^r L_m(W)^{d-r}Q(\underline{x})\,L_m(W)^r$$

A comparison with the right hand side of 5.6 may lead us to the conclusion that for all $Q\in\mathcal{QI}_m^W$ we have

$$\gamma_Q(x,\partial_x) \;=\; \Gamma Q\big((\nabla(m)\big)$$

However, examples can easily be constructed even for the simplest cases of dihedral groups where this identity fails to be true for some $W$-$m$-quasi-invariants that are not $W$-invariants.

To see how the identity in 2.41 yields the non degeneracy of the bilinear form $\langle\ ,\ \rangle_m$, we need to deal with the remarkable shift-differential operator introduced by Opdam [15]. Its definition is quite simple we set

$$O_m^W \;=\; \Gamma\big(\,\Pi_W(\nabla(m))\Pi_W(\underline{x})\big). \qquad 5.25$$

Its significance in the theory of $m$-quasi-Invariants is that if we let

$$\Omega_m^W \;=\; O_m^W O_{m-1}^W\cdots O_2^W O_1^W \qquad 5.26$$

then the operators $\gamma_Q(x,\partial_x)$ introduced by Chalykh and Veselov in [4] satisfy the commutation relation

$$\gamma_Q(x,\partial_x)\Omega_m^W \;=\; \Omega_m^W Q(\partial_x) \qquad (\text{for all } Q\in\mathcal{QI}_m^W) \qquad 5.27$$

The proof of this identity is based on an ingenious idea of Chalykh and Veselov, and although the arguments are not difficult it will take us to far out of the present context to carry them out here and we will have to refer the reader to [10] for a more leasurely detailed exposition of this chapter in the theory of $m$-quasi-invariants. Nevertheless, it will be good to see how 5.27 comes about in the simple case of the $W$-invariant $p_2(x)=\sum_{i=1}^n x_i^2$. To this end the crucial identity is given by the following

## Proposition 5.5

*For $m\geq 1$ we have*

$$p_2(\nabla(m))\Pi_W(x)f(x) \;=\; \Pi_W(x)p_2(\nabla(m-1))f(x) \qquad (\text{ for all } f\in\mathbb{Q}[x]^W) \qquad 5.28$$

## Proof

From the definition in 2.6, when $v=e_i$ (the $i^{th}$ coordinate vector), we get

$$\nabla_i(m) \;=\; \partial_{x_i} - m\,\theta_i$$

where

$$\theta_i \;=\; \sum_{\alpha\in\Phi^+}\frac{\alpha_i}{(\alpha,x)}(1-s_\alpha).$$

Thus

$$p_2(\nabla(m))\Pi_W(x)f(x) \;=\; \sum_{i=1}^n \big(\partial_{x_i}^2 - m\theta_i\partial_{x_i} - m\partial_{x_i}\theta_i + m^2\theta_i^2\big)\Pi_W(x)f(x) \;=\; A \,-\, mB \,-\, mC \,-\, m^2 D \qquad 5.29$$

where for convenience we have set

$$A=\sum_{i=1}^n \partial_{x_i}^2\,\Pi_W(x)f(x)\,,\quad B=\sum_{i=1}^n \theta_i\partial_{x_i}\,\Pi_W(x)f(x)\,,\quad C=\sum_{i=1}^n \partial_{x_i}\theta_i\,\Pi_W(x)f(x)\,,\quad D=\sum_{i=1}^n \theta_i^2\,\Pi_W(x)f(x)\,. \qquad 5.30$$

We claim that we have

$$\begin{aligned} a)\quad A &= 2\Pi_W(x)\sum_{\alpha\in\Phi^+}\frac{1}{(\alpha,x)}\partial_\alpha f(x) \;+\; \Pi_W(x)\Delta_2 f(x)\\ b)\quad B &= 0\\ c)\quad C &= 2\Pi_W(x)\sum_{\alpha\in\Phi^+}\frac{1}{(\alpha,x)}\partial_\alpha f(x)\\ d)\quad D &= 0 \end{aligned} \qquad 5.31$$

To prove a) we note that

$$\sum_{i=1}\partial_{x_i}^2\Pi_W(x)f(x) \;=\; (\Delta\Pi_W(x))f(x) \;+\; 2\sum_{i=1}\partial_{x_i}\Pi_W(x)\partial_{x_i}f(x) \;+\; \Pi_W(x)\Delta f(x). \qquad 5.32$$

Now we have

$$\Delta\Pi_W(x) \;=\; 0 \qquad 5.33$$

since $\Delta$ is a $W$-invariant operator. To deal with the second term in 5.32 we note that

$$\frac{1}{\Pi_W(x)}\partial_{x_i}\Pi_W(x) \;=\; \sum_{\alpha\in\Phi^+}\frac{1}{(\alpha,x)}\partial_{x_i}(\alpha,x).$$

Thus

$$\sum_{i=1}^n\partial_{x_i}\Pi_W(x)\partial_{x_i}f(x) \;=\; \Pi_W(x)\sum_{i=1}^n\sum_{\alpha\in\Phi^+}\frac{\alpha_i}{(\alpha,x)}\partial_{x_i}f(x) \;=\; \Pi_W(x)\sum_{\alpha\in\Phi^+}\frac{1}{(\alpha,x)}\partial_\alpha f(x)$$

Using this and 5.33 in 5.32 gives 5.31 a). Next note that

$$\begin{aligned}\sum_{i=1}^n\theta_i\partial_{x_i}\Pi_W(x)f(x) &= \sum_{i=1}^n\sum_{\alpha\in\Phi^+}\frac{\alpha_i}{(\alpha,x)}(1-s_\alpha)\partial_{x_i}\Pi_W(x)f(x)\\ &= \sum_{\alpha\in\Phi^+}\frac{1}{(\alpha,x)}\partial_\alpha\Pi_W(x)f(x) \;+\; \sum_{i=1}^n\sum_{\alpha\in\Phi^+}\frac{\alpha_i}{(\alpha,x)}s_\alpha\partial_{x_i}s_\alpha\Pi_W(x)f(x)\end{aligned} \qquad 5.34$$

Now recall that we have

$$s_\alpha\partial_{x_i}s_\alpha \;=\; \partial_{x_i} \,-\, 2\frac{\alpha_i}{(\alpha,\alpha)}\partial_\alpha$$

and using this in 5.34 gives

$$\sum_{i=1}^{n}\theta_i\partial_{x_i}\Pi_W(x)f(x) \;=\; 2\sum_{\alpha\in\Phi^+}\frac{1}{(\alpha,x)}\partial_\alpha\Pi_W(x)f(x) \;-\; 2\sum_{i=1}^{n}\sum_{\alpha\in\Phi^+}\frac{\alpha_i^2}{(\alpha,x)(\alpha,\alpha)}\partial_\alpha\Pi_W(x)f(x) \;=\; 0$$

proving 5.31 b). Next we have (by the $W$-invariance of $f(x)$)

$$\sum_{i=1}^{n}\partial_{x_i}\theta_i\,\Pi_W(x)f(x) \;=\; \sum_{i=1}^{n}\partial_{x_i}f(x)\theta_i\,\Pi_W(x) \;=\; \sum_{i=1}^{n}(\partial_{x_i}f(x))\theta_i\,\Pi_W(x) \;+\; f(x)\sum_{i=1}^{n}\partial_{x_i}\theta_i\,\Pi_W(x) \qquad 5.35$$

Now

$$\sum_{i=1}^{n}\partial_{x_i}\theta_i\,\Pi_W(x) \;=\; \sum_{i=1}^{n}\partial_{x_i}\sum_{\alpha\in\Phi^+}\frac{\alpha_i}{(\alpha,x)}(1-s_\alpha)\Pi_W(x) \;=\; 2\sum_{\alpha\in\Phi^+}\partial_\alpha\frac{1}{(\alpha,x)}\Pi_W(x)$$

but the right hand side is clearly a polynomial which alternates in sign under the action of $W$. Since its degree is less than the degree of $\Pi_W(x)$ it must identically vanish. Thus 5.35 reduces to

$$\begin{aligned}\sum_{i=1}^{n}\partial_{x_i}\theta_i\,\Pi_W(x)f(x) &= \sum_{i=1}^{n}(\partial_{x_i}f(x))\theta_i\,\Pi_W(x)\\ &= 2\sum_{i=1}^{n}(\partial_{x_i}f(x))\sum_{\alpha\in\Phi^+}\frac{\alpha_i}{(\alpha,x)}\,\Pi_W(x) \qquad 5.36\\ &= 2\Pi_W(x)\sum_{i=1}^{n}\sum_{\alpha\in\Phi^+}\frac{\alpha_i}{(\alpha,x)}\partial_{x_i}f(x) \;=\; 2\Pi_W(x)\sum_{\alpha\in\Phi^+}\frac{1}{(\alpha,x)}\partial_\alpha f(x)\end{aligned}$$

This proves 5.31 c). Finally we have

$$\begin{aligned}\sum_{i=1}^{n}\theta_i^2\,\Pi_W(x)f(x) &= f(x)\sum_{i=1}^{n}\theta_i^2\,\Pi_W(x)\\ &= 2f(x)\sum_{i=1}^{n}\Big(\sum_{\alpha\in\Phi^+}\frac{\alpha_i}{(\alpha,x)}(1-s_\alpha)\Big)\sum_{\beta\in\Phi^+}\frac{\beta_i}{(\beta,x)}\,\Pi_W(x) \qquad 5.37\\ &= 2f(x)\sum_{\alpha\beta\in\Phi^+}\frac{(\alpha,\beta)}{(\alpha,x)}(1-s_\alpha)\frac{\Pi_W(x)}{(\beta,x)} \;=\; 2f(x)\sum_{\beta\in\Phi^+}\theta_\beta\,\frac{1}{(\beta,x)}\;\Pi_W(x)\end{aligned}$$

But the expression

$$\mathcal{E} \;=\; \sum_{\beta\in\Phi^+}\theta_\beta\,\frac{1}{(\beta,x)}\;\Pi_W(x)$$

clearly evaluates to a polynomial. Moreover it is $W$-alternating since the operator

$$\sum_{\beta\in\Phi^+}\theta_\beta\,\frac{1}{(\beta,x)}$$

is $W$-invariant. Since the degree of $\mathcal{E}$ is less than the degree of $\Pi_W(x)$ it must identically vanish. This proves 5.31 d).

We can now use the identities in 5.31 and and reduce 5.29 to

$$\begin{aligned} p_2(\nabla(m))\Pi_W(x)f(x) &= 2\Pi_W(x)\sum_{\alpha\in\Phi^+}\frac{1}{(\alpha,x)}\partial_\alpha f(x) \;+\; \Pi_W(x)\Delta_2 f(x) \;-\; 2\,m\,\Pi_W(x)\sum_{\alpha\in\Phi^+}\frac{1}{(\alpha,x)}\partial_\alpha f(x) \\ &= \Pi_W(x)\Big(\Delta_2 \;-\; 2(m-1)\sum_{\alpha\in\Phi^+}\frac{1}{(\alpha,x)}\partial_\alpha\Big)f(x) \end{aligned}$$

and the identity in 5.28 is thus a consequence of Proposition 5.4. This completes our proof.

We are now in a position to derive the special case $Q(x) = p_2(x)$ of 5.27.

## Theorem 5.2

*For all $m \geq 1$ we have*

$$L_m(W)O_m^W \;=\; O_m^W L_{m-1}(W) \qquad 5.38$$

*In particular it follows that*

$$L_m(W)\Omega_m^W \;=\; \Omega_m^W \Delta \qquad 5.39$$

## Proof

Using Proposition 5.4 and the definition in 5.25 we may rewrite 5.38 in the form

$$\Big(\Gamma p_2(\nabla(m))\Big)\Gamma\,\Pi_W(\nabla(m))\Pi_W(\underline{x}) \;=\; \Big(\Gamma\,\Pi_W(\nabla(m))\Pi_W(\underline{x})\Big)\Gamma p_2(\nabla(m-1) \qquad 5.40$$

However, since the two operators

$$\Pi_W(\nabla(m))\Pi_W(\underline{x}) \qquad \text{and} \qquad p_2(\nabla(m-1))$$

are clearly $W$-invariant we can use Proposition 5.2 and derive that 5.40 holds true if and only if

$$\Gamma\,p_2(\nabla(m))\Pi_W(\nabla(m))\Pi_W(\underline{x}) \;=\; \Gamma\,\Pi_W(\nabla(m))\Pi_W(\underline{x})p_2(\nabla(m-1))$$

Now the commutativity of the Dunkl operators gives that this identity is equivalent

$$\Gamma\,\Pi_W(\nabla(m))p_2(\nabla(m))\Pi_W(\underline{x}) \;=\; \Gamma\,\Pi_W(\nabla(m))\Pi_W(\underline{x})p_2(\nabla(m-1)) \qquad 5.41$$

But from Proposition 5.3 it follows that 5.41 and therefore also 5.38 will hold true if and only if we have

$$\Pi_W(\nabla(m))p_2(\nabla(m))\Pi_W(\underline{x})f(x) \;=\; \Pi_W(\nabla(m))\Pi_W(\underline{x})p_2(\nabla(m-1))f(x) \qquad (\text{for all } f(x)\in\mathbb{Q}[x]^W)$$

This shows that 5.38 is an immediate consequence of Proposition 5.5. Finally 5.39 follows by iterations of 5.38 applied to the definition in 5.26 and noticing that $L_0(W) = \Delta$. This completes our proof.

One final ingredient that plays a crucial role in the study om $m$-quasi-invariants is the so called "*Baker-Akhiezer*" function

$$\Psi_W(x,y) \;=\; \Omega_m^W e^{(x,y)} \qquad 5.42$$

Note that we may write

$$\Psi_W(x,y) \;=\; \sum_{k\geq 0} \Psi_W^{(k)}(x,y) \tag{5.43}$$

with

$$\Psi_W^{(k)}(x,y) \;=\; \Omega_m^W\Big(\frac{\sum_{i=1}^n x_iy_i}{k!}\Big)^k \tag{5.44}$$

Thus $\Psi_W(x,y)$ may be viewed as a formal power series in in the two sets of variables $x_1,x_2,\ldots,x_n$ and $y_1,y_2,\ldots,y_n$. Since the operator $\Omega_m^W$ does not change degrees, we see that the term $\Psi_W^{(k)}(x,y)$ gives the $x,y$-homogeneous component of degree $k$ in $\Psi_W(x,y)$.

The crucial fact that connects in the theory of $m$-quasi-invariants with the Macdonald-Mehta identities may be expressed by the following

## Proposition 5.6

*Setting for each $k\geq 1$*

$$C_W(k) \;=\; d_1d_2\cdots d_n(-1)^{\sum_{j=1}(d_j-1)}\prod_{j=1}^n\prod_{1\leq i<d_j}(kd_j-i) \tag{5.45}$$

*where $d_1,d_2,\ldots .d_n$ are the degrees of the fundamental $W$-invariants, the constant term of $\Psi_W(x,y)$ is simply given by the product*

$$\Psi_W^{(0)} \;=\; C_W(m)C_W(m-1)\cdots C_W(1) \tag{5.46}$$

## Proof

Since the operator $\Omega_m^W$ preserves degrees it follows from 5.42 that

$$\Psi_W^{(0)} \;=\; \Omega_m^W\,\mathbf{1}\,.$$

Thus the definition in 5.26 gives that

$$\Psi_W^{(0)} \;=\; O_m^WO_{m-1}^W\cdots O_1^W\,\mathbf{1} \tag{5.47}$$

Now it follows from the definition in 5.25 that for each $k\geq 1$ we have

$$O_k^W\,\mathbf{1} \;=\; \big(\Gamma\,\Pi_W(\nabla(k))\Pi_W(x)\big)\mathbf{1}\,.$$

But since the constant "$\mathbf{1}$" is obviously $W$-invariant we also have

$$O_k^W\,\mathbf{1} \;=\; \Pi_W(\nabla(k))\Pi_W(x)\mathbf{1} \;=\; \Pi_W(\nabla(k))\Pi_W(x)$$

and Theorem 2.3 gives

$$O_k^W\,\mathbf{1} \;=\; C_W(k)$$

and we see that 5.46 simply follows from 5.47 by iterating this identity.

It develops that the Baker-Akhiezer function $\Psi_W(x,y)$ is essetially characterized by following result of Chalykh and Veselov

## Theorem 5.3

*For every $W$-$m$-quasi-invariant $Q(x)$ we have*

$$\gamma_Q(x,\partial_x)\Psi_W(x,y) \;=\; Q(y)\,\Psi_W(x,y) \qquad 5.48$$

We have to refer the reader to [4] or [10] for a proof. Nevertheless we should point out that 5.48 is equivalent to 5.27. In fact note that from 5.27 and the definition in 5.42 it follows that

$$\gamma_Q(x,\partial_x)\Psi_W(x,y) \,=\, \gamma_Q(x,\partial_x)\Omega_m^W e^{(x,y)} \,=\, \Omega_m^W Q(\partial_x)e^{(x,y)} \,=\, \Omega_m^W Q(y)e^{(x,y)} \;=\; Q(y)\,\Omega_m^W e^{(x,y)}$$

and this is 5.48. The converse is obtained by reversing these steps.

We can now collect a windfall of consequences of these last two basic results.

## Theorem 5.4

*For any $m \geq 1$ we have*

(1) *The bilinear form defined by setting for any two polynomials in $P, Q \in \mathcal{QI}_m[X_n]$*

$$\big\langle P\ ,\ Q\big\rangle_m \;=\; \tfrac{1}{\Psi_W^{(0)}}\,\gamma_P(x,\partial_x)Q(x)\,\big|_{x=0} \qquad 5.49$$

*is non-degenerate.*

(2) *If $\big\{\phi_k^{(d)}(x)\big\}_{k=1}^{N_d}$ is any complete orthonormal system for the homogeneous $m$-quasi-invariants of degree $d$ with respect to the form $\big\langle\ ,\ \big\rangle_m$, then*

$$\Psi_W^{(d)}(x,y) \;=\; \sum_{i=1}^{N_d}\phi_k^{(d)}(x)\phi_k^{(d)}(y) \qquad 5.50$$

*as well as*

$$\Psi_W(x,y) \;=\; \Psi_W^{(0)} \;+\; \sum_{d\geq 1}\sum_{i=1}^{N_d}\phi_k^{(d)}(x)\phi_k^{(d)}(y) \qquad 5.51$$

(3) *In particular $\Psi_W(x,y)$ is the reproducing kernel for the form $\big\langle\ ,\ \big\rangle_m$.*

## Proof

Form 5.45 we derive that for any $m$-quasi-invariant $Q(x)$ we get

$$\sum_{k\geq 0}\gamma_Q(x,\partial_x)\,\Omega_m^W\,\tfrac{(x,y)^k}{k!} \;=\; Q(y)\sum_{k\geq 0}\Omega_m^W\,\tfrac{(x,y)^k}{k!}$$

Now if $Q$ is homogeneous of of degree $d$, then operator $\gamma_Q(x,\partial_x)$ will decrease $x$-degrees by $d$. Thus it follows that by equating homgeneous components of equal degrees we get for all $k \geq d$

$$\gamma_Q(x,\partial_x)\Omega_m^W\tfrac{(x,y)^k}{k!} \;=\; Q(y)\,\Omega_m\,\tfrac{(x,y)^{k-d}}{(k-d)!}\,.$$

That is

$$\gamma_Q(x,\partial_x)\,\Psi_W^{(k)}(x,y) \;=\; Q(y)\,\Omega_m\Psi_W^{(k-d)}(x,y)$$

and setting $k=d$ we get

$$\gamma_Q(x,\partial_x)\Psi_W^{(k)}(x,y) \;=\; \Psi_W^{(0)}Q(y)$$

In other words, we have shown that

$$\big\langle Q\ ,\ \Psi_W^{(k)}\big\rangle_m \;=\; Q(y) \qquad 5.52$$

This proves the non degeneracy of the form $\big\langle\ ,\ \big\rangle_m$ since the constant in 5.46 never does vanish as long as $m$ is a positive integer.

Moreover, note that replacing $Q(x)$ by $\phi_k^{(d)}(x)$ in 5.50 gives

$$\big\langle\phi_k^{(d)}\ ,\ \Psi_W^{(k)}\big)_m \;=\; \phi_k^{(d)}(y)$$

multiplying both sides by $\phi_k^{(d)}(x)$, the completeness and orthonormality of the set $\big\{\phi_k^{(d)}(x)\big\}_{k=1}^{N_d}$ gives

$$\sum_{k=1}^{N_d}\phi_k^{(d)}(x)\phi_k^{(d)}(y) \;=\; \Psi_W^{(k)}(x,y)$$

this proves 5.50 and 5.51 immediately follows.

This completes our proof and our writing.

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