Lattice Diagram Polynomials
and
Extended Pieri Rules

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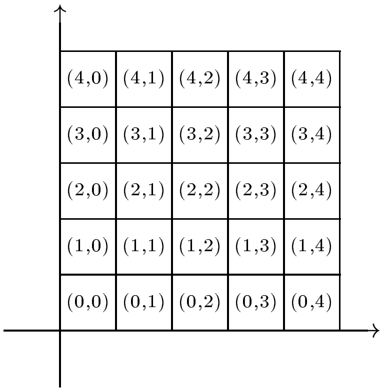
Abstract. The lattice cell in the $i+1^{st}$ row and $j+1^{st}$ column of the positive quadrant of the plane is denoted (i,j). If μ is a partition of $n+1$, we denote by μ/ij the diagram obtained by removing the cell (i,j) from the (French) Ferrers diagram of μ. We set $\Delta_{\mu/ij} = \det \| x_i^{p_j} y_i^{q_j} \|_{i,j=1}^{n}$, where $(p_1,q_1),\ldots,(p_n,q_n)$ are the cells of μ/ij, and let $\mathbf{M}_{\mu/ij}$ be the linear span of the partial derivatives of $\Delta_{\mu/ij}$. The bihomogeneity of $\Delta_{\mu/ij}$ and its alternating nature under the diagonal action of S_n gives $\mathbf{M}_{\mu/ij}$ the structure of a bigraded S_n-module. We conjecture that $\mathbf{M}_{\mu/ij}$ is always a direct sum of k left regular representations of S_n, where k is the number of cells that are weakly north and east of (i,j) in μ. We also make a number of conjectures describing the precise nature of the bivariate Frobenius characteristic of $\mathbf{M}_{\mu/ij}$ in terms of the theory of Macdonald polynomials. On the validity of these conjectures, we derive a number of surprising identities. In particular, we obtain a representation theoretical interpretation of the coefficients appearing in some Macdonald Pieri Rules.

Introduction

The lattice cells of the positive plane quadrant will be assigned coordinates $i,j \geq 0$ as indicated in the figure below.



A collection of distinct lattice cells will be briefly referred to as a "*lattice diagram.*" Given a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0)$, the lattice diagram with cells

$$\{ \, (i,j) \; : \; 0 \leq i \leq k-1 \; ; \; 0 \leq j \leq \mu_{i+1} - 1 \, \} \quad ,$$

as customary, will be called a "*Ferrers diagram.*" It will be convenient to use the symbol μ for the partition as well as its Ferrers diagram.

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Given any sequence of lattice cells

$$L \;=\; \{(p_1,q_1)\,,\,(p_2,q_2)\,,\,\ldots\,,\,(p_n,q_n)\}\ , \tag{I.1}$$

we define the "*lattice determinant*"

$$\Delta_L(x;y) \;=\; \frac{1}{p!q!}\ \det\|\, x_i^{p_j} y_i^{q_j}\,\|_{i,j=1}^{n}\ , \tag{I.2}$$

where $p! = p_1!\,p_2!\cdots p_n!$ and $q! = q_1!\,q_2!\cdots q_n!$. We can easily see that $\Delta_L(x;y)$ is a polynomial different from zero if and only if L consists of n distinct lattice cells. Note also that $\Delta_L(x;y)$ is bihomogeneous of degree $|p| = p_1+\cdots+p_n$ in x and degree $|q| = q_1+\cdots+q_n$ in y. It will be good that the definition in I.2 associates a unique polynomial to L, as a geometric object. To this end we shall require that the list of lattice cells in I.1 be given in increasing lexicographic order. This amounts to listing the cells of L in the order they are encountered as we proceed from left to right and from the lowest to the highest.

Given a polynomial $P(x;y)$, the vector space spanned by all the partial derivatives of P of all orders will be denoted $\mathcal{L}_\partial[P]$. We recall that the "*diagonal action*" of S_n on a polynomial

$$P(x;y) \;=\; P(x_1,\ldots,x_n;y_1,\ldots,y_n)$$

is defined by setting for a permutation $\sigma = (\sigma_1,\sigma_2,\ldots,\sigma_n)$

$$\sigma\, P(x;y) \;=\; P(x_{\sigma_1},x_{\sigma_2},\ldots,x_{\sigma_n};y_{\sigma_1},y_{\sigma_2},\ldots,y_{\sigma_n})\ .$$

It is easily seen from the definition I.1 that Δ_L is an alternant under the diagonal action. This given, it follows that for any lattice diagram L with n cells, the vector space

$$\mathbf{M}_L \;=\; \mathcal{L}_\partial[\Delta_L]$$

is an S_n-module. Since Δ_L is bihomogeneous, this module affords a natural bigrading. Denoting by $\mathcal{H}_{r,s}[\mathbf{M}_L]$ the subspace consisting of the bihomogeneous elements of degree r in x and degree s in y, we have the direct sum decomposition

$$\mathbf{M}_L \;=\; \bigoplus_{r=0}^{|p|}\bigoplus_{s=0}^{|q|} \mathcal{H}_{r,s}[\mathbf{M}_L]\ ,$$

and the polynomial

$$F_L(q,t) \;=\; \sum_{r=0}^{|p|}\sum_{s=0}^{|q|} t^r\, q^s\ \dim \mathcal{H}_{r,s}[\mathbf{M}_L]$$

gives the "*bigraded Hilbert series*" of $\mathbf{M}_L$. In this vein, since each of the subspaces $\mathcal{H}_{r,s}[\mathbf{M}_L]$ is necessarily an S_n-submodule, we can also set

$$C_L(x;q,t) \;=\; \sum_{r=0}^{|p|}\sum_{s=0}^{|q|} t^r\, q^s\ \mathcal{F}\,\mathrm{ch}\,\mathcal{H}_{r,s}[\mathbf{M}_L] \tag{I.3}$$

where $\mathrm{ch}\,\mathcal{H}_{r,s}[\mathbf{M}_L]$ denotes the character of $\mathcal{H}_{r,s}[\mathbf{M}_L]$ and $\mathcal{F}\,\mathrm{ch}\,\mathcal{H}_{r,s}[\mathbf{M}_L]$ denotes the image of $\mathrm{ch}\,\mathcal{H}_{r,s}[\mathbf{M}_L]$ under the Frobenius map $\mathcal{F}$ which sends the irreducible character χ^λ into the Schur function S_λ. The "x" in $C_L(x;q,t)$ is only to remind us that it is a symmetric function in the infinite alphabet $x_1, x_2, x_3, \ldots$ (as customary in [20]), and we should not confuse it with the "x" appearing in $\Delta_L(x;y)$. This may be unfortunate, but it is too much of an ingrained notation to be altered at this point. This notation should create no problems since all computations with symmetric polynomials are seldom performed in terms of the variables, but rather in terms of the classical symmetric function bases. For instance, if f is a symmetric polynomial, by writing

$$\partial_{p_1}\, f$$

we mean the symmetric polynomial obtained by expanding f in terms of the power basis and differentiating the result with respect to p_1 as if f were a polynomial in the indeterminates $p_1, p_2, p_3, \ldots$. Now it is known and easy to prove that for any Schur function S_λ we have

$$\partial_{p_1}\, S_\lambda \;=\; \sum_{\nu\to\lambda} S_\nu$$

where "$\nu\to\lambda$" is to mean that the sum is carried out over partitions ν that are obtained from λ by removing one of its corners. Since, when λ is a partition of n, we have the well-known "branching rule":

$$\chi_\lambda \downarrow^{S_n}_{Sn-1} \;=\; \sum_{\nu\to\lambda} \chi^\nu \;,$$

we see that we must have

$$\partial_{p_1}\, C_L(x;q,t) \;=\; \sum_{r=0}^{|p|}\sum_{s=0}^{|q|} t^r\, q^s\, \mathcal{F}\left(\mathrm{ch}\,\mathcal{H}_{r,s}[\mathbf{M}_L] \downarrow^{S_n}_{Sn-1}\right) \;.$$

In other words, $\partial_{p_1}\, C_L(x;q,t)$ gives the bigraded Frobenius characteristic of $\mathbf{M}_L$ restricted to S_{n-1}. In particular we see that we must necessarily have (for any lattice diagram L with n cells)

$$F_L(q,t) \;=\; \partial_{p_1}^n\, C_L(x;q,t) \;. \qquad \text{I.4}$$

Computer experimentation with a limited number of cases suggests that the following may hold true:

Conjecture I.1

For any Lattice diagram L with n cells, the module $\mathbf{M}_L$ decomposes into a direct sum of left regular representations of S_n.

Unfortunately, the complexity of computing $C_L(x;q,t)$ for large lattice diagrams prevents us from gathering sufficiently strong evidence in support of this conjecture. However, the situation is quite different for lattice diagrams obtained by removing a single cell from a partition diagram. It develops that in this case we have tools at our disposal which allow us to convert our experimental evidence into a collection of conjectures asserting that the Frobenius characteristics $C_L(x;q,t)$ satisfy some truly remarkable recurrences. Since the latter may be expressed as very precise and explicit symmetric function identities, we have been in a position to obtain overwhelming computational and theoretical evidence in their support. To see how this comes about we need to state some auxiliary results whose proofs will be found in the next section. To begin with we have the following useful fact:

Proposition I.1

Let $L=\{(p_1,q_1)\,,\,(p_2,q_2)\,,\,\ldots\,,\,(p_n,q_n)\,\}$ be a lattice diagram. Then for any integers $h,k\geq 0$ (with $h+k\geq 1$) we have

$$\sum_{i=1}^{n} \partial_{x_i}^h\, \partial_{y_i}^k\, \Delta_L(x;y) \;=\; \sum_{i=1}^{n} \epsilon(L\downarrow_{hk}^i)\, \Delta_{L\downarrow_{hk}^i}(x;y)$$

where

$$L\downarrow_{hk}^i=\{(p_1,q_1)\,,\,\ldots(p_i-h,q_i-k)\,,\,\ldots\,,\,(p_n,q_n)\,\} \qquad \text{I.5}$$

and the coefficient $\epsilon(L\downarrow_{hk}^i)$ is different from zero only if (p_i-h,q_i-k) is in the positive quadrant and $L\downarrow_{hk}^i$ consists of n distinct cells, in which case it is given by the sign of the permutation that rearranges the pairs in I.5 in increasing lexicographic order.

If μ is a partition of $n+1$, we shall denote by μ/ij the lattice diagram obtained by removing the cell (i,j) from the diagram of μ. We shall refer to the cell (i,j) as the "*hole*" of μ/ij. We can easily see that the Proposition I.1 has the following immediate corollary:

Proposition I.2

For any partition μ and $(i,j)\in\mu$ we have

$$\sum_{i=1}^{n} \partial_{x_i}^h\, \partial_{y_i}^k\, \Delta_{\mu/ij}(x;y) \;=\; \begin{cases} \pm\,\Delta_{\mu/i+h,j+k}(x;y) & \text{if } (i+h,j+k)\in\mu \\ 0 & \text{otherwise} \end{cases}$$

where the sign is "+" if there is an odd number of cells (in the lex order) between (i,j) and $(i+h,j+k)$ and is "−" otherwise.

It will be convenient to write $(i,j)\leq(i',j')$ meaning $\{i\leq i'\ \&\ j\leq j'\}$. This given, the collection of cells

$$\{(i',j')\in\mu\;:\;(i,j)\leq(i',j')\,\}$$

will be called the "*shadow*" of (i,j) in μ. It is a translation of the Ferrers diagram of a partition. Let us also set

$$D_x \;=\; \sum_{i=1}^{n}\partial_{x_i}\,,\quad D_y \;=\; \sum_{i=1}^{n}\partial_{y_i} \qquad \text{and} \qquad D_{hk} \;=\; \sum_{i=1}^{n}\partial_{x_i}^h\partial_{y_i}^k\;.$$

Now we have the following important consequences of Proposition I.2:

Proposition I.3

Let μ be a partition of $n+1$. Then for any pair of cells (i,j) , $(i+h,j+k)\in\mu$ we have

$$D_x^h D_y^k\,\mathbf{M}_{\mu/ij} \;=\; D_{hk}\,\mathbf{M}_{\mu/ij} \;=\; \mathbf{M}_{\mu/i+h,j+k} \qquad \text{I.6}$$

meaning that both $D_x^h D_y^k$ and D_{hk} are surjective linear maps. In particular we have the inclusion

$$\mathbf{M}_{\mu/i'j'} \;\subseteq\; \mathbf{M}_{\mu/ij} \qquad \text{I.7}$$

for all cells (i',j') in the shadow of (i,j) .

Proposition I.4

The collection of polynomials

$$\{\ \Delta_{\mu/i'j'}(x;y)\ :\ (i',j')\in\mu \text{ and } (i',j')\geq(i,j)\ \}$$

form a basis for the submodule of alternants of $\mathbf{M}_{\mu/ij}$.

Note that Conjecture I.1, combined with this result, leads us to a more precise statement concerning our modules $\mathbf{M}_{\mu/ij}$:

Conjecture I.2

For any $\mu\vdash n+1$ and any $(i,j)\in\mu$, the S_n-module $\mathbf{M}_{\mu/ij}$ decomposes into the direct sum of m left regular representations of S_n , where m gives the number of cells in the shadow of (i,j).

This may be viewed as an extension of the conjecture made in [7] that for any $\mu\vdash n$ the module $\mathbf{M}_\mu$ gives a bigraded version of the left regular representation of S_n. It was also conjectured in [7] that the bivariate Frobenius characteristic of $\mathbf{M}_\mu$ is given by the the symmetric polynomial

$$\tilde{H}_\mu(x;q,t) \;=\; \sum_{\lambda\vdash n} S_\lambda(x)\,\tilde{K}_{\lambda\mu}(q,t)\ , \qquad \text{I.8}$$

where the coefficients $\tilde{K}_{\lambda\mu}(q,t)$ are related to the Macdonald [19] q,t-Kostka coefficients $K_{\lambda\mu}(q,t)$ by the formula

$$\tilde{K}_{\lambda\mu}(q,t) \;=\; t^{n(\mu)}\,K_{\lambda\mu}(q,1/t)\ .$$

Here as in [20], for any partition μ we set

$$n(\mu) \;=\; \sum_i (i-1)\,\mu_i\ . \qquad \text{I.9}$$

In the present notation, the latter conjecture may be expressed by writing

$$C_\mu(x;q,t) \;=\; \tilde{H}_\mu(x;q,t)\ . \qquad \text{I.10}$$

For this reason, we shall refer to this equality as the $C=\tilde{H}$ conjecture. Macdonald conjectured in [19] that $K_{\lambda\mu}(q,t)$ is always a polynomial in q,t with positive integer coefficients. Though recently in [12], [13], [15], [16] and [18] it was shown that they are polynomials with integer coefficients, the positivity still remains to be proved. Of course, the equality in I.10 would completely settle the positivity conjecture. It follows from Macdonald's work that

$$\tilde{K}_{\lambda\mu}(1,1) \;=\; f_\lambda \;=\; \#\{\text{ standard tableaux of shape } \lambda \ \} \,.$$

Thus I.10 is consistent with the statement that $\mathbf{M}_\mu$ is a bigraded version of the left regular representation of S_n. Now it develops that there is also a way of extending the $C = \tilde{H}$ conjecture to the lattice diagrams μ/ij. The point of departure here is the following remarkable fact.

Proposition I.5

For any $\mu \vdash n+1$ we have

$$C_{\mu/00}(x;q,t) \;=\; \sum_{r=0}^{|p|}\sum_{s=0}^{|q|} t^r\, q^s\, \mathcal{F}\left(\mathrm{ch}\,\mathcal{H}_{r,s}[\mathbf{M}_\mu]\downarrow_{S_n}^{S_{n+1}}\right) \;=\; \partial_{p_1}\, C_\mu(x;q,t) \ . \qquad \text{I.11}$$

Thus on the $C = \tilde{H}$ conjecture we should have

$$C_{\mu/00}(x;q,t) \;=\; \partial_{p_1}\, \tilde{H}_\mu(x;q,t) \ . \qquad \text{I.12}$$

Since the operator ∂_{p_1} is the adjoint of multiplication by the elementary symmetric function e_1 with respect to the Hall scalar product, it may be derived from one of the Macdonald Pieri rules (see [6]) that we have

$$\partial_{p_1}\, \tilde{H}_\mu(x;q,t) \;=\; \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\, \tilde{H}_\nu(x;q,t) \qquad \text{I.13}$$

with

$$c_{\mu\nu}(q,t) \;=\; \prod_{s\in\mathcal{R}_{\mu/\nu}} \frac{t^{l_\mu(s)} - q^{a_\mu(s)+1}}{t^{l_\mu(s)} - q^{a_\mu(s)}} \prod_{s\in\mathcal{C}_{\mu/\nu}} \frac{q^{a_\mu(s)} - t^{l_\mu(s)+1}}{q^{a_\mu(s)} - t^{l_\mu(s)}} \ . \qquad \text{I.14}$$

Here $\mathcal{R}_{\mu/\nu}$ (resp. $\mathcal{C}_{\mu/\nu}$) denotes the set of lattice squares of ν that are in the same row (resp. same column) as the cell we must remove from μ to obtain ν and for any cell $s \in \mu$, the parameter $l_\mu(s)$ gives the number of cells of μ that are strictly north of s and $a_\mu(s)$ gives the number of cells that are strictly east. In view of I.13, we may rewrite I.11 in the form

$$C_{\mu/00}(x;q,t) \;=\; \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\, \tilde{H}_\nu(x;q,t) \ . \qquad \text{I.15}$$

Now extensive computations with the modules $\mathbf{M}_{\mu/ij}$ have revealed that a truly remarkable analogue of this formula may hold true for all the Frobenius characteristics $C_{\mu/ij}(x;q,t)$; we can state it as follows:

Conjecture I.3

For any $(i,j) \in \mu$ we have

$$C_{\mu/ij}(x;q,t) \;=\; \sum_{\rho\to\tau} c_{\tau\rho}(q,t)\, \tilde{H}_{\mu-\tau+\rho}\,(x;q,t) \ , \qquad \text{I.16}$$

where τ denotes the Ferrers diagram contained in the shadow of (i,j) and the symbol "$\mu - \tau + \rho$" is to represent replacing τ by ρ in the shadow of (i,j).

The following result not only reveals the true nature of I.16, but sheds some surprising light on the Macdonald Pieri rule corresponding to the identity in I.13.

Theorem I.1

The validity of I.16 for all $(i,j) \in \mu$ is equivalent to

(a) the four term recursion

$$C_{\mu/ij} \;=\; \frac{t^l - q^{a+1}}{t^l - q^a}\, C_{\mu/i,j+1} \;+\; \frac{t^{l+1} - q^a}{t^l - q^a}\, C_{\mu/i+1,j} \;-\; \frac{t^{l+1} - q^{a+1}}{t^l - q^a}\, C_{\mu/i+1,j+1} \;, \qquad \text{I.17}$$

where l and a give the number of cells that are respectively north and east of (i,j) in μ,

(b) together with the boundary conditions that the terms $C_{\mu/i,j+1}$, $C_{\mu/i,j+1}$ or $C_{\mu/i,j+1}$ are equal to zero when the corresponding cells $(i,j+1)$, $(i+1,j)$ or $(i+1,j+1)$ fall outside of μ , and are equal to $\tilde{H}_{\mu/i,j+1}$, $\tilde{H}_{\mu/i,j+1}$ or $\tilde{H}_{\mu/i,j+1}$ when any of the corresponding cells is a corner of μ.

Now a crucial development here is that I.17 has a representation theoretical interpretation that strongly suggests an inductive argument for proving both Conjectures I.2 and I.3. To present it we must introduce some notation. For a given $(i,j) \in \mu$, let $\mathbf{K}^x_{ij}$ denote the kernel of the operator D_x as a map of $\mathbf{M}_{ij}$ onto $\mathbf{M}_{i+1,j}$. Similarly, let $\mathbf{K}^y_{ij}$ be the kernel of D_y as a map of $\mathbf{M}_{ij}$ onto $\mathbf{M}_{i,j+1}$. It will also be convenient to denote by K^x_{ij} and K^y_{ij} the corresponding bivariate Frobenius characteristics. Note that since $\mathbf{M}_{i,j+1} \subseteq \mathbf{M}_{i,j}$ and $\mathbf{M}_{i+1,j} \subseteq \mathbf{M}_{i,j}$ we see that we must have

$$\mathbf{K}^x_{i,j+1} \subseteq \mathbf{K}^x_{ij} \qquad \text{as well as} \qquad \mathbf{K}^y_{i+1,j} \subseteq \mathbf{K}^y_{ij} \;.$$

Note further that if $\mu \vdash n+1$ all of these vector spaces are S_n-invariant and the quotients

$$\mathbf{A}^x_{ij} \;=\; \mathbf{K}^x_{ij} \,/\mathbf{K}^x_{i,j+1} \qquad \text{and} \qquad \mathbf{A}^y_{ij} \;=\; \mathbf{K}^y_{ij} \,/\mathbf{K}^y_{i+1,j} \qquad \text{I.18}$$

are well-defined bigraded S_n-modules. Let A^x_{ij} and A^y_{ij} denote their respective Frobenius characteristics. This given, a simple linear algebra argument gives that we have the following relations:

Proposition I.6

$$\begin{aligned} &a) \quad K^x_{ij} \;=\; C_{\mu/ij} - t\, C_{\mu/i+1,j} \;, \qquad K^y_{ij} \;=\; C_{\mu/ij} - q\, C_{\mu/i,j+1} \\ &b) \quad A^x_{ij} \;=\; K^x_{ij} - K^x_{i,j+1} \;, \qquad A^y_{ij} \;=\; K^y_{ij} - K^y_{i+1,j} \end{aligned} \qquad \text{I.19}$$

In particular, the recurrence in I.17 may be rewritten in the simple form

$$t^l\, A^x_{ij} \;=\; q^a\, A^y_{ij} \;. \qquad \text{I.20}$$

It develops that I.20 encapsulates a great deal of combinatorial and representation theoretical information. Indeed, a proof of this identity may turn out to be the single most important result in the present theory and in the theory of Macdonald polynomials. For this reason we shall here and after refer to I.20 as the "*crucial identity*."

To be precise, we shall show in Section 1 that I.20 is more than sufficient to imply the validity of Conjectures I.2 and I.3 and the q,t-Kostka positivity conjecture. The argument also shows that for $\mu \vdash n+1$ the modules $\mathbf{A}^x_{ij}$ and $\mathbf{A}^y_{ij}$ are all left regular representations of S_n. It will then result that in some sense the modules $\mathbf{A}^x_{i'j'}$ and $\mathbf{A}^y_{i'j'}$ with $(i',j') \geq (i,j)$, yield what may be viewed as an "atomic" decomposition of $\mathbf{M}_{\mu/ij}$ into a direct sum of left regular representations of S_n.

This given, our basic goal here is to understand the representation theoretical significance of I.20 in the hope that it may lead to the construction of a proof. Now it develops that the methods introduced in [1] can be extended to the present situation to yield some very precise information concerning the behavior of the Frobenius characteristics $\mathbf{A}^x_{ij}$ and $\mathbf{A}^y_{ij}$ as (i,j) varies in μ. One of the main results in [1], translated into the present language, is an algorithm for decomposing $\mathbf{M}_{\mu/00}$ as a direct sum of appropriate intersections of the modules $\mathbf{M}_\alpha$ with $\alpha \to \mu$. This algorithm is based on a package of assumptions which have come to be referred to as the "*SF-heuristics*." We shall show here that the SF-heuristics can be extended to yield a similar decomposition for all the modules $\mathbf{M}_{\mu/ij}$. We should mention that, as was the case in [1], all these decompositions, combined with the $C = \tilde{H}$ conjecture, yield (via the Frobenius map) a variety of symmetric function identities for which we have overwhelming experimental and theoretical confirmation through the theory of Macdonald polynomials.

To state our results we need to review and extend some of the notation introduced in [1]. The reader is referred to [1] for the motivation underlying these definitions.

Here and after, if $P(x;y) = P(x_1,\ldots,x_n;y_1,\ldots,y_n)$ is a polynomial, we will let $P(\partial_x;\partial_y)$, or even simply $P(\partial)$, denote the differential operator obtained by replacing, for each i and j, x_i by ∂_{x_i} and y_j by ∂_{y_j}. This given, we shall set for any two polynomials $P(x;y)$ and $Q(x;y)$

$$\langle P \,,\, Q \rangle \;=\; P(\partial_x;\partial_y)\, Q(x;y)\,\big|_{x=y=0} \;. \qquad \text{I.21}$$

It easily seen that this defines a scalar product which is invariant under the diagonal action of S_n. That is, for each $\sigma \in S_n$ we have

$$\langle \sigma P \,,\, Q \rangle \;=\; \langle P \,,\, \sigma^{-1} Q \rangle \;. \qquad \text{I.22}$$

Moreover, since the monomials $\{x^p y^q\}_{p,q}$ form an orthogonal set under this scalar product, pairs of polynomials of different bidegree will necessarily be orthogonal to each other.

If $\Delta(x;y)$ is any diagonally alternating polynomial, the space $\mathbf{M}_\Delta = \mathcal{L}[\partial_x^p \partial_y^q \Delta(x;y)]$ spanned by all partial derivatives of $\Delta(x;y)$ will necessarily be S_n-invariant. If Δ is bihomogeneous of bidegree (r_0,s_0), then $\mathbf{M}_\Delta$ has a sign-twisting, bidegree-complementing isomorphism we shall denote by $\mathbf{flip}_\Delta$, which may be defined by setting for each $P \in \mathbf{M}_\Delta$

$$\mathbf{flip}_\Delta \; P(x;y) \;=\; P(\partial_x;\partial_y)\, \Delta(x;y) \;. \qquad \text{I.23}$$

In particular, this implies that the bivariate Frobenius characteristic $\Phi_\Delta(x;q,t)$ of $\mathbf{M}_\Delta$ will necessarily satisfy the identity

$$\Phi_\Delta(x;q,t) \;=\; t^{r_0}q^{s_0}\;\omega\Phi_\Delta(x;1/q,1/t)$$

where ω, as customary, denotes the involution that sends the Schur function S_λ into $S_{\lambda'}$. It will be convenient to set, for any symmetric polynomial $\Phi(x;q,t)$ with coefficients rational functions of q and t:

$$\downarrow \Phi(x;q,t) \;=\; \omega\Phi(x;1/q,1/t)\ . \qquad \text{I.24}$$

It can also be seen that if $\mathbf{M}_1 \subseteq \mathbf{M}_\Delta$ is any bigraded S_n-invariant submodule of $\mathbf{M}_\Delta$ with bivariate Frobenius characteristic $\Phi_1(x;q,t)$ then the subspace

$$\mathbf{flip}_\Delta\ \mathbf{M}_1 \;=\; \{\ \mathbf{flip}_\Delta\ P\ :\ P\in\mathbf{M}_1\ \}$$

is also S_n-invariant, bigraded, and its bivariate Frobenius characteristic is given by the formula

$$\mathcal{F}\,\mathrm{ch}\,\mathbf{flip}_\Delta\ \mathbf{M}_1 \;=\; t^{r_0}q^{s_0}\ \downarrow\Phi_1(x;q,t)\ . \qquad \text{I.25}$$

Both the flip map and our scalar product have a number of easily verified properties that will be used in our development. To begin with, we should note that the orthogonal complement $\mathbf{M}_\Delta^\perp$ of $\mathbf{M}_\Delta$ with respect to $\langle\ ,\ \rangle$, that is the space

$$\mathbf{M}_\Delta^\perp = \{\ Q(x;y)\ :\ \langle P,Q\rangle = 0 \quad \forall \quad P\in\mathbf{M}_\Delta\ \}\ ,$$

consists of all the polynomial differential operators that kill $\Delta(x;y)$. More precisely,

$$\mathbf{M}_\Delta^\perp \;=\; \{\ Q(x;y)\ :\ Q(\partial_x;\partial_y)\,\Delta(x;y) \;=\; 0\ \}\ . \qquad \text{I.26}$$

Note that since

$$\langle P\ ,\ \mathbf{flip}_\Delta\,Q\rangle \;=\; P(\partial_x;\partial_y)Q(\partial_x;\partial_y)\Delta(x;y)\,\Big|_{x=y=0}\ ,$$

we see that $\mathbf{flip}_\Delta$ is self-adjoint. That is, for all P and Q, we have

$$\langle\, \mathbf{flip}_\Delta\,P\ ,\ Q\rangle \;=\; \langle P\ ,\ \mathbf{flip}_\Delta\,Q\rangle\ . \qquad \text{I.27}$$

For an element $P\in\mathbf{M}_\Delta$, the (necessarily) unique $P_1\in\mathbf{M}_\Delta$ such that

$$P(x;y) \;=\; P_1(\partial_x;\partial_y)\,\Delta(x;y)$$

will be denoted by $\mathbf{flip}_\Delta^{-1}\,P$. The following result will play a basic role in our development:

Proposition I.7

Let $D(x;y)$ be a polynomial, $\Delta(x;y)$ be an alternant, and set $\widetilde\Delta(x;y) = D(\partial_x;\partial_y)\Delta(x;y)$. Let $\mathbf{M}_\Delta$ (resp., $\mathbf{M}_{\widetilde\Delta}$) be the module spanned by all partial derivatives of Δ (resp., $\widetilde\Delta$). Then $\mathbf{M}_{\widetilde\Delta}$ is a submodule of $\mathbf{M}_\Delta$ and $D(\partial_x;\partial_y)$ is a surjective map from $\mathbf{M}_\Delta$ to $\mathbf{M}_{\widetilde\Delta}$. Letting $\mathbf{K}$ denote the kernel of this map, we have that

$$\mathbf{M}_\Delta \cap \mathbf{M}_{\widetilde\Delta}^\perp \;=\; \mathbf{flip}_\Delta^{-1}\ \mathbf{K}\ . \qquad \text{I.28}$$

This gives the direct sum decompositions

$$\begin{aligned} a)\quad \mathbf{M}_\Delta &= \mathbf{M}_{\widetilde{\Delta}} \oplus_\perp \mathbf{flip}_\Delta^{-1}\ \mathbf{K}, \\ b)\quad \mathbf{M}_\Delta &= \mathbf{flip}_\Delta \mathbf{M}_{\widetilde{\Delta}} \oplus \mathbf{K} \quad . \end{aligned} \qquad \text{I.29}$$

where the symbol "$\oplus$" denotes the direct sum of disjoint spaces, and "$\oplus_\perp$" further denotes that these spaces are orthogonal to each other.

Now let μ be a fixed partition of $n+1$ and let

$$\mathcal{P}red(\mu) \;=\; \{\, \nu^{(1)}\,,\, \nu^{(2)}\,,\, \ldots\,,\, \nu^{(d)} \,\} \qquad \text{I.30}$$

be the collection of partitions obtained by removing one of the corners of μ. For a pair $\nu\to\mu$, it will be convenient to denote by μ/ν the corner cell we must remove from μ to get ν. To be specific, we shall assume that the partitions in I.30 are ordered so that the corner $\mu/\nu^{(k)}$ is northwest of the corner $\mu/\nu^{(k+1)}$. Similarly, for a given cell $(i,j)\subseteq\mu$ let

$$\mathcal{P}red_{ij}(\mu) \;=\; \{\, \alpha^{(1)}\,,\, \alpha^{(2)}\,,\, \ldots\,,\, \alpha^{(m)} \,\} \qquad \text{I.31}$$

be the subset of $\mathcal{P}red(\mu)$ consisting of the $\nu^{(k)}$ such that $\mu/\nu^{(k)}$ is in the shadow of (i,j). We again assume that the $\alpha^{(i)}$ are labelled so that, for $i=1,\ldots,m-1$, the corner $\mu/\alpha^{(i)}$ is northwest of the corner $\mu/\alpha^{(i+1)}$.

Following Macdonald [20] we call the "*coleg*" and "*coarm*" of a lattice cell $s\in\mu$ the numbers $l'_\mu(s)$, and $a'_\mu(s)$ of cells that are respectively strictly south and strictly west of s in μ. In our notation, if $s=(i,j)$ then $l'_\mu(s)=i$ and $a'_\mu(s)=j$. We shall call the monomial $w(s)=t^{l'_\mu(s)}q^{a'_\mu(s)}$ the "*weight*" of s. For any lattice diagram L we set

$$T_L=\prod_{s\in L} w(s) \quad .$$

We shall also denote by ∇ the linear operator defined by setting for every partition μ

$$\nabla\, \tilde H_\mu(x;q,t) \;=\; T_\mu\, \tilde H_\mu(x;q,t) \quad . \qquad \text{I.32}$$

Since the polynomials $\tilde H_\mu(x;q,t)$ form a symmetric function basis, I.32 defines ∇ as an operator acting on all symmetric polynomials. For two subsets $T\subseteq S\subseteq\mathcal{P}red(\mu)$ set

$$\mathbf{M}_S^T \;=\; \Big(\bigcap_{\alpha\in T}\mathbf{M}_\alpha\Big)\cap\Bigg(\Big(\sum_{\beta\in S-T}\mathbf{M}_\beta\Big)\cap\Big(\bigcap_{\alpha\in T}\mathbf{M}_\alpha\Big)\Bigg)^{\perp} \qquad \text{I.33}$$

where the symbols "$\bigcap$" and "$\sum$" denote intersection and sum (not usually direct) of vector spaces, and "$\perp$" denotes the operation of taking orthogonal complements with respect to the scalar product defined in I.21. Since this scalar product is invariant under the diagonal action of S_n , we see that

$\mathbf{M}_S^T$ is a well-defined S_n-module, and its bivariate Frobenius characteristic will be denoted by ϕ_S^T. One of the assertions of the SF-heuristics is that in the linear span

$$\mathcal{L}[\ \tilde{H}_\alpha\ :\ \alpha \in S\ ]$$

we have $m = |S|$ Schur positive symmetric polynomials

$$\phi_S^{(1)}\ ,\ \phi_S^{(2)}\ ,\ \ldots\ ,\ \phi_S^{(m)}$$

such that for any $T \subseteq S$ of cardinality k we have

$$\phi_S^T \;=\; \frac{\phi_S^{(k)}}{\prod_{\alpha\in S-T} T_\alpha}\ . \qquad \text{I.34}$$

It is also a consequence of the SF-heuristics that for $k = 1, \ldots, m-1$ we can set

$$\phi_S^{(k)} \;=\; (-\nabla)^{m-k}\, \phi_S^{(m)}\ , \qquad \text{I.35}$$

while $\phi_S^{(m)}$ itself can be computed from the formula

$$\phi_S^{(m)} \;=\; \sum_{\alpha\in S}\Big(\prod_{\beta\in S/\{\alpha\}} \frac{1}{1-T_\alpha/T_\beta}\Big)\,\tilde{H}_\alpha \;=\; \sum_{\alpha\in S}\Big(\prod_{\beta\in S/\{\alpha\}} \frac{1}{1-\nabla/T_\beta}\Big)\,\tilde{H}_\alpha\ . \qquad \text{I.36}$$

To be consistent with the notation we adopted in [1] we shall use the symbols ϕ_μ or $\phi_\mu^{(k)}$ to denote $\phi_S^{(m)}$ or $\phi_S^{(k)}$ when S consists of all the predecessors of μ. In this vein, it will also be convenient to set, for any subset $S \subseteq \mathcal{P}red(\mu)$,

$$^cS \;=\; \mathcal{P}red(\mu) - S\ .$$

By comparing the expansion of $\phi_S^{(m)}$ with that of $\phi_\mu = \phi_{S'}^{(m')}$ (where $S' = \mathcal{P}red(\mu)$ has cardinality m') in I.36, it follows that

$$\phi_S^{(m)} \;=\; \Big(\prod_{\beta\in\ ^cS}\Big(1-\frac{\nabla}{T_\beta}\Big)\Big)\,\phi_\mu\ . \qquad \text{I.37}$$

In particular, when S consists of a single partition $\nu^{(i)} \in \mathcal{P}red(\mu)$, this reduces to

$$\tilde{H}_{\nu^{(i)}}(x;q,t) \;=\; \Big(\prod_{j=1\,;\,j\neq i}^{d}\Big(1-\frac{\nabla}{T_{\nu^{(j)}}}\Big)\Big)\,\phi_\mu\ , \qquad \text{I.38}$$

which may also be rewritten in the form (see 3.19 of [1])

$$\tilde{H}_{\nu^{(i)}}(x;q,t) \;=\; \sum_{k=1}^{d} \phi_\mu^{(k)}\, e_{d-k}\Big[\frac{1}{T_{\nu^{(1)}}}+\frac{1}{T_{\nu^{(2)}}}+\cdots+\frac{1}{T_{\nu^{(d)}}} \;-\; \frac{1}{T_{\nu^{(i)}}}\Big]\ . \qquad \text{I.39}$$

Finally note that if $\nu^{(i)} = \alpha \in S$ then by combining I.37 and I.38 we can also write

$$\tilde{H}_\alpha(x;q,t) \;=\; \prod_{\beta\in S\,;\,\beta\neq\alpha}\Big(1-\frac{\nabla}{T_\beta}\Big)\prod_{\beta\in\ ^cS}\Big(1-\frac{\nabla}{T_\beta}\Big)\,\phi_\mu \;=\; \prod_{\beta\in S\,;\,\beta\neq\alpha}\Big(1-\frac{\nabla}{T_\beta}\Big)\,\phi_S^{(m)} \qquad \text{I.40}$$

or equivalently, for $S = \{\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(m)}\}$ and $\alpha = \alpha^{(i)}$

$$\tilde{H}_{\alpha^{(i)}}(x;q,t) \;=\; \sum_{k=1}^{m} \phi_S^{(k)} e_{m-k}\Big[\frac{1}{T_{\alpha^{(1)}}} + \frac{1}{T_{\alpha^{(2)}}} + \cdots + \frac{1}{T_{\alpha^{(m)}}} \;-\; \frac{1}{T_{\alpha^{(i)}}}\Big] \;. \qquad \text{I.41}$$

To complete our notation we need to recall that in [13] the weights of the corners

$$\mu/\nu^{(1)} \;,\; \mu/\nu^{(2)} \;,\; \ldots \;,\; \mu/\nu^{(d)}$$

were respectively called

$$x_1 \;,\; x_2 \;,\; \ldots \;,\; x_d \;.$$

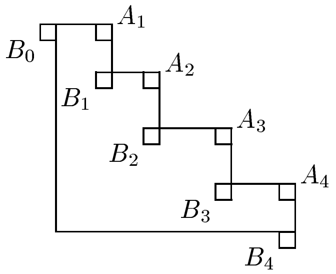
Moreover, if $x_i = t^{l'_i} q^{a'_i}$ then we also let

$$u_i \;=\; t^{l'_{i+1}} q^{a'_i} \qquad (\text{ for } \; i = 1, 2, \ldots, m-1\,) \qquad \text{I.42}$$

be the weights of what we might refer to as the "*inner corners*" of μ. The picture is completed by setting

$$u_0 \;=\; t^{l'_1}/q \quad , \quad u_m \;=\; q^{a'_m}/t \quad \text{and} \quad x_0 \;=\; 1/tq \;. \qquad \text{I.43}$$

To appreciate the geometric significance of these weights, in the figure below we illustrate a 4-corner case with corner cells labelled A_1, A_2, A_3, A_4 and inner corner cells labelled B_0, B_1, B_2, B_3, B_4.



It was shown in [13] that the products in I.14 giving the coefficients $c_{\mu\nu}(q,t)$ undergo massive cancellations which reduce them to relatively simpler expressions in terms of the corner weights. This results in the formula

$$c_{\mu\nu^{(i)}} \;=\; \frac{1}{M}\,\frac{1}{x_i}\,\frac{\prod_{j=0}^{d}\,(x_i - u_j)}{\prod_{j=1\,;\,j\neq i}^{d}(x_i - x_j)} \qquad \text{I.44}$$

where for convenience we have set

$$M = (1-1/t)(1-1/q) \;. \qquad \text{I.45}$$

Taking account of the fact that $x_i\, T_{\nu^{(i)}} = T_\mu$, formula I.38 can also be written in the form

$$\tilde{H}_{\nu^{(i)}}(x;q,t) \;=\; \prod_{j=1\,;\,j\neq i}^{d} \Big(\,1 - \nabla \frac{x_i}{T_\mu}\Big)\,\phi_\mu \;. \qquad \text{I.46}$$

It was shown in [1] (Theorem 3.3) that using I.44 and I.46 in I.13 yields the following beautiful identities:

$$\begin{aligned} a)\quad \partial_{p_1}\tilde H_\mu &= \frac{1}{M}\frac{T_\mu}{\nabla}\ \Big(\prod_{s=0}^{d}\Big(1-\nabla\frac{u_s}{T_\mu}\Big)\Big)\ \phi_\mu \\ b)\quad \partial_{p_1}\tilde H_\mu &= \sum_{k=1}^{d}\frac{\phi_S^{(k)}}{T_\mu^{m-k}}\ \frac{e_{m+1-k}[x_0+\cdots+x_d]\ -\ e_{m+1-k}[u_0+\cdots+u_d]}{M}\ . \end{aligned} \qquad \text{I.47}$$

It develops that using the same argument we can obtain analogous identities for I.16. To state them we need some notation. Let

$$S\ =\ \mathcal{P}red_{ij}(\mu)\ =\ \{\ \alpha^{(1)}\ ,\ \alpha^{(2)}\ ,\ \dots\ ,\ \alpha^{(m)}\ \}\ , \qquad \text{I.48}$$

and let τ denote the partition that corresponds to the shadow of (i,j) in μ. That is,

$$\tau\ =\ (\mu_{i+1}-j,\mu_{i+2}-j,\dots,\mu_{i+1+l}-j)\ ,$$

where l gives the number of cells above (i,j) in μ. Finally, let x_s^{ij} and u_s^{ij} (for $0\le s\le m$) be the corner weights of τ. This given, we can rewrite I.16 in either of the two forms

Proposition I.8

$$\begin{aligned} a)\quad C_{\mu/ij} &= \frac{1}{M}\frac{T_{\mu/ij}}{\nabla}\ \Big(\prod_{s=0}^{m}\Big(1-\nabla\frac{u_s^{ij}}{T_{\mu/ij}}\Big)\Big)\ \phi_S^{(m)} \\ b)\quad C_{\mu/ij} &= \sum_{k=1}^{m}\frac{\phi_S^{(k)}}{T_{\mu/ij}^{m-k}}\ \frac{e_{m+1-k}[x_0^{ij}+\cdots+x_m^{ij}]\ -\ e_{m+1-k}[u_0^{ij}+\cdots+u_m^{ij}]}{M}\ . \end{aligned} \qquad \text{I.49}$$

Formula I.49 a) enables us to obtain completely explicit expressions for the bivariate Frobenius characteristics of the modules $\mathbf{A}_{ij}^x$.

Theorem I.2

Letting l and a be the leg and arm of (i,j) and assuming I.48, with the above conventions, we have

$$A_{ij}^x/q^a\ =\ A_{ij}^y/t^l\ =\ \Big(\prod_{s=1}^{m-1}\Big(1-\nabla\frac{u_s^{ij}}{T_{\mu/ij}}\Big)\Big)\ \phi_S^{(m)}\ . \qquad \text{I.50}$$

This result has a truly surprising consequence. For a moment let $\mathcal{P}red(\mu)$ be as in I.30 and let the weight of $\mu/\nu^{(i)}$ be $t^{l_i'}q^{a_i'}$. For any pair $i,j\in[1,m]$ set

$$R_{i,j}\ =\ \{\ s\in\mu\ :\ a_{i-1}'<a'(s)\le a_i'\ \ ;\ \ l_{j+1}'<l'(s)\le l_j'\ \}\ , \qquad \text{I.51}$$

where for convenience we set $a_0'=l_{m+1}'=-1$. In words R_{i_0,j_0} is the subrectangle of μ consisting of the cells which have in their shadow only the corner cells

$$(l_i'\ ,\ a_i')\qquad \text{for}\qquad i_0\le i\le j_0\ .$$

This given, from I.50 we immediately deduce the following.

Theorem I.3

The bigraded modules $\mathbf{A}^x_{i'j'}$ *and* $\mathbf{A}^y_{i'j'}$ *, up to a bidegree shift, remain isomorphic as the cell* (i',j') *varies in a rectangle* $R_{i,j}$.

This paper is divided into five sections. In Section 1 we prove all the propositions and theorems stated in the Introduction. Some of these proofs rely on material presented in [1]. The reader will be well advised to have a copy of that paper at hand in reading the present work. The main goal in Section 2 is to give a representation theoretical interpretation of the "crucial identity" (I.20). The basic tool there is an algorithm for constructing bases for all our modules $\mathbf{M}_{\mu/ij}$. Since this algorithm is based on the heuristics proposed in [1], its validity depends on the validity of those heuristics, which at the present time are still conjectural. Nevertheless it will be seen that the symmetric function identities implied by the validity of the algorithm are in complete agreement with massive computational evidence provided by the theory of Macdonald polynomials. In Section 3 we treat in full detail the case when μ is a "hook" shape and show that all our conjectures are indeed correct in this case to the finest detail. In Section 4 we give a combinatorial argument proving that for all $\mu \vdash n$, each of the modules $\mathbf{M}_{\mu/ij}$ has dimension bounded above by $n!$ times the number of cells in the shadow of (i,j). Finally, in Section 5 we show that some of the modules whose existence was conjectured in [8] have a natural setting within the theory of "atoms" we have developed in the present work. In particular we are able to explain the origin of some puzzling identities derived in [8].

1. Basic properties of our lattice modules.

This section is dedicated to proving all the propositions and theorems we stated in the introduction.

Proof of Proposition I.1

For $L=\{(p_1,q_1)\,,\,(p_2,q_2)\,,\,\ldots\,,\,(p_n,q_n)\,\}$ we can write

$$\Delta_L(x;y) \;=\; \frac{1}{p!q!}\sum_{\sigma\in S_n} \mathrm{sign}(\sigma)\, x_{\sigma_1}^{p_1}y_{\sigma_1}^{q_1}\, x_{\sigma_2}^{p_2}y_{\sigma_2}^{q_2}\,\cdots\, x_{\sigma_n}^{p_n}y_{\sigma_n}^{q_n}\;. \qquad 1.1$$

Thus using the diagonal symmetry of the operator D_{hk} we have

$$D_{hk}\,\Delta_L \;=\; \sum_{i=1}^n \frac{1}{p!q!}\sum_{\sigma\in S_n}\mathrm{sign}(\sigma)\,\partial^h_{x_{\sigma_i}}\partial^k_{y_{\sigma_i}}\,\left(x_{\sigma_1}^{p_1}y_{\sigma_1}^{q_1}\, x_{\sigma_2}^{p_2}y_{\sigma_2}^{q_2}\,\cdots\, x_{\sigma_n}^{p_n}y_{\sigma_n}^{q_n}\right)\;. \qquad 1.2$$

Now,

$$\partial^h_{x_{\sigma_i}}\partial^k_{y_{\sigma_i}}\left(x_{\sigma_1}^{p_1}y_{\sigma_1}^{q_1}\cdots x_{\sigma_n}^{p_n}y_{\sigma_n}^{q_n}\right)=\begin{cases}(p_i)_h(q_i)_k\,x_{\sigma_1}^{p_1}y_{\sigma_1}^{q_1}\,\cdots\,x_{\sigma_i}^{p_i-h}y_{\sigma_i}^{q_i-k}\,\cdots\,x_{\sigma_n}^{p_n}y_{\sigma_n}^{q_n} & \text{if } h\le p_i \text{ and } k\le q_i\,,\\ 0 & \text{otherwise,}\end{cases}$$

where for two integers $h\le p$ we set $(p)_h=p(p-1)\cdots(p-h+1)$.

Moreover, we can easily see that the determinant

$$\sum_{\sigma\in S_n} \mathrm{sign}(\sigma)\, x_{\sigma_1}^{p_1} y_{\sigma_1}^{q_1} \;\cdots x_{\sigma_i}^{p_i-h} y_{\sigma_i}^{q_i-k} \;\cdots\; x_{\sigma_n}^{p_n} y_{\sigma_n}^{q_n} \tag{1.3}$$

fails to vanish if and only if the biexponent pairs

$$(p_1,q_1)\,,\;\ldots\;,\,(p_{i-1},q_{i-1})\,,\,(p_i-h,q_i-k)\,,\,(p_{i-1},q_{i-1})\,,\;\ldots\;,\,(p_n,q_n) \tag{1.4}$$

are all distinct. Putting all this together, formula I.5 follows from our conventions concerning lattice determinants.

Proof of Proposition I.2

What we assert there is just a special case of Proposition I.1.

Proof of Proposition I.3

Note that from Proposition I.2 it immediately follows that

$$\pm D_x^h D_y^k\, \Delta_{\mu/ij} \;=\; \pm D_{hk}\, \Delta_{\mu/ij} \;=\; \Delta_{\mu/i+h,j+k}\;, \tag{1.5}$$

and this is easily seen to imply I.6 and I.7. To show the stated surjectivity, we use the nonsingularity of the **flip** map and write every element $Q\in \mathbf{M}_{\mu/i+h,j+k}$ in the form $Q=P(\partial_x,\partial_y)\Delta_{\mu/i+h,j+k}$ with $P(x,y)$ a uniquely determined element of $\mathbf{M}_{\mu/i+h,j+k}$. Now, we see from 1.5 that we also have

$$Q \;=\; \pm D_x^h D_y^k\, P(\partial_x,\partial_y)\Delta_{\mu/ij} \;=\; \pm D_{hk}\, P(\partial_x,\partial_y)\Delta_{\mu/ij}\;.$$

This shows that both $D_x^h D_y^k$ and D_{hk} map the subspace

$$\{\, P(\partial_x,\partial_y)\Delta_{\mu/ij} \;:\; P\in \mathbf{M}_{\mu/i+h,j+k} \,\} \;=\; \mathbf{flip}_{\Delta_{\mu/ij}}\; \mathbf{M}_{\mu/i+h,j+k} \;\subseteq\; \mathbf{M}_{\mu/ij}$$

isomorphically onto $\mathbf{M}_{\mu/i+h,j+k}$. This completes our proof.

Remark 1.1

We get a better picture of what is going on here if we make use of Proposition I.7. For instance, if we let $\mathbf{K}_{ij}^{hk}$ denote the kernel of D_{hk} as a map of $\mathbf{M}_{ij}$ onto $\mathbf{M}_{i+h,j+k}$, then I.29 b), with $\Delta=\Delta_{\mu/ij}$ and $\tilde{\Delta}=\Delta_{\mu/i+h,j+k}$, gives the direct sum decomposition

$$\mathbf{M}_{\mu/ij} \;=\; \mathbf{flip}_{\Delta_{\mu/ij}} \mathbf{M}_{\mu/i+h,j+k} \;\oplus\; \mathbf{K}_{ij}^{hk}\;. \tag{1.6}$$

Moreover, since D_{hk}, (up to a bidegree shift of $(-h,-k)$), gives also an isomorphism of bigraded S_n-modules of $\mathbf{flip}_{\Delta_{\mu/ij}} \mathbf{M}_{\mu/i+h,j+k}$ onto $\mathbf{M}_{\mu/i+h,j+k}$, we see from 1.6 that the bigraded Frobenius characteristic $K_{ij}^{hk}(x;q,t)$ of $\mathbf{K}_{ij}^{hk}$ must be given by the formula

$$K_{ij}^{hk} \;=\; C_{ij}(x;q,t) \;-\; t^h q^k\, C_{i+h,j+k}(x;q,t)\;. \tag{1.7}$$

Proof of Proposition I.4

Our proof proceeds by induction with respect to the partial order $(i,j) \leq (i',j')$. We know from [10] that, up to a scalar factor, $\Delta_\alpha(x;y)$ is the only alternant in $\mathbf{M}_\alpha$. This can also be seen from the following reasoning. Note that all the monomials

$$x_1^{p_1} y_1^{q_1} x_2^{p_2} y_2^{q_2} \cdots x_n^{p_n} y_n^{q_n} \qquad 1.8$$

occurring in $\Delta_\alpha(x;y)$ consist of factors $x_i^{p_i} y_i^{q_i}$ with $(p_i,q_i) \in \alpha$. Since α has only n cells, all the monomials contained in any derivative of $\Delta_\alpha(x;y)$ will have at least one pair of equal biexponents. This forces the vanishing of the antisymmetrization of every derivative of $\Delta_\alpha(x;y)$. This proves the assertion when (i,j) is a corner cell of μ and $\alpha = \mu/ij$. So let us assume that the assertion is true for any $(i',j') > (i,j)$. This given, note that every bihomogeneous alternating polynomial $\Delta(x;y) \in \mathbf{M}_{\mu/ij}$ can be written in the form

$$\Delta(x;y) \;=\; P(\partial_x;\partial_y)\, \Delta_{\mu/ij}(x;y) \qquad 1.9$$

with P bihomogeneous and invariant under the diagonal action. Now it is well known (see [24]) that the ideal generated by the diagonal invariant polynomials with vanishing constant term is also generated by the polynomials

$$\sum_{i=1}^{n} x_i^h\, y_i^k \qquad \text{with} \quad 1 \leq h+k \leq n \ .$$

Thus, if $P(x;y)$ is not a constant, we may express it in the form

$$P(x;y) \;=\; \sum_{1\leq h+k\leq n} A_{hk}(x;y) \sum_{i=1}^{n} x_i^h\, y_i^k \ . \qquad 1.10$$

Substituting this into 1.9 and using 1.5 gives

$$\Delta(x;y) = \sum_{1\leq h+k\leq n} A_{hk}(\partial_x;\partial_y)\, D_{hk}\, \Delta_{\mu/ij}(x;y) = \sum_{\substack{(i+h,j+k)\,\in\,\mu \\ 1\leq h+k\leq n}} \pm A_{hk}(\partial_x;\partial_y)\, \Delta_{\mu/i+h,j+k}(x;y) \ . \qquad 1.11$$

Thus, from the induction hypothesis we derive that any bihomogeneous alternant of $\mathbf{M}_{\mu/ij}$, with lesser total degree than $\Delta_{\mu/ij}$, must be a linear combination of the $\Delta_{\mu/i'j'}$ with $(i',j') > (i,j)$. This completes the induction since the only elements of $\mathbf{M}_{\mu/ij}$ of the same total degree as $\Delta_{\mu/ij}$ are its scalar multiples.

Proof of Proposition I.5

From Proposition I.1 it immediately follows that for any $\mu \vdash n+1$ we have

$$\sum_{i=1}^{n+1} \partial_{x_i}^h \partial_{y_i}^k\, \Delta_\mu(x;y) \;=\; 0 \qquad (\forall \quad h+k \geq 1\,) \ . \qquad 1.12$$

In particular, if D_x and D_y are as given in I.6, we deduce that

$$\begin{aligned} \partial_{x_{n+1}} \Delta_\mu(x;y) &= -D_x \, \Delta_\mu(x;y) \quad , \\ \partial_{y_{n+1}} \Delta_\mu(x;y) &= -D_y \, \Delta_\mu(x;y) \quad . \end{aligned} \tag{1.13}$$

This means that in constructing a basis for $\mathbf{M}_\mu$ of the form

$$\mathcal{B}_\mu \;=\; \{\, b(\partial_x;\partial_y)\, \Delta_\mu(x;y) \;:\; b \in \mathcal{C} \,\} \;, \tag{1.14}$$

the polynomials in $\mathcal{C}$ need not contain any of the variables x_{n+1} , y_{n+1}. Now we have the following

Lemma 1.1

If $\mathcal{C}$ is a collection of polynomials in $\mathbf{Q}[x_1,\ldots,x_n \,;\, y_1,\ldots,y_n]$ then the collection $\mathcal{B}_\mu$ given in 1.14 is a basis for $\mathbf{M}_\mu$ if and only if the collection

$$\mathcal{B}_{\mu/00} \;=\; \{\, b(\partial_x;\partial_y)\, \Delta_{\mu/00}(x;y) \;:\; b \in \mathcal{C} \,\} \tag{1.15}$$

is a basis for $\mathbf{M}_{\mu/00}$.

Proof

The Laplace expansion of the determinant giving Δ_μ, with respect to the last row, gives that

$$\Delta_\mu(x;y) \;=\; \sum_{(i,j)\in\mu} x_{n+1}^i y_{n+1}^j \;\epsilon_{ij}\; \Delta_{\mu/ij}(x_1,\ldots,x_n;y_1,\ldots,y_n) \tag{1.16}$$

with $\epsilon_{ij} = \pm 1$. Note then that for $f \in \mathbf{Q}[x_1,\ldots,x_n \,;\, y_1,\ldots,y_n]$ we necessarily have

$$a) \quad f(\partial_x;\partial_y)\, \Delta_\mu(x;y) \;=\; 0 \qquad \longleftrightarrow \qquad b) \quad f(\partial_x;\partial_y)\, \Delta_{\mu/00}(x;y) \;=\; 0 \;.$$

In fact, we see from 1.16 that b) immediately follows from a) by setting $x_{n+1} = y_{n+1} = 0$. Conversely, if b) holds true then by applying to it the operator $D_{i,j}$ we obtain that

$$f(\partial_x;\partial_y)\, \Delta_{\mu/ij}(x;y) \;=\; 0$$

must hold as well for all $(i,j) \in \mu$ and then a) again follows by applying $f(\partial_x;\partial_y)$ to both sides of 1.16. We thus derive that, for a given collection $\mathcal{C}$, $\mathcal{B}_\mu$ is an independent set if and only if $\mathcal{B}_{\mu/00}$ is. In particular, both spaces $\mathbf{M}_\mu$ and $\mathbf{M}_{\mu/00}$ must have the same dimension. **Q.E.D.**

This given, I.11 follows by choosing $\mathcal{C}$ so that both $\mathcal{B}_\mu$ and $\mathcal{B}_{\mu/00}$ are bihomogeneous bases and noting that (because of Lemma 1.1) the action of S_n on corresponding bihomogeneous components of $\mathcal{B}_\mu$ and $\mathcal{B}_{\mu/00}$ are given by the same matrices. This completes the proof of Proposition I.5.

We should note that a useful consequence of Lemma 1.1 is the following.

Proposition 1.1

If $\mathcal{B}^*_\mu(x_1,\ldots,x_n,x_{n+1};y_1,\ldots,y_n,y_{n+1})$ is a basis for $\mathbf{M}_\mu$ then $\mathcal{B}^*_\mu(x_1,\ldots,x_n,0;y_1,\ldots,y_n,0)$ is a basis for $\mathbf{M}_{\mu/00}$.

Proof

Let $\mathcal{C} \subseteq \mathbf{Q}[x_1,\ldots,x_n;y_1,\ldots,y_n]$ be chosen so that both $\mathcal{B}_\mu$ and $\mathcal{B}_{\mu/00}$, (as given by 1.14 and 1.15) are bases for $\mathbf{M}_\mu$ and $\mathbf{M}_{\mu/00}$ respectively. By assumption, for every element of $b \in \mathcal{C}$ we have the expansion

$$\begin{aligned} b(\partial_{x_1},\ldots,\partial_{x_n};\partial_{y_1},\ldots,\partial_{y_n})\ \Delta_\mu(x_1,\ldots,x_n,x_{n+1};y_1,\ldots,y_n,y_{n+1}) \\ = \sum_{b^*\in\mathcal{B}^*_\mu} c_{b^*}\ b^*(x_1,\ldots,x_n,x_{n+1};y_1,\ldots,y_n,y_{n+1})\ . \end{aligned}$$

However, setting $x_{n+1}=y_{n+1}=0$ here (and using 1.16) gives the identity

$$\begin{aligned} b(\partial_{x_1},\ldots,\partial_{x_n};\partial_{y_1},\ldots,\partial_{y_n})\ \Delta_{\mu/00}(x_1,\ldots,x_n;y_1,\ldots,y_n) \\ = \sum_{b^*\in\mathcal{B}^*_\mu} c_{b^*}\ b^*(x_1,\ldots,x_n,0;y_1,\ldots,y_n,0)\ . \end{aligned}$$

This shows that $\mathcal{B}^*_\mu(x_1,\ldots,x_n,0;y_1,\ldots,y_n,0)$ spans $\mathbf{M}_{\mu/00}$. However, it must be a basis since its cardinality is no larger than the dimension of $\mathbf{M}_\mu$ and the latter has the same dimension as $\mathbf{M}_{\mu/00}$.

Proof of Theorem I.1

For a given cell $(i,j)\in\mu$ we are to determine if there are constants x, y and z such that

$$C_{\mu/ij}\ -\ x\,C_{\mu/i,j+1}\ -\ y\,C_{\mu/i+1,j}\ +\ z\,C_{\mu/i+1,j+1}\ =\ 0\ . \tag{1.17}$$

Let us begin with the generic case, that is when the shadows of the four cells (i,j), $(i,j+1)$, $(i+1,j)$, $(i+1,j+1)$ contain the same corners of μ. To this end, let τ be the partition contained in the shadow of (i,j) and ρ be one of the predecessors of τ. Denoting by c_ρ^{ij}, $c_\rho^{i,j+1}$, $c_\rho^{i+1,j}$ and $c_\rho^{i+1,j+1}$ the coefficients of $\tilde H_{\mu-\tau+\rho}(x;q,t)$ in $C_{\mu/ij}(x;q,t)$, $C_{\mu/i,j+1}(x;q,t)$, $C_{\mu/i+1,j}(x;q,t)$ and $C_{\mu/i+1,j+1}(x;q,t)$ respectively, it is not difficult to derive from I.16 and the definition I.14 that we must have

$$\begin{aligned} c_\rho^{ij} &= \frac{t^{l_1}-q^{a_1+1}}{t^{l_1}-q^{a_1}}\ \frac{q^{a_2}-t^{l_2+1}}{q^{a_2}-t^{l_2}}\ c_\rho^{i+1,j+1}\ ,\\ c_\rho^{i,j+1} &= \frac{q^{a_2}-t^{l_2+1}}{q^{a_2}-t^{l_2}}\ c_\rho^{i+1,j+1}\ ,\\ c_\rho^{i+1,j} &= \frac{t^{l_1}-q^{a_1+1}}{t^{l_1}-q^{a_1}}\ c_\rho^{i+1,j+1}\ , \end{aligned}$$

with

$$l_1=l+i-l'\ ,\quad a_1=a'-j\ ,\quad l_2=l'-i\ ,\quad a_2=a+j-a'\ , \tag{1.18}$$

where l and a give the leg and arm of (i,j) and l' and a' give the coleg and coarm of the cell $\mu/\mu-\tau+\rho$.

This given, equating to zero the coefficient of $\tilde H_{\mu-\tau+\rho}(x;q,t)$ in 1.17 yields the equation

$$\left(\frac{t^{l_1}-q^{a_1+1}}{t^{l_1}-q^{a_1}}\ \frac{q^{a_2}-t^{l_2+1}}{q^{a_2}-t^{l_2}}\ -\ x\ \frac{q^{a_2}-t^{l_2+1}}{q^{a_2}-t^{l_2}}\ -\ y\ \frac{t^{l_1}-q^{a_1+1}}{t^{l_1}-q^{a_1}}\ +\ z\ \right) c_\rho^{i+1,j+1}\ =\ 0$$

Since by definition the coefficients $c_{\mu\nu}$ are never zero, we see that 1.17 will hold true if and only if we can find x, y and z **independent** of ρ such that

$$(t_1 - q\,q_1)(q_2 - t\,t_2) - x\,(q_2 - t\,t_2)(t_1 - q_1) - y\,(t_1 - q\,q_1)(q_2 - t_2) + z\,(t_1 - q_1)(q_2 - t_2) = 0\,, \quad 1.19$$

where for convenience we have set

$$t_1 = t^{l_1}\ , \quad t_2 = t^{l_2}\ , \quad q_1 = q^{a_1}\ , \quad q_2 = q^{a_2}\ .$$

Setting $T = t^l$ and $Q = q^a$, from 1.18 we deduce that $t_2 = T/t_1$, $q_2 = Q/q_1$. Thus, making these substitutions and multiplying by $t_1\,q_1$, reduces 1.19 to

$$Q\,(x{+}y{-}z{-}1)\,t_1^2 \;-\; \big(x(tT{+}Q){+}y(T{+}qQ){-}z(T{+}Q){-}(tT{+}qQ)\big)\,q_1 t_1 \;+\; T\,(xt{+}yq{-}z{-}tq)\,q_1^2 \;=\; 0\ .$$

Now this is most fortunate since the coefficients of t_1^2, $t_1 q_1$ and q_1^2 are independent of ρ.

Setting to zero these coefficients yields the system

$$\begin{array}{rcrcrcl} x & + & y & - & z & = & 1 \\ (tT+Q)\,x & + & (T+qQ)\,y & - & (T+Q)\,z & = & tT+qQ \\ t\,x & + & q\,y & - & z & = & tq \end{array}$$

whose unique solution

$$x \;=\; \frac{T - q\,Q}{T-Q}\ , \qquad y \;=\; \frac{t\,T - Q}{T-Q}\ , \qquad z \;=\; \frac{t\,T - q\,Q}{T-Q}$$

establishes the identity in I.17, in this case.

Let us deal next with the case when the leftmost corner of τ is in the shadow of $(i+1, j)$ but not in the shadow of $(i, j+1)$ and $(i+1, j+1)$. Let ρ_1 be the partition obtained by removing this corner from τ. This given we derive from I.16 that neither $C_{\mu/i,j+1}(x; q, t)$ nor $C_{\mu/i+1,j+1}(x; q, t)$ will contain a term involving $\tilde{H}_{\mu-\tau+\rho_1}(x; q, t)$ in their expansion. So, taking the coefficient of this polynomial in 1.17 reduces it to

$$c_{\rho_1}^{ij} \;-\; y\,c_{\rho_1}^{i+1,j} \;=\; 0\ .$$

Now, using again the same notation, we may write

$$c_{\rho_1}^{ij} \;=\; \frac{q^{a_2} - t^{l_2+1}}{q^{a_2} - t^{l_2}}\ c_{\rho_1}^{i+1,j}\ .$$

These two equations give that

$$y \;=\; \frac{q^{a_2} - t^{l_2+1}}{q^{a_2} - t^{l_2}}\ .$$

However, in this case it is easily seen that $l_2 = l' - i = l$ and $a' = j$, giving $a_2 = a + j - a' = a$, and we are led again to the solution

$$y \;=\; \frac{Q - t\,T}{Q - T}\ .$$

The remaining cases can be easily checked to yield the same values of x and y. This completes the proof of Theorem I.1 since the other assertions are immediate consequences of I.16.

Proof of Proposition I.6

There is very little left to do here since (see Remark 1.1) both equations in I.19 a) are but particular cases of 1.7 and the equations in I.19 b) as well as I.20 are immediate consequences of the definitions.

Proof of Proposition I.7

By the properties of the map $\mathbf{flip}_{\widetilde{\Delta}}$, a polynomial $Q(x,y)$ in $\mathbf{M}_{\widetilde{\Delta}}$ may be written in the form

$$Q(x,y) = P(\partial_x;\partial_y)\widetilde{\Delta}(x;y) \ , \qquad \text{with} \quad P(x;y) \in \mathbf{M}_{\widetilde{\Delta}} \ .$$

Since $\widetilde{\Delta}(x;y) = D(\partial_x;\partial_y)\Delta(x;y)$, we may also write $Q(x,y)$ in the form

$$Q(x,y) = P(\partial_x;\partial_y)D(\partial_x;\partial_y)\Delta(x;y) = D(\partial_x;\partial_y)P(\partial_x;\partial_y)\Delta(x;y)$$

with

$$P(\partial_x;\partial_y)\Delta(x;y) \in \mathbf{M}_{\Delta} \ .$$

This establishes surjectivity and the containment $\mathbf{M}_{\widetilde{\Delta}} \subseteq \mathbf{M}_{\Delta}$. In fact, this argument shows that $D(\partial_x;\partial_y)$ maps the space

$$\mathbf{flip}_{\Delta} \ \mathbf{M}_{\widetilde{\Delta}} \ = \ \left\{ \ P(\partial_x;\partial_y)\,\Delta \ : \ P \in \mathbf{M}_{\widetilde{\Delta}} \ \right\}$$

surjectively onto $\mathbf{M}_{\widetilde{\Delta}}$.

Now we establish I.28, the description of the kernel. To this end note that the polynomial $f = P(\partial_x;\partial_y)\Delta(x;y) = \mathbf{flip}_{\Delta}\, P$ is in $\mathbf{K}$ if and only if $0 = D(\partial_x;\partial_y)f(x;y) = P(\partial_x;\partial_y)\widetilde{\Delta}(x;y)$, or equivalently, $P \in \mathbf{M}_{\widetilde{\Delta}}^{\perp}$. Thus we may write

$$\begin{aligned}
\mathbf{K} \ &= \ \left\{ \ f = P(\partial_x;\partial_y)\Delta(x;y) \ : \ P \in \mathbf{M}_{\Delta} \quad \& \quad P(\partial_x;\partial_y)\widetilde{\Delta}(x;y) = 0 \ \right\} \\
&= \ \mathbf{flip}_{\Delta} \left\{ \ P \in \mathbf{M}_{\Delta} \ : \ P(\partial_x;\partial_y)\widetilde{\Delta}(x;y) = 0 \ \right\} \\
&= \ \mathbf{flip}_{\Delta} \ \mathbf{M}_{\Delta} \cap \mathbf{M}_{\widetilde{\Delta}}^{\perp} \ ,
\end{aligned}$$

and I.28 follows by an application of $\mathbf{flip}_{\Delta}^{-1}$ to both sides of this relation. This shows that the orthogonal decomposition

$$\mathbf{M}_{\Delta} \ = \ \mathbf{M}_{\widetilde{\Delta}} \ \oplus_{\perp} \ \mathbf{M}_{\Delta} \cap \mathbf{M}_{\widetilde{\Delta}}^{\perp}$$

in this case can be written in the form

$$\mathbf{M}_{\Delta} \ = \ \mathbf{M}_{\widetilde{\Delta}} \ \oplus_{\perp} \ \mathbf{flip}_{\Delta}^{-1} \ \mathbf{K} \ ,$$

establishing I.29 a). Applying $\mathbf{flip}_{\Delta}$ to both sides gives I.29 b), completing our proof.

Proof of Proposition I.8

Our point of departure is formula I.16. So let τ be the partition in the shadow of (i,j) and let $x_0^{ij},\ldots,x_m^{ij}$; $u_0^{ij},\ldots,u_m^{ij}$ be the corner weights of τ. Let $\rho^{(1)},\rho^{(2)},\ldots,\rho^{(m)}$ be the predecessors of τ ordered from left to right so that $x_1^{ij},\ldots,x_m^{ij}$ are the respective weights of the cells $\tau/\rho^{(1)},\ldots,\tau/\rho^{(m)}$. This given, using formula I.44 with μ replaced by τ and $\nu^{(i)}$ replaced by $\rho^{(s)}$, formula I.16 becomes

$$C_{\mu/ij}(x;q,t) \;=\; \frac{1}{M}\sum_{s=1}^{m}\frac{1}{x_s^{ij}}\,\frac{\prod_{r=0}^{m}\big(x_s^{ij}-u_r^{ij}\big)}{\prod_{r=1\,;\,r\neq s}^{m}\big(x_s^{ij}-x_r^{ij}\big)}\;\tilde H_{\mu-\tau+\rho^{(s)}}\ . \qquad 1.20$$

For convenience set $\mu-\tau+\rho^{(s)}=\alpha^{(s)}$, so that as in I.48 we have

$$S \;=\; \mathcal{P}red_{ij}(\mu) \;=\; \big\{\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(m)}\big\}\ .$$

Now, formula I.40 for $\alpha=\alpha^{(s)}$ may be written as

$$\tilde H_{\alpha^{(s)}}(x;q,t) \;=\; \prod_{r=1\,;\,r\neq s}^{m}\Big(1-\frac{\nabla}{T_{\alpha^{(r)}}}\Big)\,\phi_S^{(m)}\ . \qquad 1.21$$

Note next that from the definition of μ/ij it follows that $T_\mu=t^iq^jT_{\mu/ij}$, and since $t^iq^jx_s^{ij}$ is the weight of the cell $\mu/\mu-\tau+\rho^{(s)}=\mu/\alpha^{(s)}$, we also have $t^iq^jx_s^{ij}T_{\alpha^{(s)}}=T_\mu$. In conclusion we see that

$$\frac{1}{T_{\alpha^{(s)}}} \;=\; \frac{x_s^{ij}}{T_{\mu/ij}}\ . \qquad 1.22$$

Using this in 1.21 and substituting the resulting expression in 1.20 we finally obtain

$$C_{\mu/ij}(x;q,t) \;=\; \frac{1}{M}\sum_{s=1}^{m}\frac{1}{x_s^{ij}}\,\frac{\prod_{r=0}^{m}\big(x_s^{ij}-u_r^{ij}\big)}{\prod_{r=1\,;\,r\neq s}^{m}\big(x_s^{ij}-x_r^{ij}\big)}\;\prod_{r=1\,;\,r\neq s}^{m}\Big(1-\nabla\frac{x_r^{ij}}{T_{\mu/ij}}\Big)\,\phi_S^{(m)}\ . \qquad 1.23$$

Now it develops that we have the following identity.

Lemma 1.2

If $x_0,x_1,\ldots,x_m$ and $u_0,u_1,\ldots,u_m$ are any quantities such that

$$x_0x_1\cdots x_m=u_0u_1\cdots u_m\ , \qquad 1.24$$

then for all z we have

$$\sum_{s=0}^{m}\frac{1}{x_s}\,\frac{\prod_{r=0}^{m}(x_s-u_r)}{\prod_{r=1\,;\,r\neq s}^{m}(x_s-x_r)}\;\prod_{r=1\,;\,r\neq s}^{m}(1-zx_r) \;=\; \frac{1}{z}\bigg(\prod_{s=0}^{m}(1-zu_s) \;-\; \prod_{s=0}^{m}(1-zx_s)\bigg)\ . \qquad 1.25$$

Proof

Note that because of 1.24 the expression on the right hand side of 1.25 evaluates to a polynomial of degree at most $m-1$. We can thus apply the Lagrange interpolation formula at the points

$$1/x_1\ ,\ 1/x_2\ ,\ \ldots\ ,\ 1/x_m\ ,$$

and obtain that

$$\frac{1}{z}\left(\prod_{s=0}^{m}(1-zu_s) \;-\; \prod_{s=0}^{m}(1-zx_s)\right) \;=\; \sum_{s=0}^{m} x_s \prod_{r=0}^{m}(1-\frac{u_r}{x_s}) \prod_{r=1\,;\,r\neq s}^{m} \frac{(z-\frac{1}{x_r})}{(\frac{1}{x_s}-\frac{1}{x_r})} \;.$$

Clearly this is just another way of writing 1.25.

This given, evaluating 1.25 at

$$x_s = x_s^{ij} \quad , \quad u_s = u_s^{ij} \ \text{ and } \ z = \frac{\nabla}{T_{\mu/ij}} \;,$$

applying both sides to $\phi_S^{(m)}$ and using 1.23, we finally obtain that

$$C_{\mu/ij} \;=\; \frac{1}{M}\frac{T_{\mu/ij}}{\nabla} \left[\prod_{s=0}^{m}\left(1-\nabla\frac{u_s^{ij}}{T_{\mu/ij}}\right) - \prod_{s=0}^{m}\left(1-\nabla\frac{x_s^{ij}}{T_{\mu/ij}}\right)\right] \phi_S^{(m)} \;. \qquad 1.26$$

We claim that this formula contains both I.49 a) and b). In fact, expanding the products, collecting terms according to powers of ∇, and using the identity

$$u_0^{ij}u_1^{ij}\cdots u_m^{ij} \;=\; x_0^{ij}x_1^{ij}\cdots x_m^{ij} \qquad 1.27$$

gives

$$C_{\mu/ij} \;=\; \sum_{k=1}^{m} \frac{(-\nabla)^{m-k}\phi_S^{(m)}}{T_{\mu/ij}^{m-k}} \; \frac{e_{m+1-k}[x_0^{ij}+\cdots+x_d^{ij}] \;-\; e_{m+1-k}[u_0^{ij}+\cdots+u_d^{ij}]}{M} \;.$$

In view of I.35, we see that this just another way of writing I.49 b). Note next that, using 1.22, we may write

$$\prod_{s=0}^{m}\left(1-\nabla\frac{x_s^{ij}}{T_{\mu/ij}}\right) \;=\; \left(1-\nabla\frac{x_0^{ij}}{T_{\mu/ij}}\right)\prod_{s=1}^{m}\left(1-\frac{\nabla}{T_{\alpha^{(s)}}}\right) \;.$$

However, using I.36, we derive that

$$\prod_{s=1}^{m}\left(1-\frac{\nabla}{T_{\alpha^{(s)}}}\right)\phi_S^{(m)} \;=\; \sum_{\alpha\in S}\left(\prod_{\beta\in S/\{\alpha\}}\frac{1}{1-T_\alpha/T_\beta}\right)\prod_{s=1}^{m}\left(1-\frac{T_\alpha}{T_{\alpha^{(s)}}}\right)\tilde{H}_\alpha \;=\; 0 \;. \qquad 1.28$$

Thus the second product in 1.26 is entirely superfluous and we see that 1.26 is also another way of writing I.49 a). This completes the proof of Proposition I.8.

Proof of Theorem I.2

We see from the recurrences in I.19 that we may write

$$\begin{aligned} a)\quad A_{ij}^{x} \;&=\; C_{\mu/ij} \;-\; t\,C_{\mu/i+1,j} \;-\; C_{\mu/i,j+1} \;+\; t\,C_{\mu/i+1,j+1} \;, \\ b)\quad A_{ij}^{y} \;&=\; C_{\mu/ij} \;-\; q\,C_{\mu/i,j+1} \;-\; C_{\mu/i+1,j} \;+\; q\,C_{\mu/i+1,j+1} \;. \end{aligned} \qquad 1.29$$

Let us, for a moment, assume (as in the proof of Theorem I.1) that the shadows of the four cells (i,j), $(i+1,j)$, $(i,j+1)$, $(i+1,j+1)$ contain the same corners of μ. This means that if

$$\mathcal{P}red_{i,j}(\mu) \;=\; S \;=\; \{\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(m)}\}$$

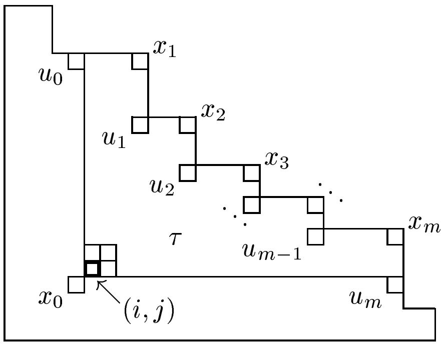
then we also have

$$S \;=\; \mathcal{P}red_{i+1,j}(\mu) \;=\; \mathcal{P}red_{i,j+1}(\mu) \;=\; \mathcal{P}red_{i+1,j+1}(\mu) \ .$$

Thus formula I.49 a) applied to each of the four cells gives

$$\begin{aligned} C_{\mu/ij} &= \frac{1}{M}\frac{T_{\mu/ij}}{\nabla}\ \Big(\prod_{s=0}^{m}\Big(1-\nabla\frac{u_s^{ij}}{T_{\mu/ij}}\Big)\Big)\ \phi_S^{(m)}\ ,\\ C_{\mu/i+1,j} &= \frac{1}{M}\frac{T_{\mu/i+1,j}}{\nabla}\ \Big(\prod_{s=0}^{m}\Big(1-\nabla\frac{u_s^{i+1,j}}{T_{\mu/i+1,j}}\Big)\Big)\ \phi_S^{(m)}\ ,\\ C_{\mu/i,j+1} &= \frac{1}{M}\frac{T_{\mu/i,j+1}}{\nabla}\ \Big(\prod_{s=0}^{m}\Big(1-\nabla\frac{u_s^{i,j+1}}{T_{\mu/i,j+1}}\Big)\Big)\ \phi_S^{(m)}\ ,\\ C_{\mu/i+1,j+1} &= \frac{1}{M}\frac{T_{\mu/i+1,j+1}}{\nabla}\ \Big(\prod_{s=0}^{m}\Big(1-\nabla\frac{u_s^{i+1,j+1}}{T_{\mu/i+1,j+1}}\Big)\Big)\ \phi_S^{(m)}\ . \end{aligned} \qquad 1.30$$

In the figure below we have depicted the generic situation we are dealing with.

$\mu \;=$ 

We have the partition τ that is in the shadow of (i,j), its corner cells, the corresponding corner weights, the cell (i,j) and the adjacent cells $(i+1,j)$, $(i,j+1)$, $(i+1,j+1)$. Now a look at the figure should reveal that in this case we have the identities

$$\frac{u_s^{ij}}{T_{\mu/ij}} \;=\; \frac{u_s^{i+1,j}}{T_{\mu/i+1,j}} \;=\; \frac{u_s^{i,j+1}}{T_{\mu/i,j+1}} \;=\; \frac{u_s^{i+1,j+1}}{T_{\mu/i+1,j+1}} \qquad (\text{ for } \ 1\le s\le m-1\ )\ .$$

This implies that $C_{\mu/ij}$, $C_{\mu/i+1,j}$, $C_{\mu/i,j+1}$ and $C_{\mu/i+1,j+1}$ have the common factor

$$CF \;=\; \frac{1}{M\,\nabla}\ \prod_{s=1}^{m-1}\Big(1-\frac{u_s^{ij}}{T_{\mu/ij}}\nabla\Big)\ . \qquad 1.31$$

Note further that, from the figure and the definition of the corner weights, we see that we must also have

$$
\begin{aligned}
u_0^{ij} &= t\,u_0^{i+1,j} = u_0^{i,j+1} = t\,u_0^{i+1,j+1}\ ,\\
u_m^{ij} &= u_m^{i+1,j} = q\,u_m^{i,j+1} = q\,u_m^{i+1,j+1}\ ,\\
T_{\mu/ij} &= t\,T_{\mu/i+1,j} = q\,T_{\mu/i,j+1} = t\,q\,T_{\mu/i+1,j+1}\ .
\end{aligned}
$$

Thus, setting for convenience

$$z_0^{ij} = \frac{u_0^{ij}}{T_{\mu/ij}}\,\nabla \qquad \text{and} \qquad z_m^{ij} = \frac{u_m^{ij}}{T_{\mu/ij}}\,\nabla\ ,$$

we can rewrite the identities in 1.30 in the form

$$
\begin{aligned}
C_{\mu/ij} &= CF\ T_{\mu/ij}\ \Big(1-z_0^{ij}\Big)\Big(1-z_m^{ij}\Big)\,\phi_S^{(m)}\ ,\\
C_{\mu/i+1,j} &= CF\ \frac{T_{\mu/ij}}{t}\ \Big(1-z_0^{ij}\Big)\Big(1-t\,z_m^{ij}\Big)\,\phi_S^{(m)}\ ,\\
C_{\mu/i,j+1} &= CF\ \frac{T_{\mu/ij}}{q}\ \Big(1-q\,z_0^{ij}\Big)\Big(1-z_m^{ij}\Big)\,\phi_S^{(m)}\ ,\\
C_{\mu/i+1,j+1} &= CF\ \frac{T_{\mu/ij}}{tq}\ \Big(1-q\,z_0^{ij}\Big)\Big(1-t\,z_m^{ij}\Big)\,\phi_S^{(m)}\ .
\end{aligned}
\qquad 1.32
$$

Substituting these expressions in 1.29 a) and grouping terms we get

$$
\begin{aligned}
A_{ij}^x &= CF\cdot T_{\mu/ij}\Big[(1-z_0^{ij})\big(1-z_m^{ij}-1+t\,z_m^{ij}\big) - \frac{1}{q}(1-q\,z_0^{ij})\big(1-z_m^{ij}-1+t\,z_m^{ij}\big)\Big]\,\phi_S^{(m)}\\
&= CF\cdot T_{\mu/ij}\Big[(1-z_0^{ij})\,z_m^{ij}\,(t-1) - \frac{1}{q}(1-q\,z_0^{ij})\,z_m^{ij}\,(t-1)\Big]\,\phi_S^{(m)}\\
&= CF\cdot T_{\mu/ij}\,z_m^{ij}\,(t-1)\,\big(1-z_0^{ij} - \frac{1}{q}+z_0^{ij}\big) = CF\cdot q^a\,\nabla\,(1-1/t)\,(1-1/q)\,\phi_S^{(m)}\ ,
\end{aligned}
$$

where the last equality is due to the fact that we have $u_m^{ij} = q^a/t$ with a the arm of the cell (i,j). Using 1.31 yields our desired formula

$$A_{ij}^x = q^a\ \prod_{s=1}^{m-1}\Big(1-\frac{u_s^{ij}}{T_{\mu/ij}}\,\nabla\Big)\,\phi_S^{(m)}\ . \qquad 1.33$$

Similarly, starting from 1.29 b) we derive that

$$
\begin{aligned}
A_{ij}^y &= CF\cdot T_{\mu/ij}\Big[(1-z_m^{ij})\big(1-z_0^{ij}-1+q\,z_0^{ij}\big) - \frac{1}{t}(1-t\,z_m^{ij})\big(1-z_0^{ij}-1+q\,z_0^{ij}\big)\Big]\,\phi_S^{(m)}\\
&= CF\cdot T_{\mu/ij}\Big[1-z_m^{ij}-\frac{1}{t}+z_m^{ij}\Big]\,z_0^{ij}\,(q-1)\,\phi_S^{(m)}\\
&= CF\cdot T_{\mu/ij}\,(1-1/t)\ z_0^{ij}\,(q-1) = CF\cdot t^l\,\nabla\,(1-1/t)\,(1-1/q)\,\phi_S^{(m)}\ ,
\end{aligned}
$$

where we have set $u_0^{ij} = t^l/q$ with l the leg of (i,j). This gives

$$\frac{A_{ij}^x}{q^a} = \frac{A_{ij}^y}{t^l} \qquad 1.34$$

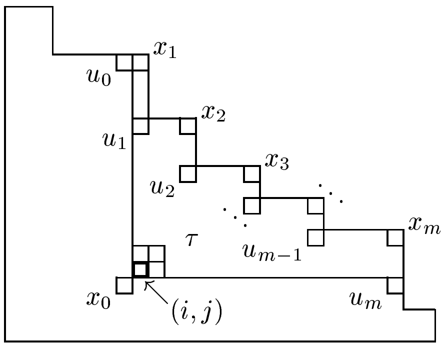
as desired.

Let us assume next that the shadows of (i,j) and $(i+1,j)$ contain the same corners of μ with

$$\mathcal{P}red_{i,j}(\mu) \;=\; \mathcal{P}red_{i+1,j}(\mu) \;=\; S \;=\; \{\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(m)}\}$$

but (see figure below) the shadows of $(i,j+1)$ and $(i+1,j+1)$ miss the corner $\mu/\alpha^{(1)}$. Thus

$$\mathcal{P}red_{i,j+1}(\mu) \;=\; \mathcal{P}red_{i+1,j+1}(\mu) \;=\; S^* \;=\; \{\alpha^{(2)},\ldots,\alpha^{(m)}\} \;=\; S\;/\;\{\alpha_1\}\ . \qquad 1.35$$

$\mu \;=$ 

Remarkably it develops that all the relations in 1.32 do hold true also in this case so that the final conclusions in 1.33 and 1.34 still remain unchanged. To see how this comes about note first that, since the situation is the same as before as far as (i,j) and $(i+1,j)$ are concerned, there is no change in the first two equations of 1.30 and 1.32. On the other hand, in this case, the remaining two equations in 1.30 become

$$\begin{aligned} a)\quad C_{\mu/i,j+1} \;&=\; \frac{1}{M}\frac{T_{\mu/i,j+1}}{\nabla}\;\Big(\prod_{s=0}^{m-1}\Big(1-\nabla\frac{u_s^{i,j+1}}{T_{\mu/i,j+1}}\Big)\Big)\,\phi_{S^*}^{(m-1)}\ ,\\ b)\quad C_{\mu/i+1,j+1} \;&=\; \frac{1}{M}\frac{T_{\mu/i+1,j+1}}{\nabla}\;\Big(\prod_{s=0}^{m-1}\Big(1-\nabla\frac{u_s^{i+1,j+1}}{T_{\mu/i+1,j+1}}\Big)\Big)\,\phi_{S^*}^{(m-1)}\ . \end{aligned} \qquad 1.36$$

Now, using 1.35, from I.37 we get

$$\phi_{S^*}^{(m-1)} \;=\; \Big(1-\frac{\nabla}{T_{\alpha^{(1)}}}\Big)\,\phi_S^{(m)}\ .$$

However, since $T_{\alpha^{(1)}}\,x_1^{ij}=T_{\mu/ij}$ and in this case $x_1^{ij}=q\,u_0^{ij}$, this may be rewritten in the form

$$\phi_{S^*}^{(m-1)} \;=\; \Big(1-\frac{q\,u_0^{ij}\nabla}{T_{\mu/ij}}\Big)\,\phi_S^{(m)}\ . \qquad 1.37$$

Note further that we also have

$$u_s^{i,j+1} \;=\; \frac{1}{q}\;u_{s+1}^{ij} \qquad\qquad (\text{for } s=0,\ldots,m-1)\ . \qquad 1.38$$

Since $T_{\mu/i,j+1}=T_{\mu/ij}/q$, substituting 1.37 and 1.38 in 1.36 a) gives

$$C_{\mu/i,j+1} \;=\; \frac{1}{M}\frac{T_{\mu/ij}}{q\nabla}\;\Big(\prod_{s=1}^{m}\Big(1-\nabla\frac{u_s^{ij}}{T_{\mu/ij}}\Big)\Big)\;\Big(1-\frac{q\,u_0^{ij}\nabla}{T_{\mu/ij}}\Big)\;\phi_S^{(m)}\;. \qquad 1.39$$

Setting again $z_0^{ij}=u_0^{ij}\nabla/T_{\mu/ij}$ and $z_m^{ij}=u_m^{ij}\nabla/T_{\mu/ij}$, we see that 1.39 may be written as

$$C_{\mu/i,j+1} \;=\; CF\;\frac{T_{\mu/ij}}{q}\;(1-q\,z_0^{ij})(1-z_m^{ij}\,)\;\phi_S^{(m)}\;,$$

which is in perfect agreement with 1.32.

Similarly, using 1.37 and the identities

$$u_s^{i+1,j+1}=\frac{1}{qt}\,u_{s+1}^{ij}\quad(\text{for } s=0,\ldots,m-2)\;,\qquad u_{m-1}^{i+1,j+1}=\frac{1}{q}\,u_m^{ij}\qquad\text{and}\qquad T_{\mu/i+1,j+1}=T_{\mu/ij}/qt$$

we may write 1.36 b) as

$$C_{\mu/i+1,j+1} \;=\; \frac{1}{M}\frac{T_{\mu/ij}}{qt\nabla}\;\Big(\prod_{s=1}^{m-1}\Big(1-\nabla\frac{u_s^{ij}}{T_{\mu/ij}}\Big)\Big)\;\Big(1-\frac{t\,u_m^{ij}\nabla}{T_{\mu/ij}}\Big)\;\Big(1-\frac{q\,u_0^{ij}\nabla}{T_{\mu/ij}}\Big)\;\phi_S^{(m)}\;,$$

which is easily seen to be again in perfect agreement with 1.32. The case we have just considered should be sufficient evidence that we have an underlying mechanism here that forces the same final answer to come out in all the possible cases, completing our proof of Theorem I.1.

An immediate consequence of I.50 is the following remarkable fact

Theorem 1.1

Under the SF-heuristic and the $n!$ conjecture, all the modules $\mathbf{A}_{ij}^x$, $\mathbf{A}_{ij}^y$, for $\mu\vdash n+1$, are bigraded versions of the left regular representation of S_n.

Proof

Note that formula I.50 may also be written in the form

$$A_{ij}^x/q^a \;=\; A_{ij}^y/t^l \;=\; \sum_{k=1}^{m}\frac{(-\nabla)^{m-k}\phi_S^{(m)}}{T_{\mu/ij}^{m-k}}\,e_{m-k}[\,u_1^{ij}+u_2^{ij}+\cdots+u_{m-1}^{ij}\,]\;\;;$$

thus, using I.35 we get

$$A_{ij}^x/q^a \;=\; A_{ij}^y/t^l \;=\; \sum_{k=1}^{m}\frac{\phi_S^{(k)}}{T_{\mu/ij}^{m-k}}\,e_{m-k}[\,u_1^{ij}+u_2^{ij}+\cdots+u_{m-1}^{ij}\,]\;\;. \qquad 1.40$$

To compare this formula with I.41 we should set there, for every $\alpha_i\in S$,

$$T_{\alpha_i}=T_\mu/x_i$$

and obtain

$$\tilde H_{\alpha_i} \;=\; \sum_{k=1}^{m}\frac{\phi_S^{(k)}}{T_\mu^{m-k}}\,e_{m-k}[\,x_1+x_2+\cdots+x_m\;\;-\;\;x_i\,]\;\;. \qquad 1.41$$

Since $\alpha_i \to \mu$ gives that $\alpha_i \vdash n$, we have the expansion

$$\tilde{H}_{\alpha_i} \;=\; \sum_{\lambda \vdash n} S_\lambda(x)\, \tilde{K}_{\lambda,\alpha_i}(q,t)$$

which, together with the Macdonald result $\tilde{K}_{\lambda,\alpha_i}(1,1) = f_\lambda$, yields that

$$\tilde{H}_{\alpha_i}(x;1,1) \;=\; \sum_{\lambda \vdash n} S_\lambda(x)\, f_\lambda \;=\; h_1^n(x) \;.$$

Thus, placing $q = t = 1$ in 1.40 and 1.41 we obtain that

$$A^x_{ij}(x;1,1) \;=\; A^y_{ij}(x;1,1) \;=\; \sum_{k=1}^{m} \phi_S^{(k)}(x;1,1) \binom{m-1}{k-1} \;=\; \tilde{H}_{\alpha_i}(x;1,1) \;=\; h_1^n(x) \;,$$

which proves that both A^x_{ij} and A^y_{ij} are Frobenius characteristics of bigraded left regular representations.

Another interesting identity relating the characteristics A^x_{ij} and A^y_{ij} may be derived from the theory of Macdonald polynomials as well as the present heuristics:

Theorem 1.2

$$T_{\mu/ij} \downarrow A^x_{ij} \;=\; A^y_{ij} \;. \tag{1.42}$$

Proof

Due to the fact that $C_{\mu/ij}(x;q,t)$ is the bivariate Frobenius characteristic of $\mathbf{M}_{\mu/ij} = \mathcal{L}_\partial[\Delta_{\mu/ij}]$, from I.25 with $\Delta = \Delta_{\mu/ij}$ we get that

$$T_{\mu/ij} \downarrow C_{\mu/ij} \;=\; C_{\mu/ij} \;. \tag{1.43}$$

Now I.19 a) and b) give

$$\begin{aligned} a)\quad A^x_{ij} &= C_{\mu/ij} - t\, C_{\mu/i+1,j} - C_{\mu/i,j+1} + t\, C_{\mu/i+1,j+1} \;, \\ b)\quad A^y_{ij} &= C_{\mu/ij} - q\, C_{\mu/i,j+1} - C_{\mu/i+1,j} + q\, C_{\mu/i+1,j+1} \;. \end{aligned} \tag{1.44}$$

Using 1.43 on each term of the expansion in 1.44 a) yields

$$\downarrow A^x_{ij} \;=\; \frac{1}{T_{\mu/ij}}\, C_{\mu/ij} - \frac{1}{t\, T_{\mu/i+1,j}}\, C_{\mu/i+1,j} - \frac{1}{T_{\mu/i,j+1}}\, C_{\mu/i,j+1} + \frac{1}{t\, T_{\mu/i+1,j+1}}\, C_{\mu/i+1,j+1} \;.$$

Multiplying both sides by $T_{\mu/ij}$ and using the identities

$$T_{\mu/ij} \;=\; t\, T_{\mu/i+1,j} \;=\; q\, T_{\mu/i,j+1} \;=\; tq\, T_{\mu/i+1,j+1}$$

we finally obtain

$$T_{\mu/ij} \downarrow A^x_{ij} \;=\; C_{\mu/ij} - C_{\mu/i,j+1} - q\, C_{\mu/i,j+1} + q\, C_{\mu/i+1,j+1}$$

whose right-hand side is seen to be a rearrangement of the right-hand side of 1.44 b). This proves 1.42.

Remark 1.2

We should emphasize at this point that each symmetric function identity we write down here may be studied from two different viewpoints. On one hand it can be viewed as an identity involving Macdonald polynomials and may be verified using purely symmetric function manipulations. On the other hand, if we view it as an identity relating two bigraded Frobenius characteristics, we may try to give it a representation theoretical proof. It develops that 1.42, which here and after we shall refer to as the "*flip identity*," may also be shown in this manner. It is significant that the "crucial identity," which on the surface appears quite similar, nevertheless turns out so much more difficult to prove.

Our point of departure is the introduction of a bilinear form in each of the spaces $\mathbf{M}_{\mu/ij}$, which is defined by setting

$$\langle\langle P\ ,\ Q\rangle\rangle \;=\; \Big\langle\ \mathbf{flip}_{ij}^{-1}\, P\ ,\ Q\ \Big\rangle\ , \qquad 1.45$$

where for convenience, we set

$$\mathbf{flip}_{ij}^{-1} \;=\; \mathbf{flip}_{\Delta_{\mu/ij}}^{-1} \quad .$$

In other words, for any polynomial $(x;y)$, $\mathbf{flip}_{ij}^{-1}\, P$ denotes the unique polynomial $P_1 \in \mathbf{M}_{\mu/ij}$ such that $P = P_1(\partial)\Delta_{\mu/ij}$. In particular, we see that if $P_1 = \mathbf{flip}_{ij}^{-1}\, P$ and $Q_1 = \mathbf{flip}_{ij}^{-1}\, Q$ then 1.45 may also be rewritten as

$$\langle\langle P\ ,\ Q\rangle\rangle \;=\; P_1(\partial)Q_1(\partial)\ \Delta_{\mu/ij}\ \Big|_{x=y=0}\ ,$$

yielding that $\langle\langle\ ,\ \rangle\rangle$ is a symmetric bilinear form. Now it develops that this form may be used to construct a nondegenerate pairing of $\mathbf{A}_{ij}^{x}$ with $\mathbf{A}_{ij}^{y}$ that forces the identity in 1.42. More precisely, we have the following remarkable result.

Proposition 1.2

The two spaces $\mathbf{A}_{ij}^{x}$ and $\mathbf{A}_{ij}^{y}$ have the same dimension and we can construct two bihomogeneous bases $\mathcal{B}_{ij}^{x} = \{f_1^x, f_2^x, \ldots, f_N^x\}$ and $\mathcal{B}_{ij}^{y} = \{f_1^y, f_2^y, \ldots, f_N^y\}$ for $\mathbf{A}_{ij}^{x}$ and $\mathbf{A}_{ij}^{y}$ respectively such that

$$\langle\langle f_r^x\ ,\ f_s^y\rangle\rangle \;=\; \begin{cases} 1 & \text{if } r = s\ , \\ 0 & \text{if } r \neq s\ . \end{cases} \qquad 1.46$$

In particular, we must have

$$\text{weight}(f_r^x)\ \times\ \text{weight}(f_r^y) \;=\; T_{\mu/ij}\ , \qquad 1.47$$

where for convenience for a bihomogeneous polynomial P of bidegree (h,k) we set

$$\text{weight}(P) = t^h q^k \quad .$$

Moreover, if for all $\sigma \in S_n$ we have

$$\sigma f_s^x \;=\; \sum_{r=1}^{N} f_r^x\ a_{r,s}(\sigma) \qquad 1.48$$

then

$$\sigma f_s^y \;=\; \mathrm{sign}(\sigma)\;\sum_{r=1}^{N} f_r^y \;\, a_{s,r}(\sigma^{-1}) \;\; . \qquad 1.49$$

Proof

Note that I.29 gives the orthogonal decompositions

$$\mathbf{M}_{\mu/ij} = \mathbf{M}_{\mu/i,j+1} \;\oplus_\perp\; \mathbf{flip}_{ij}^{-1}\;\mathbf{K}_{ij}^y \quad , \qquad \mathbf{M}_{\mu/ij} = \mathbf{M}_{\mu/i+1,j} \;\oplus_\perp\; \mathbf{flip}_{ij}^{-1}\;\mathbf{K}_{ij}^x \;\; . \qquad 1.50$$

In particular this means that, for $P \in \mathbf{M}_{\mu/ij}$, we have

$$\big\langle\, P \;,\; Q \,\big\rangle \;=\; 0$$

for all $Q \in \mathbf{flip}_{ij}^{-1}\;\mathbf{K}_{ij}^y$ if and only if

$$P \in \mathbf{M}_{\mu/i,j+1} \;\; .$$

We thus deduce the equivalences

$$P \in \mathbf{K}_{ij}^x \quad \text{and} \quad P \in (\mathbf{flip}_{ij}^{-1})^{\perp}\;\mathbf{K}_{ij}^y \qquad \Longleftrightarrow \qquad P \in \mathbf{K}_{i,j+1}^x \;\; . \qquad 1.51$$

Similarly we derive that

$$P \in \mathbf{K}_{ij}^y \quad \text{and} \quad P \in (\mathbf{flip}_{ij}^{-1})^{\perp}\;\mathbf{K}_{ij}^x \qquad \Longleftrightarrow \qquad P \in \mathbf{K}_{i+1,j}^y \;\; . \qquad 1.52$$

In view of our definition 1.45 of the form $\langle\langle\;,\;\rangle\rangle$ we deduce from 1.51 and 1.52 that if $P_1, P_2 \in \mathbf{K}_{ij}^x$ and $Q_1, Q_2 \in \mathbf{K}_{ij}^y$, with $P_1 - P_2 \in \mathbf{K}_{i,j+1}^x$ and $Q_1 - Q_2 \in \mathbf{K}_{i+1,j}^y$, then

$$\langle\langle P_1 - P_2 \;,\; Q_1 \rangle\rangle = 0 \quad \text{and} \quad \langle\langle P_2 \;,\; Q_1 - Q_2 \rangle\rangle = 0 \;\; .$$

Thus

$$\langle\langle P_1 \;,\; Q_1 \rangle\rangle = \langle\langle P_1 - P_2 \;,\; Q_1 \rangle\rangle + \langle\langle P_2 \;,\; Q_1 \rangle\rangle \;=\; \langle\langle P_2 \;,\; Q_1 \rangle\rangle$$

and similarly

$$\langle\langle P_2 \;,\; Q_1 \rangle\rangle = \langle\langle P_2 \;,\; Q_1 - Q_2 \rangle\rangle + \langle\langle P_2 \;,\; Q_2 \rangle\rangle \;=\; \langle\langle P_2 \;,\; Q_2 \rangle\rangle \;\; ,$$

yielding

$$\langle\langle P_1 \;,\; Q_1 \rangle\rangle \;=\; \langle\langle P_2 \;,\; Q_2 \rangle\rangle \;\; . \qquad 1.53$$

This shows that $\langle\langle\;,\;\rangle\rangle$ is a well-defined pairing of $\mathbf{A}_{ij}^x = \mathbf{K}_{ij}^x/\mathbf{K}_{i,j+1}^x$ with $\mathbf{A}_{ij}^y = \mathbf{K}_{ij}^y/\mathbf{K}_{i+1,j}^x$. We are left to show that it is nondegenerate. To this end suppose that for some representative element $P \in \mathbf{K}_{ij}^x$ of $\mathbf{K}_{ij}^x/\mathbf{K}_{i,j+1}^x$ we have

$$\langle\langle P \;,\; Q \rangle\rangle = \big\langle\; \mathbf{flip}_{ij}^{-1}\, Q \;,\; P \,\big\rangle \;=\; 0 \qquad 1.54$$

for all representatives $Q \in \mathbf{K}_{ij}^y$ of $\mathbf{K}_{ij}^y/\mathbf{K}_{i+1,j}^x$. In view of 1.53, the relation in 1.54 must hold true for all $Q \in \mathbf{K}_{ij}^y$, but then the first equation in 1.50 yields that $P \in \mathbf{M}_{\mu/i,j+1}$ and this, together with

$P \in \mathbf{K}^x_{ij}$, forces $P \in \mathbf{K}^x_{i,j+1}$. In other words, P is equal to zero in the quotient $\mathbf{K}^x_{ij}/\mathbf{K}^x_{i,j+1}$. Similarly we show that 1.54 for all $P \in \mathbf{A}^x_{ij}$ can only hold for $Q = 0$ in $\mathbf{A}^y_{ij}$. Thus $\langle\langle\ ,\ \rangle\rangle$ is nondegenerate.

Now let $\{f_1, f_2, \ldots, f_N\}$ and $\{g_1, g_2, \ldots, g_M\}$ be bihomogeneous bases for $\mathbf{A}^x_{ij}$ and $\mathbf{A}^y_{ij}$, and set

$$R_{i,j} \;=\; \langle\langle f_i, g_j\rangle\rangle \ .$$

Note that we cannot have $N < M$ for otherwise we would be able to construct a nontrivial solution $c_1, c_2, \ldots, c_M$ of the homogeneous system of equations

$$c_1 R_{i,1} + c_2 R_{i,2} + \cdots + c_M R_{i,M} \;=\; 0 \qquad (\text{ for } \quad i = 1, 2, \ldots, N\ )$$

and this would contradict the nondegeneracy of $\langle\langle\ ,\ \rangle\rangle$. For the same reason we can't have $N > M$ nor $M = N$ with $R = \|R_{i,j}\|_{i=1}^N$ a singular matrix. Thus $\mathbf{A}^x_{ij}$ and $\mathbf{A}^y_{ij}$ have the same dimension and R must be invertible. This given, the two bases $\{f^x_1, f^x_2, \ldots, f^x_N\}$ with the asserted properties may be obtained by setting

$$\{f^x_1, f^x_2, \ldots, f^x_N\} \;=\; \{f_1, f_2, \ldots, f_N\}$$

and

$$\{f^y_1, f^y_2, \ldots, f^y_N\} \;=\; \{f^x_1, f^x_2, \ldots, f^x_N\} \times R^{-1} \ .$$

With this choice, 1.46 is immediate and then 1.47 follows from the fact that if for two bihomogeneous polynomials P, Q we have

$$P(\partial)Q(\partial)\Delta_{\mu/ij} = 1$$

then necessarily their bidegrees must add up to the bidegree of $\Delta_{\mu/ij}$. Finally, to show that 1.49 follows from 1.48 note first that from 1.46 we derive that the expansion of any element $Q \in \mathbf{A}^y_{ij}$ in terms of the basis $\{f^y_1, f^y_2, \ldots, f^y_N\}$ may be written in the form

$$Q \;=\; \sum_{r=1}^N f^y_r \ \langle\langle Q, f^x_r\rangle\rangle \ .$$

Thus we may write

$$\sigma f^y_s \;=\; \sum_{r=1}^N f^y_r \ \langle\langle \sigma f^y_s \ , \ f^x_r\rangle\rangle \ . \tag{1.55}$$

However, we see that

$$\begin{aligned}\langle\langle \sigma f^y_s \ , \ f^x_r\rangle\rangle &= \left\langle \mathbf{flip}^{-1}_{ij}\, \sigma f^y_s \ , \ f^x_r \right\rangle = \mathrm{sign}(\sigma) \left\langle \sigma \ \mathbf{flip}^{-1}_{ij}\, f^y_s \ , \ f^x_r\right\rangle \\ &= \mathrm{sign}(\sigma) \left\langle \mathbf{flip}^{-1}_{ij}\, f^y_s \ , \ \sigma^{-1} f^x_r\right\rangle = \mathrm{sign}(\sigma)\, \langle\langle f^y_s \ , \ \sigma^{-1} f^x_r\rangle\rangle \ .\end{aligned}$$

Substituting this in 1.55 gives

$$\sigma f^y_s \;=\; \mathrm{sign}(\sigma) \sum_{r=1}^N f^y_r \ \langle\langle f^y_s \ , \ \sigma^{-1} f^x_r\rangle\rangle \ . \tag{1.56}$$

Now, from 1.48 for σ^{-1} and r,s interchanged we derive that

$$\sigma^{-1} f_r^x \;=\; \sum_{s=1}^{N} f_s^x \;\, a_{s,r}(\sigma^{-1})$$

and 1.46 then gives that

$$\langle\langle f_s^y \,,\, \sigma^{-1} f_r^x \rangle\rangle \;=\; a_{s,r}(\sigma^{-1}) \;.$$

Substituting in 1.56 yields 1.49 as desired, completing our proof.

Remark 1.3

We should note that the fact that $\mathbf{A}_{ij}^x$ and $\mathbf{A}_{ij}^y$ have the same dimension is also an immediate consequence of 1.44 a) and b). In fact, setting $q=t$ there yields the stronger result that these two modules (graded by total degree) have identical Frobenius characteristics. We should also emphasize that this argument as well as the proof of Proposition 1.2 makes no use of any of our conjectures nor any identification of the polynomials $C_{\mu/ij}$ with expressions (such as in 1.20) involving the Macdonald polynomials $\tilde{H}_\mu$.

Remark 1.4

Proposition 1.2 leads to an alternate proof of Theorem 1.2 and a direct representation theoretical interpretation of the identity in 1.42. To see this note that 1.48 yields that the bigraded characters of $\mathbf{A}_{ij}^x$ and $\mathbf{A}_{ij}^x$, are respectively given by the expressions

$$\big(\mathrm{ch}\,\mathbf{A}_{ij}^x\big)(\sigma;q,t) = \sum_{s=1}^{N} \text{weight}(f_r^x)\; a_{r,r}(\sigma) \;, \quad \big(\mathrm{ch}\,\mathbf{A}_{ij}^y\big)(\sigma;q,t) = \text{sign}(\sigma) \sum_{s=1}^{N} \text{weight}(f_r^y)\; a_{r,r}(\sigma^{-1}) \;. \tag{1.57}$$

Now, 1.46 gives

$$\text{weight}(f_r^y) \;=\; T_{\mu/ij} \,/\, \text{weight}(f_r^x) \;,$$

and from 1.57 we derive that

$$\big(\mathrm{ch}\,\mathbf{A}_{ij}^y\big)(\sigma;q,t) = T_{\mu/ij}\;\text{sign}(\sigma) \sum_{s=1}^{N} a_{r,r}(\sigma^{-1}) / \,\text{weight}(f_r^x) = T_{\mu/ij}\;\text{sign}(\sigma)\;\big(\mathrm{ch}\,\mathbf{A}_{ij}^x\big)(\sigma;1/q,1/t) \;.$$

Equating the Frobenius images of both sides yields 1.42.

We shall terminate this section by showing that a proof of the "crucial identity" would in one stroke establish Conjecture I.2 as well as all the conjectured expansions in I.16. This implication is based on a result of M. Haiman in [14] which asserts that a proof of the $n!$ conjecture for a given μ yields that the bigraded Frobenius characteristic of $\mathbf{M}_\mu$ for that same μ must be given by the polynomial $\tilde{H}_\mu(x;q,t)$. Since the $n!$ conjecture has been verified by computer for all $|\mu| \le 8$, the argument can proceed by induction on $|\mu|$. So let us assume that for a given $\mu \vdash n$ we have $\dim \mathbf{M}_\nu = (n-1)!$ for all $\nu{\to}\mu$. The Haiman result then yields that for all $\nu{\to}\mu$ the bigraded Frobenius characteristic of $\mathbf{M}_\nu$ is $\tilde{H}_\nu$. Since I.20 is just another way of writing the four term recursion in I.17, its validity implies (by Theorem I.1) that I.16 must hold true as well. Now, as we have seen, I.16, for $(i,j)=(0,0)$ states (via the Macdonald identity in I.13) that

$$C_{\mu/00}(x;q,t) \;=\; \partial_{p_1}\tilde{H}_\mu(x;q,t)\ .$$

Combining this with Proposition I.5 gives

$$\partial_{p_1}C_\mu(x;q,t) \;=\; \partial_{p_1}\tilde{H}_\mu(x;q,t)\ .$$

In particular, applying $\partial_{p_1}^{n-1}$ to both sides we get that the bigraded Hilbert series of the module $\mathbf{M}_\mu$ is given by the polynomial

$$F_\mu(q,t) \;=\; \partial_{p_1}^n\tilde{H}_\mu(x;q,t) \;=\; \sum_{\lambda\vdash n} f_\lambda\,\tilde{K}_{\lambda\mu}(q,t)\ .$$

Here, the last equality follows from I.8. But now, the Macdonald result that $\tilde{K}_{\lambda\mu}(1,1)=f_\lambda$ yields that

$$\dim \mathbf{M}_\mu \;=\; F_\mu(1,1) \;=\; \sum_{\lambda\vdash n} f_\lambda^2 \;=\; n!\ ,$$

completing the induction. Then of course we can combine this with Haiman's result and obtain that the $\tilde{K}_{\lambda\mu}(q,t)$ are polynomials with positive integer coefficients. To show that I.20 implies Conjecture I.2, we use I.16 with the $c_{\tau\rho}(q,t)$ given by I.14 and obtain

$$C_{\mu/ij}(x;q,t) \;=\; \sum_{\rho\to\tau}\ \prod_{s\in\mathcal{R}_{\tau/\rho}} \frac{t^{l_\tau(s)}-q^{a_\tau(s)+1}}{t^{l_\tau(s)}-q^{a_\tau(s)}} \prod_{s\in\mathcal{C}_{\tau/\rho}} \frac{q^{a_\tau(s)}-t^{l_\tau(s)+1}}{q^{a_\tau(s)}-t^{l_\tau(s)}}\ \ \tilde{H}_{\mu-\tau+\rho}(x;q,t)\ .$$

Now this identity, for $t=1/q$, may be rewritten as

$$\begin{aligned} C_{\mu/ij}(x;q,1/q) \;&=\; \sum_{\rho\to\tau}\Big(\prod_{s\in R_{\tau/\rho}\cup\, C_{\tau/\rho}} \frac{1-q^{h_\tau(s)}}{1-q^{h_\rho(s)}}\Big)\frac{1}{q^{|C_{\tau/\rho}|}}\ \ \tilde{H}_{\mu-\tau+\rho}(x;q,1/q) \\ &=\; \sum_{\rho\to\tau}\frac{1}{1-q}\frac{\prod_{s\in\tau}(1-q^{h_\tau(s)})}{\prod_{s\in\rho}(1-q^{h_\rho(s)})}\ \frac{1}{q^{|C_{\tau/\rho}|}}\ \ \tilde{H}_{\mu-\tau+\rho}(x;q,1/q)\ . \end{aligned} \qquad 1.58$$

Here, the symbols $h_\tau(s),h_\rho(s)$ denote the hook lengths of the cell s with respect to the two partitions τ and ρ. Using the fact that $\tilde{H}_{\mu-\tau+\rho}(x;1,1)=h_1^{n-1}$, we see that letting $q\to 1$ reduces 1.58 to

$$C_{\mu/ij}(x;1,1) \;=\; \Big(\sum_{\rho\to\tau}\frac{\prod_{s\in\tau}h_\tau(s)}{\prod_{s\in\rho}h_\rho(s)}\Big)h_1^{n-1}\ . \qquad 1.59$$

Now the classical recursion for the number of standard tableaux gives

$$\sum_{\rho\to\tau}\frac{\prod_{s\in\tau}h_\tau(s)}{\prod_{s\in\rho}h_\rho(s)} \;=\; \frac{|\tau|}{f_\tau}\sum_{\rho\to\tau} f_\rho \;=\; |\tau|\ .$$

Thus, 1.59 may be rewritten as

$$C_{\mu/ij}(x;1,1) \;=\; |\tau|\ h_1^{n-1}\ ,$$

which establishes that $\mathbf{M}_{\mu/ij}$ consists of $|\tau|$ occurrences of the left regular representation of S_{n-1}, precisely as asserted by Conjecture I.2.

2. Conjectural Bases and the "crucial identity".

As we have seen in the previous section, the proof of Conjecture I.3 is reduced to establishing the "crucial identity"

$$t^l\, A^x_{ij} \;=\; q^a\, A^y_{ij} \;. \tag{2.1}$$

Although at the moment we are unable to prove this identity except in some special cases (see the next section), we can nevertheless search for the underlying representation theoretical mechanism that produces it. Our main goal in this section is to provide such a mechanism. This will be obtained by means of an algorithm for constructing bases for all of our spaces $\mathbf{M}_{\mu/ij}$ which is an extension of an algorithm given in [1]. All our constructs here, as in [1], are heavily dependent on the SF-heuristic, and as such they are still conjectural. Nevertheless, the validity of our arguments is strongly supported by amply verifiable theoretical and numerical consequences. Remarkably, these heuristics not only yield 2.1 but reveal that both 2.1 and the "flip" identity

$$A^x_{ij} \;=\; T_{\mu/ij} \downarrow A^y_{ij} \tag{2.2}$$

are consequences of considerably more refined versions. Before we can state these results we need to introduce some notation. Given that

$$\mathcal{P}red(\mu) \;=\; \{\, \nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(d)} \,\} \;,$$

It will be convenient here to use the shorter symbol S_{ij} to represent the subset $\mathcal{P}red_{ij}(\mu)$ define in I.31. That is, we are setting

$$S_{ij} \;=\; \{\, \nu^{(i)} \;:\; \mu/\nu^{(i)} \text{ is in the shadow of } (i,j) \,\} \tag{2.3}$$

Given that $S_{ij} = \{\nu^{(i_1)}, \nu^{(i_2)}, \ldots, \nu^{(i_m)}\}$, with $i_1 < i_2 < \cdots < i_m$, here and after we shall identify a subset T of S_{ij} with its corresponding $0,1$-word $\epsilon(T) = \epsilon_1 \cdots \epsilon_m$ defined by setting $\epsilon_s = 1$ or 0 according as $\nu^{(i_s)} \in T$ or $\nu^{(i_s)} \notin T$. Conversely, given such a word ϵ, we shall set

$$T(\epsilon) \;=\; \{\, \nu^{(i_s)} \;:\; \epsilon_s = 1 \,\} \;.$$

This given, recalling the definition in I.33, we shall also set (when $|S_{ij}| = m$)

$$\mathbf{M}^{\epsilon_1 \cdots \epsilon_m}_{ij} \;=\; \mathbf{M}^{T(\epsilon)}_{S_{ij}} \;. \tag{2.4}$$

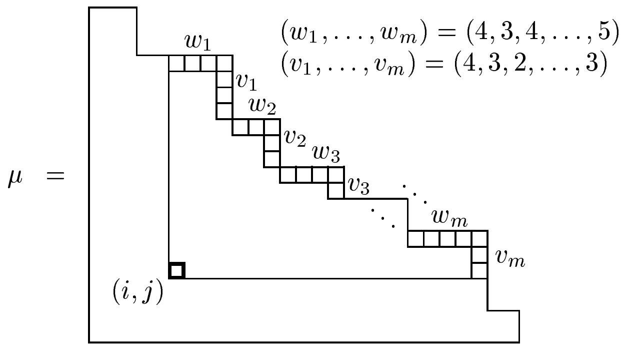
Assuming that the corners of μ in the shadow of (i,j) have weights

$$x^{ij}_r \;=\; t^{l'_r} q^{a'_r} \qquad (\text{ for } \quad r = 1 \ldots m\,)$$

we shall set

$$w^{ij}_r = a'_r - a'_{r-1} \qquad \text{and} \qquad v^{ij}_r = l'_r - l'_{r+1} \;.$$

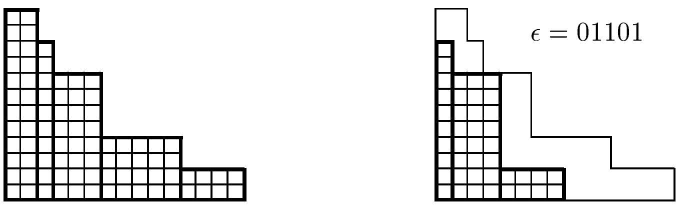
Of course when dealing with a fixed pair (i,j) we shall drop the superscripts ij and simply write x_r, w_r, v_r. In the figure below we have illustrated the geometric meaning of the parameters w_r and v_r as representing the exposed "*width*" of corner r and the vertical "*drop*" immediately after it.



Given a subset $T = T(\epsilon) \subseteq S_{ij}$ we shall let $D_{ij}(T)$ denote the subdiagram of μ obtained by the following construction:

Divide the shadow of (i,j) in μ into m rectangles, of widths $w_1,\ldots,w_m$ from left to right, by dropping vertical lines from each of its corners. Then delete the r^{th} rectangle if $\epsilon_r = 0$, and slide the remaining rectangles horizontally left to fill the gaps, setting the southwest corner of the leftmost rectangle at (i,j). This done, the cells covered by the resulting rectangles form $D_{ij}(T)$.

In the figure below we illustrate this construction when $\mu = (15,15,11,11,6,6,6,6,3,3,2,2)$ $m = 5$, $i = j = 0$ and $T = \{2,3,5\}$ or $\epsilon = 01101$.



We need one further convention before we can present our basic construction. In some of the formulas that follow it will be more illuminating to use the symbol "$\mathbf{M}_1(\partial)\Delta$" rather than "$\mathbf{flip}_\Delta\ \mathbf{M}_1$" to denote the image of $\mathbf{M}_1$ by $\mathbf{flip}_\Delta$. In other words, we are setting

$$\mathbf{M}_1(\partial)\Delta \;=\; \left\{\, P(\partial)\Delta \;:\; P \in \mathbf{M}_1 \,\right\}\ . \tag{2.5}$$

This given, extensive numerical and theoretical evidence strongly suggests that

Conjecture 2.1

The space $\mathbf{M}_{\mu/ij}$ affords the following direct sum decomposition:

$$\mathbf{M}_{\mu/ij} \;=\; \bigoplus_{\emptyset \neq T \subseteq S_{ij}} \;\; \bigoplus_{(i',j') \in D_{ij}(T)} \;\; \mathbf{M}^{T}_{S_{ij}}(\partial)\Delta_{\mu/i'j'}\ . \tag{2.6}$$

The constructions underlying this identity are of course heavily dependent on the "Science Fiction Conjecture" (see [1]) which states that the modules $\mathbf{M}_{\nu_1}, \mathbf{M}_{\nu_2}, \ldots, \mathbf{M}_{\nu_d}$ generate a distributive lattice under span and intersection. Under this assumption, 2.6 yields an algorithm for constructing bihomogeneous bases for the modules $\mathbf{M}_{\mu/ij}$. To be explicit, this algorithm consists in preconstructing bihomogeneous bases $\mathcal{B}_S^T$ for all the subspaces $\mathbf{M}_S^T$ given in I.33 and for all pairs

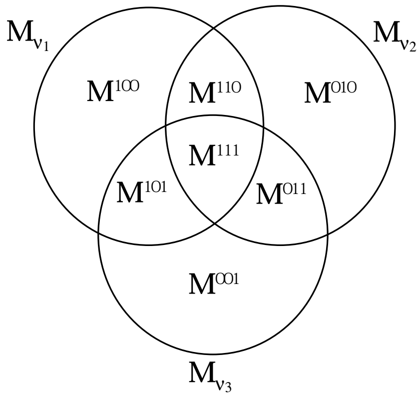
$$\left\{ \ (T,S) \ : \ \emptyset \neq T \subseteq S \subseteq \{1,2,\ldots,d\} \ \right\} \ .$$

This done, for all $(i,j) \in \mu$ a basis for $\mathbf{M}_{\mu/ij}$ should be given by the collection

$$\mathcal{B}_{\mu/ij} \ = \ \sum_{\emptyset \neq T \subseteq S_{ij}} \ \sum_{(i',j') \in D_{ij}(T)} \ \mathbf{flip}_{i'j'} \ \mathcal{B}_{S_{ij}}^T \qquad 2.7$$

where for convenience we have set $\mathbf{flip}_{i'j'} = \mathbf{flip}_{\Delta_{\mu/i'j'}}$.

Before we proceed any further it will be good to see what 2.6 yields in at least one concrete example. We shall illustrate it in the case $\mu = (3,2,1)$ and $(i,j) = (0,0)$. To this end, we begin by noting that under the SF hypotheses, in any three-corner case, the module $\mathbf{M}_{\nu_1} + \mathbf{M}_{\nu_2} + \mathbf{M}_{\nu_3}$ decomposes into the direct sum of the submodules $\mathbf{M}^{\epsilon_1 \epsilon_2 \epsilon_3}$ as indicated by the following figure.



2.8

For $\mu = (3,2,1)$ we have $\nu_1 = (3,2)$, $\nu_2 = (3,1,1)$, $\nu_3 = (2,2,1)$. Accordingly we have the direct sum decompositions

$$\begin{aligned} \mathbf{M}_{32} \ &= \ \mathbf{M}^{100} \oplus \mathbf{M}^{110} \oplus \mathbf{M}^{101} \oplus \mathbf{M}^{111} \\ \mathbf{M}_{311} \ &= \ \mathbf{M}^{010} \oplus \mathbf{M}^{110} \oplus \mathbf{M}^{011} \oplus \mathbf{M}^{111} \\ \mathbf{M}_{221} \ &= \ \mathbf{M}^{001} \oplus \mathbf{M}^{101} \oplus \mathbf{M}^{011} \oplus \mathbf{M}^{111} \ . \end{aligned} \qquad 2.9$$

After constructing bases $\mathcal{B}^{\epsilon_1 \epsilon_2 \epsilon_3}$ for each of the submodules $\mathbf{M}^{\epsilon_1 \epsilon_2 \epsilon_3}$ appearing above, the result of applying the recipe in 2.7 with $(i,j) = (0,0)$ and $S_{00} = \{1,2,3\}$ may be described by the following diagrams:

$$\epsilon(T)=100 \longrightarrow \begin{array}{|c|ccc} \hline \mathcal{B}^{100} & \multicolumn{1}{|c}{} & & \\ \cline{1-2} \mathcal{B}^{100} & \multicolumn{1}{|c|}{0} & & \\ \cline{1-3} \mathcal{B}^{100} & \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{0} & \\ \cline{1-3} \end{array} \qquad \epsilon(T)=101 \longrightarrow \begin{array}{|c|ccc} \hline \mathcal{B}^{101} & \multicolumn{1}{|c}{} & & \\ \cline{1-2} \mathcal{B}^{101} & \multicolumn{1}{|c|}{0} & & \\ \cline{1-3} \mathcal{B}^{101} & \multicolumn{1}{|c|}{\mathcal{B}^{101}} & \multicolumn{1}{c|}{0} & \\ \cline{1-3} \end{array}$$

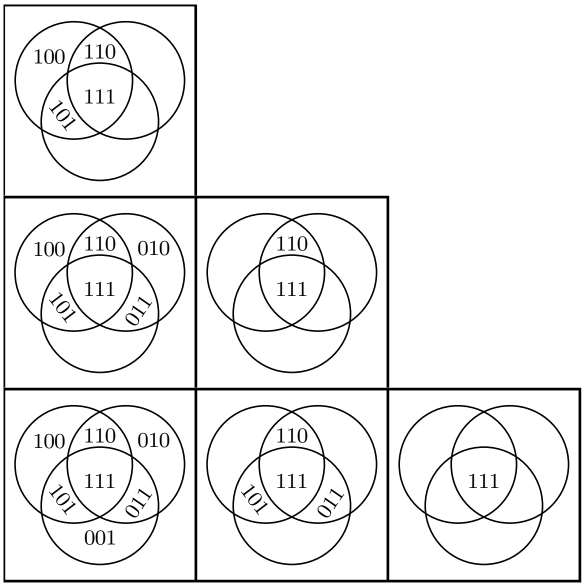
$$\epsilon(T)=111 \longrightarrow \begin{array}{|c|ccc} \hline \mathcal{B}^{111} & \multicolumn{1}{|c}{} & & \\ \cline{1-2} \mathcal{B}^{111} & \multicolumn{1}{|c|}{\mathcal{B}^{111}} & & \\ \cline{1-3} \mathcal{B}^{111} & \multicolumn{1}{|c|}{\mathcal{B}^{111}} & \multicolumn{1}{c|}{\mathcal{B}^{111}} & \\ \cline{1-3} \end{array} \qquad \epsilon(T)=110 \longrightarrow \begin{array}{|c|ccc} \hline \mathcal{B}^{110} & \multicolumn{1}{|c}{} & & \\ \cline{1-2} \mathcal{B}^{110} & \multicolumn{1}{|c|}{\mathcal{B}^{110}} & & \\ \cline{1-3} \mathcal{B}^{110} & \multicolumn{1}{|c|}{\mathcal{B}^{101}} & \multicolumn{1}{c|}{0} & \\ \cline{1-3} \end{array}$$

$$\epsilon(T)=010 \longrightarrow \begin{array}{|c|ccc} \hline 0 & \multicolumn{1}{|c}{} & & \\ \cline{1-2} \mathcal{B}^{010} & \multicolumn{1}{|c|}{0} & & \\ \cline{1-3} \mathcal{B}^{010} & \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{0} & \\ \cline{1-3} \end{array} \qquad \epsilon(T)=011 \longrightarrow \begin{array}{|c|ccc} \hline 0 & \multicolumn{1}{|c}{} & & \\ \cline{1-2} \mathcal{B}^{011} & \multicolumn{1}{|c|}{0} & & \\ \cline{1-3} \mathcal{B}^{011} & \multicolumn{1}{|c|}{\mathcal{B}^{011}} & \multicolumn{1}{c|}{0} & \\ \cline{1-3} \end{array}$$

$$\epsilon(T)=001 \longrightarrow \begin{array}{|c|ccc} \hline 0 & \multicolumn{1}{|c}{} & & \\ \cline{1-2} 0 & \multicolumn{1}{|c|}{0} & & \\ \cline{1-3} \mathcal{B}^{001} & \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{0} & \\ \cline{1-3} \end{array}$$

Placing a basis $\mathcal{B}^{\epsilon_1\epsilon_2\epsilon_3}$ in cell (i',j') means that in the construction of the basis for the module $\mathbf{M}_{321/00}$ we are to apply each element of $\mathcal{B}^{\epsilon_1\epsilon_2\epsilon_3}$ as a differential operator on the polynomial $\Delta_{321/i'j'}$.

In the same vein all the diagrams above may be combined in the single diagram given below.



2.10

We should note that from the Venn diagram in 2.8 or the expansions in 2.9, we get from this picture that a basis for the module $\mathbf{M}_{321/00}$ is given by the following collection:

$$\begin{aligned}\mathcal{B}_{321/00} \;=\; & \mathbf{flip}_{20}\ \mathcal{B}(\mathbf{M}_{32}) \;+\; \mathbf{flip}_{10}\ \mathcal{B}(\mathbf{M}_{32}+\mathbf{M}_{311}) \;+\; \mathbf{flip}_{11}\ \mathcal{B}(\mathbf{M}_{32}\cap\mathbf{M}_{311}) \\ & +\; \mathbf{flip}_{00}\ \mathcal{B}(\mathbf{M}_{32}+\mathbf{M}_{311}+\mathbf{M}_{221}) \;+\; \mathbf{flip}_{01}\Big(\mathcal{B}(\mathbf{M}_{32}\cap\mathbf{M}_{311})+\mathcal{B}^{101}+\mathcal{B}^{011}\Big) \\ & +\; \mathbf{flip}_{02}\ \mathcal{B}(\mathbf{M}_{32}\cap\mathbf{M}_{311}\cap\mathbf{M}_{221})\ .\end{aligned}$$

Here we have used the symbol $\mathcal{B}(\mathbf{M})$ to denote a bihomogeneous basis for a module $\mathbf{M}$. If we compare this result with the developments in Section 2 of [1] we see that although the algorithm described there involved a recursive process rather than an assignment of bases to cells, the results are identical. This is in fact a theorem that we shall soon establish. But before we get into that, it will be good to see how the present construction leads to a representation theoretical explanation of the crucial identity.

We should note that Conjecture 2.1 may also be stated in a manner which interchanges the roles of x and y. This "dual" version requires that the subdiagram $D_{ij}(T)$ be replaced by one obtained by dividing the shadow of (i,j) into m rectangles, of heights $v_1, v_2, \ldots, v_m$ from bottom to top, by drawing horizontal lines from each of the corners of μ, then deleting the r^{th} rectangle if $\epsilon_r = 0$ and vertically dropping the remaining rectangles to fill the gaps. This given, any of the constructions and proofs that follow have dual versions which can be routinely derived from their counterparts. We leave it to the reader to fill the gaps that result from our systematically dealing with only one of the versions. With this proviso our basic result here may be stated as follows.

Theorem 2.1

Let $(i,j) \in \mu$ and $|S_{ij}| = m$. Then on the validity of Conjecture 2.1, the following are isomorphic *as bigraded S_n-modules to $\mathbf{K}^x_{ij}$, $\mathbf{K}^y_{ij}$, $\mathbf{A}^x_{ij}$ and $\mathbf{A}^y_{ij}$ respectively:*

$$\widetilde{\mathbf{K}}^x_{ij} \;=\; \bigoplus_{\substack{\epsilon_1\cdots\epsilon_m \\ \epsilon_m=1}} \bigoplus_{\substack{(i,j')\in\mu \\ j'\geq j}} \mathbf{M}^{T(\epsilon)}_{S_{ij}}(\partial)\Delta_{i+\epsilon_c v_c+\cdots+\epsilon_m v_m-1,j'} \qquad 2.11$$

$$\widetilde{\mathbf{K}}^y_{ij} \;=\; \bigoplus_{\substack{\epsilon_1\cdots\epsilon_m \\ \epsilon_1=1}} \bigoplus_{\substack{(i',j)\in\mu \\ i'\geq i}} \mathbf{M}^{T(\epsilon)}_{S_{ij}}(\partial)\Delta_{i',j+\epsilon_1 w_1+\cdots+\epsilon_r w_r-1} \qquad 2.12$$

$$\widetilde{\mathbf{A}}^x_{ij} \;=\; \bigoplus_{\epsilon_1\cdots\epsilon_m\,:\,\epsilon_m=1} \mathbf{M}^{T(\epsilon)}_{S_{ij}}(\partial)\Delta_{i+\epsilon_1 v_1+\cdots+\epsilon_m v_m-1,j} \qquad 2.13$$

$$\widetilde{\mathbf{A}}^y_{ij} \;=\; \bigoplus_{\epsilon_1\cdots\epsilon_m\,:\,\epsilon_1=1} \mathbf{M}^{T(\epsilon)}_{S_{ij}}(\partial)\Delta_{i,j+\epsilon_1 w_1+\cdots+\epsilon_m w_m-1} \qquad 2.14$$

where r in 2.12 is determined so that within each term, the lowest corner weakly above (i',j') is the r^{th} (that is, $l'_{r+1} < i'-i \leq l'_r$), and in 2.11, the leftmost corner weakly right of (i',j') is the c^{th} (that is, $a'_{c-1} < j'-j \leq a'_c$).

Proof

We shall prove the relations for $\widetilde{\mathbf{K}}^y_{ij}$ and $\widetilde{\mathbf{A}}^y_{ij}$. The relations for the other two are proved similarly by means of the dual version of 2.6.

The kernel $\mathbf{K}^y_{ij}$ is isomorphic to

$$\widetilde{\mathbf{K}}^y_{ij} \;=\; \mathbf{M}_{ij}/D_y^{-1}(\mathbf{M}_{i,j+1}) \qquad 2.15$$

where $D_y^{-1}(\mathbf{M}_{i,j+1})$ denotes any submodule of $\mathbf{M}_{ij}$ whatsoever that is in one-to-one correspondence with $\mathbf{M}_{i,j+1}$ via D_y. We shall choose a preimage obtained by shifting each contribution to 2.6 one cell to the left, noting that

$$D_y\mathbf{M}^T_{S_{ij}}(\partial)\Delta_{\mu/i'j'} \;=\; \mathbf{M}^T_{S_{ij}}(\partial)\Delta_{\mu/i',j'+1} \;\; .$$

This given we may set

$$D_y^{-1}(\mathbf{M}_{i,j+1}) \;=\; \bigoplus_{\emptyset\neq T\subseteq S_{i,j+1}} \bigoplus_{(i',j')\in D_{i,j+1}(T)} \mathbf{M}^T_{S_{i,j+1}}(\partial)\Delta_{\mu/i',j'-1} \;\; . \qquad 2.16$$

For simplicity we shall only deal with the case when the shadows of (i,j) and $(i,j+1)$ contain the same corners of μ. In this case we may set $S_{i,j+1}=S_{ij}$ in 2.16 and obtain

$$D_y^{-1}(\mathbf{M}_{i,j+1}) \;=\; \bigoplus_{\emptyset\neq T\subseteq S_{ij}} \bigoplus_{(i',j')\in D_{i,j+1}(T)} \mathbf{M}^T_{S_{ij}}(\partial)\Delta_{\mu/i',j'-1} \;\; .$$

This may also be rewritten in the form

$$D_y^{-1}(\mathbf{M}_{i,j+1}) \;=\; \bigoplus_{\emptyset\neq T\subseteq S_{ij}} \bigoplus_{(i',j')\in D^{\leftarrow}_{i,j+1}(T)} \mathbf{M}^T_{S_{ij}}(\partial)\Delta_{\mu/i'j'} \;\; , \qquad 2.17$$

where the symbol "$D^{\leftarrow}_{i,j+1}(T)$" is to represent the subdiagram of μ obtained by shifting all cells of $D_{i,j+1}(T)$ one unit to the left. Now note that when $\epsilon_1(T)=0$, the diagram $D_{i,j+1}(T)$ is identical in shape with $D_{ij}(T)$ but shifted one column to the right. Thus in this case $D^{\leftarrow}_{i,j+1}(T)=D_{i,j}(T)$. On the other hand, when $\epsilon_1(T)=1$ then $D_{i,j+1}(T)$ is $D_{i,j}(T)$ with the leftmost column removed and then the difference $D_{i,j}(T)-D^{\leftarrow}_{i,j+1}(T)$ is obtained by picking the rightmost cell from each of the rows of $D_{i,j}(T)$. In any case, in view of 2.15, we may write

$$\widetilde{\mathbf{K}}^y_{ij} \;=\; \bigoplus_{\emptyset\neq T\subseteq S_{ij}} \;\bigoplus_{(i',j')\in D_{ij}(T)-D^{\leftarrow}_{i,j+1}(T)} \mathbf{M}^T_{S_{ij}}(\partial)\Delta_{\mu/i'j'}\ , \tag{2.18}$$

which is easily seen to be another way of writing 2.12.

To prove 2.14 we shall assume that the shadows of $(i,j),(i+1,j),(i,j+1)$ and $(i+1,j+1)$ contain the same corners of μ, so that in this case we can also write

$$\widetilde{\mathbf{K}}^y_{i+1,j} \;=\; \bigoplus_{\emptyset\neq T\subseteq S_{ij}} \;\bigoplus_{(i',j')\in D_{i+1,j}(T)-D^{\leftarrow}_{i+1,j+1}(T)} \mathbf{M}^T_{S_{ij}}(\partial)\Delta_{\mu/i'j'}\ .$$

Moreover, we see that under these assumptions the diagrams $D_{i+1,j}(T)$ and $D^{\leftarrow}_{i+1,j+1}(T)$ are simply $D_{i,j}(T)$ and $D^{\leftarrow}_{i,j+1}(T)$ with the bottom row removed, thus the difference

$$SED_{ij}(T)=\big(D_{i,j}(T)-D^{\leftarrow}_{i,j+1}(T)\big)\ -\ \big(D_{i+1,j}(T)-D^{\leftarrow}_{i+1,j+1}(T)\big)$$

reduces to the southeast corner cell of $D_{i,j}(T)$ when $\epsilon_1=1$ and is otherwise empty when $\epsilon_1=0$. In any case we may write

$$\widetilde{\mathbf{A}}^y_{ij} \;=\; \widetilde{\mathbf{K}}^y_{1j}/\widetilde{\mathbf{K}}^y_{i+1,j} \;=\; \bigoplus_{\emptyset\neq T\subseteq S_{ij}} \;\bigoplus_{(i',j')\in SED_{ij}(T)} \mathbf{M}^T_{S_{ij}}(\partial)\Delta_{\mu/i'j'}\ . \tag{2.19}$$

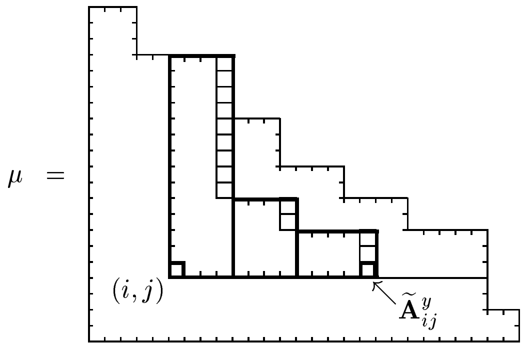
Since when $\epsilon_1=1$, we have $SED_{ij}(T) \;=\; \big\{(i,\epsilon_1 w_1+\cdots+\epsilon_m w_m-1)\big\}$, we see that 2.19 is just another way of writing 2.14.

The cases we have omitted here are a bit more tedious to deal with if we stick with the convention of making the set S_{ij} vary with (i,j). A way to deal with all cases at the same time is to fix $S=\{\alpha^{(1)},\ldots,\alpha^{(m)}\}$ to be a set of predecessors of μ obtained by removing some consecutive corners from left to right. Suppose that corners $\mu/\alpha^{(b)},\ldots,\mu/\alpha^{(c)}$ are the ones in the shadow of (i,j). In 2.6 we would have $\mathbf{M}^{T(\epsilon)}_{S_{ij}}$ with (up to renumbering) $\epsilon=\epsilon_b\cdots\epsilon_c$; however, this decomposes further into the sum of $\mathbf{M}^{T(\epsilon_1\cdots\epsilon_m)}_S$ where $\epsilon_1\cdots\epsilon_{b-1}$ and $\epsilon_{c+1}\cdots\epsilon_m$ vary freely. Setting $w_s=0$ and $v_s=0$ for each corner s where $\mu/\alpha^{(s)}$ is not in the shadow of (i,j), the only dependence on $T(\epsilon)$ in our construction is on $\epsilon_b,\ldots,\epsilon_c$. In particular, if we use the same set S in our decompositions of $\mathbf{M}_{ij}$, $\mathbf{M}_{i+1,j}$, $\mathbf{M}_{i,j+1}$, and $\mathbf{M}_{i+1,j+1}$, the the above reasoning works even in the omitted cases.

In the figure below we have illustrated 2.12 and 2.14 in the case

$$\mu=(27^2,25^5,20^2,16^2,12^3,9^4,3^3)\quad,\quad (i,j)=(4,5)\quad,\quad \epsilon=10011\quad,\quad m=5\ .$$

Here the vertical rectangles in bold lines give $D_{ij}(T(\epsilon))$, and the drawn individual cells along the righthand edge give the contribution of $T(\epsilon)$ to $\mathbf{K}^y_{ij}$ with the lowest giving the contribution to $\widetilde{\mathbf{A}}^y_{ij}$.



To proceed we need the following identity satisfied by the characteristics defined by I.35.

Proposition 2.1

$$\downarrow \phi_S^{(k)} \;=\; \frac{\phi_S^{(m+1-k)}}{\prod_{\beta\in S} T_\alpha}$$

Proof

Combining I.35 and I.36 we derive that

$$\phi_S^{(k)} = \sum_{\alpha\in S}\Big(\prod_{\beta\in S/\{\alpha\}}\frac{1}{1-T_\alpha/T_\beta}\Big)(-\nabla)^{m-k}\tilde H_\alpha = \sum_{\alpha\in S}\Big(\prod_{\beta\in S/\{\alpha\}}\frac{1}{1-T_\alpha/T_\beta}\Big)(-T_\alpha)^{m-k}\tilde H_\alpha \;.$$

Thus (since $\downarrow \tilde H_\alpha = \tilde H_\alpha/T_\alpha$):

$$\begin{aligned}\downarrow \phi_S^{(k)} &= \sum_{\alpha\in S}\Big(\prod_{\beta\in S/\{\alpha\}}\frac{T_\alpha}{T_\alpha-T_\beta}\Big)\Big(\frac{-1}{T_\alpha}\Big)^{m-k}\tilde H_\alpha\\ &= \sum_{\alpha\in S}\Big(\prod_{\beta\in S/\{\alpha\}}\frac{1}{1-T_\alpha/T_\beta}\Big)\frac{(-T_\alpha)^{k-1}}{\prod_{\beta\in S}T_\beta}\tilde H_\alpha \;=\; \frac{\phi_S^{(m+1-k)}}{\prod_{\beta\in S}T_\beta}\;.\end{aligned}$$

The last equality results from I.35 with k replaced by $m+1-k$. **Q.E.D.**

We are now ready to show that both 2.1 and 2.2 may be derived from geometric properties of lattice diagrams. To see how this comes about, for given $0,1$-words $\epsilon=\epsilon_1\cdots\epsilon_m$ and $\eta=\eta_1\cdots\eta_m$ set

$$\widetilde{\mathbf{A}}^x_{ij}(\epsilon) \;=\; \mathbf{M}^{T(\epsilon)}_{S_{ij}}(\partial)\Delta_{\mu/i+\epsilon_1 v_1+\cdots+\epsilon_m v_m-1,j} \tag{2.20}$$

and

$$\widetilde{\mathbf{A}}^y_{ij}(\eta) \;=\; \mathbf{M}^{T(\eta)}_{S_{ij}}(\partial)\Delta_{\mu/i,j+\eta_1 w_1+\cdots+\eta_m w_m-1}\;. \tag{2.21}$$

We see from 2.13 and 2.14 that

$$\widetilde{\mathbf{A}}^x_{ij} = \bigoplus_{\epsilon_1\cdots\epsilon_m\,:\,\epsilon_m=1} \widetilde{\mathbf{A}}^x_{ij}(\epsilon) \qquad \text{and} \qquad \widetilde{\mathbf{A}}^y_{ij} = \bigoplus_{\eta_1\cdots\eta_m\,:\,\eta_1=1} \widetilde{\mathbf{A}}^y_{ij}(\eta) \ .$$

This given we have the following refinements of the crucial and flip identities.

Theorem 2.2

For $\epsilon = (\epsilon_1, \ldots, \epsilon_{m-1}, 1)$ and $\eta = (1, \epsilon_1, \ldots, \epsilon_{m-1})$ we have (with the same l and a as in 2.1)

$$t^l \ \mathcal{F}\,\mathrm{ch}\,\widetilde{\mathbf{A}}^x_{ij}(\epsilon) \ = \ q^a \ \mathcal{F}\,\mathrm{ch}\,\widetilde{\mathbf{A}}^y_{ij}(\eta) \ , \tag{2.22}$$

while for $\eta = (1, 1-\epsilon_1, \ldots, 1-\epsilon_{m-1})$ we have

$$\mathcal{F}\,\mathrm{ch}\,\widetilde{\mathbf{A}}^x_{ij}(\epsilon) \ = \ T_{\mu/ij} \ \downarrow \mathcal{F}\,\mathrm{ch}\,\widetilde{\mathbf{A}}^y_{ij}(\eta) \ . \tag{2.23}$$

Proof

We first determine the Frobenius characteristic of $\widetilde{\mathbf{A}}^x_{ij}(\epsilon)$, and then that of $\widetilde{\mathbf{A}}^y_{ij}(\eta)$. Let $\eta = (1, \epsilon_1, \ldots, \epsilon_{m-1})$. Set $\epsilon_1 + \cdots + \epsilon_m = k$ and $V(\epsilon) = \epsilon_1 v_1 + \cdots + \epsilon_m v_m - 1$. Then from 2.20 we get that

$$\mathcal{F}\,\mathrm{ch}\,\widetilde{\mathbf{A}}^x_{ij}(\epsilon) \ = \ \frac{T_\mu}{t^i q^j t^{V(\epsilon)}} \downarrow \mathcal{F}\,\mathrm{ch}\,\mathbf{M}^{T(\epsilon)}_{S_{ij}} \ . \tag{2.24}$$

Setting $\nu^{(i_s)} = \alpha^{(s)}$, that is

$$S_{ij} \ = \ \left\{\, \alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(m)} \right\} \ ,$$

the definition I.34 gives

$$\mathcal{F}\,\mathrm{ch}\,\mathbf{M}^{T(\epsilon)}_{S_{ij}} \ = \ \frac{\phi^{(k)}_{S_{ij}}}{\prod_{s=1}^m T^{1-\epsilon_s}_{\alpha^{(s)}}} \ ,$$

and Proposition 2.1 yields

$$\downarrow \mathcal{F}\,\mathrm{ch}\,\mathbf{M}^{T(\epsilon)}_{S_{ij}} \ = \ \frac{\phi^{(m+1-k)}_{S_{ij}}}{\prod_{s=1}^m T^{\epsilon_s}_{\alpha^{(s)}}} \ .$$

This reduces 2.24 to

$$\mathcal{F}\,\mathrm{ch}\,\widetilde{\mathbf{A}}^x_{ij}(\epsilon) \ = \ \frac{T_\mu}{t^i q^j t^{V(\epsilon)}} \ \frac{\phi^{(m+1-k)}_{S_{ij}}}{\prod_{s=1}^m T^{\epsilon_s}_{\alpha^{(s)}}} \ .$$

Recalling the definition of $V(\epsilon)$, we may write

$$\mathcal{F}\,\mathrm{ch}\,\widetilde{\mathbf{A}}^x_{ij}(\epsilon) \ = \ \frac{T_\mu}{t^{i-1} q^j} \ \frac{\phi^{(m+1-k)}_{S_{ij}}}{\prod_{s=1}^m \left(T_{\alpha^{(s)}} t^{v_s}\right)^{\epsilon_s}} \ . \tag{2.25}$$

In a similar way, for $\eta_1 + \cdots + \eta_m = k$, we derive that

$$\mathcal{F}\,\mathrm{ch}\,\widetilde{\mathbf{A}}^y_{ij}(\eta) \ = \ \frac{T_\mu}{t^i q^{j-1}} \ \frac{\phi^{(m+1-k)}_{S_{ij}}}{\prod_{s=1}^m \left(T_{\alpha^{(s)}} q^{w_s}\right)^{\eta_s}} \ . \tag{2.26}$$

In conclusion, 2.25 and 2.26 yield the identity

$$\Big(\frac{1}{t}\,\prod_{s=1}^{m}\big(T_{\alpha^{(s)}}t^{v_s}\big)^{\epsilon_s}\Big)\ \mathcal{F}\,\mathrm{ch}\,\widetilde{\mathbf{A}}^{x}_{ij}(\epsilon) \;=\; \Big(\frac{1}{q}\,\prod_{s=1}^{m}\big(T_{\alpha^{(s)}}t^{w_s}\big)^{\eta_s}\Big)\ \mathcal{F}\,\mathrm{ch}\,\widetilde{\mathbf{A}}^{y}_{ij}(\eta) \qquad 2.27$$

To see that this is 2.22, note that the definition of the coefficients v_s and w_s gives

$$u^{ij}_s = x^{ij}_s/t^{v_s} = x^{ij}_{s+1}/q^{w_s}\ , \qquad 2.28$$

and since $T_{\alpha^{(s)}} = T_{\mu/ij}/x^{ij}_s$ we may write (using $\epsilon_m = 1$)

$$\frac{1}{t}\,\prod_{s=1}^{m}\big(T_{\alpha^{(s)}}t^{v_s}\big)^{\epsilon_s} \;=\; \frac{1}{t}\,\prod_{s=1}^{m}\left(\frac{T_{\mu/ij}}{x^{ij}_s}\,t^{v_s}\right)^{\epsilon_s} \;=\; \frac{T^{k}_{\mu/ij}}{t\,u^{ij}_m}\,\prod_{s=1}^{m-1}\left(\frac{1}{u^{ij}_s}\right)^{\epsilon_s}\ . \qquad 2.29$$

Similarly when $\eta_1 = 1$ we obtain that

$$\frac{1}{q}\,\prod_{s=1}^{m}\,(T_{\alpha^{(s)}}q^{w_s})^{\epsilon_s} \;=\; \frac{T^{k}_{\mu/ij}}{q\,u^{ij}_0}\,\prod_{s=2}^{m}\left(\frac{1}{u^{ij}_s}\right)^{\eta_s}\ .$$

Thus, taking account of 2.28, we see that when $\eta_{s+1} = \epsilon_s$ for $1 \le s \le m-1$, this last expression may be written in the form

$$\frac{T^{k}_{\mu/ij}}{q\,u^{ij}_0}\,\prod_{s=1}^{m-1}\left(\frac{1}{u^{ij}_s}\right)^{\epsilon_s}\ .$$

Comparing with 2.29 we finally derive that, after making the approriate cancellations, 2.27 reduces to

$$\frac{1}{t\,u^{ij}_m}\ \mathcal{F}\,\mathrm{ch}\,\widetilde{\mathbf{A}}^{x}_{ij}(\epsilon) \;=\; \frac{1}{q\,u^{ij}_0}\ \mathcal{F}\,\mathrm{ch}\,\widetilde{\mathbf{A}}^{y}_{ij}(\eta)$$

which is another way of writing 2.22 since $t\,u^{ij}_m = q^a$ and $q\,u^{ij}_0 = t^l$.

Next let us assume that $\eta = (1, 1-\epsilon_1, \ldots, 1-\epsilon_{m-1})$. Since $\epsilon_1 + \cdots + \epsilon_{m-1} = k-1$, this choice gives $\sum_{s=1}^{m}\eta_s = m+1-k$ and in this case $A^{y}_{ij}(\eta)$ is given by 2.26 with k replaced by $m+1-k$. Thus we may write

$$A^{y}_{ij}(\eta) \;=\; q\,T_{\mu/ij}\ \frac{\phi^{(k)}_{S_{ij}}}{\prod_{s=1}^{m}\big(T_{\alpha^{(s)}}q^{w_s}\big)^{\eta_s}}\ .$$

Now using 2.20 again we obtain

$$\begin{aligned}
\downarrow A^{y}_{ij}(\eta) \;&=\; \frac{1}{q\,T_{\mu/ij}}\frac{\phi^{(m+1-k)}_{S_{ij}}}{\prod_{s=1}^{m}T_{\alpha^{(s)}}}\,\prod_{s=1}^{m}\big(T_{\alpha^{(s)}}q^{w_s}\big)^{\eta_s}\\
&=\; \frac{1}{q\,T_{\mu/ij}}\frac{\phi^{(m+1-k)}_{S_{ij}}\ \prod_{s=1}^{m}q^{w_s}}{\prod_{s=1}^{m}\big(T_{\alpha^{(s)}}q^{w_s}\big)}\,\prod_{s=1}^{m}\big(T_{\alpha^{(s)}}q^{w_s}\big)^{\eta_s} \qquad 2.30\\
&=\; \frac{1}{q\,T_{\mu/ij}}\frac{\phi^{(m+1-k)}_{S_{ij}}\ \prod_{s=1}^{m}q^{w_s}}{\prod_{s=1}^{m}\big(T_{\alpha^{(s)}}q^{w_s}\big)^{1-\eta_s}}
\end{aligned}$$

Taking account of 2.28 and recalling that here $\eta = (1, 1-\epsilon_1, \ldots, 1-\epsilon_{m-1})$, we see that we have

$$T_{\mu/ij} \prod_{s=1}^{m} \left(T_{\alpha^{(s)}} q^{w_s}\right)^{1-\eta_s} \;=\; T^k_{\mu/ij} \prod_{s=2}^{m} \left(\frac{q^{w_s}}{x_s^{ij}}\right)^{\epsilon_{s-1}} \;=\; T^k_{\mu/ij} \prod_{s=1}^{m-1} \left(\frac{1}{u_s^{ij}}\right)^{\epsilon_s} .$$

Thus since $\sum_{s=1}^{m} w_s = a+1$, the identity in 2.30 reduces to

$$\downarrow A^y_{ij}(\eta) \;=\; \frac{q^a \, \phi^{(m+1-k)}_{S_{ij}}}{T^k_{\mu/ij}} \prod_{s=1}^{m-1} \left(u_s^{ij}\right)^{\epsilon_s} . \qquad 2.31$$

On the other hand, using 2.28 again, we can also rewrite 2.25 in the form

$$A^x_{ij}(\epsilon) \;=\; \frac{t\, T_{\mu/ij} \, \phi^{(m+1-k)}_{S_{ij}} \, u_m^{ij}}{T^k_{\mu/ij}} \prod_{s=1}^{m-1} \left(u_s^{ij}\right)^{\epsilon_s} .$$

Comparing with 2.31, we see that

$$A^x_{ij}(\epsilon) \;=\; \frac{t\, u_m^{ij}}{q^a} \; T_{\mu/ij} \downarrow A^y_{ij}(\eta) \;,$$

and this is 2.23 since, as we have seen, $tu_s^{ij} = q^a$. This completes our proof.

The refined crucial identity 2.22 and flip identity 2.23 each relate a term 2.20 in the direct sum decomposition 2.13 of the x-atom to a term 2.21 in the direct sum decomposition 2.14 of the y-atom. We illustrate this in the figure that follows in the case

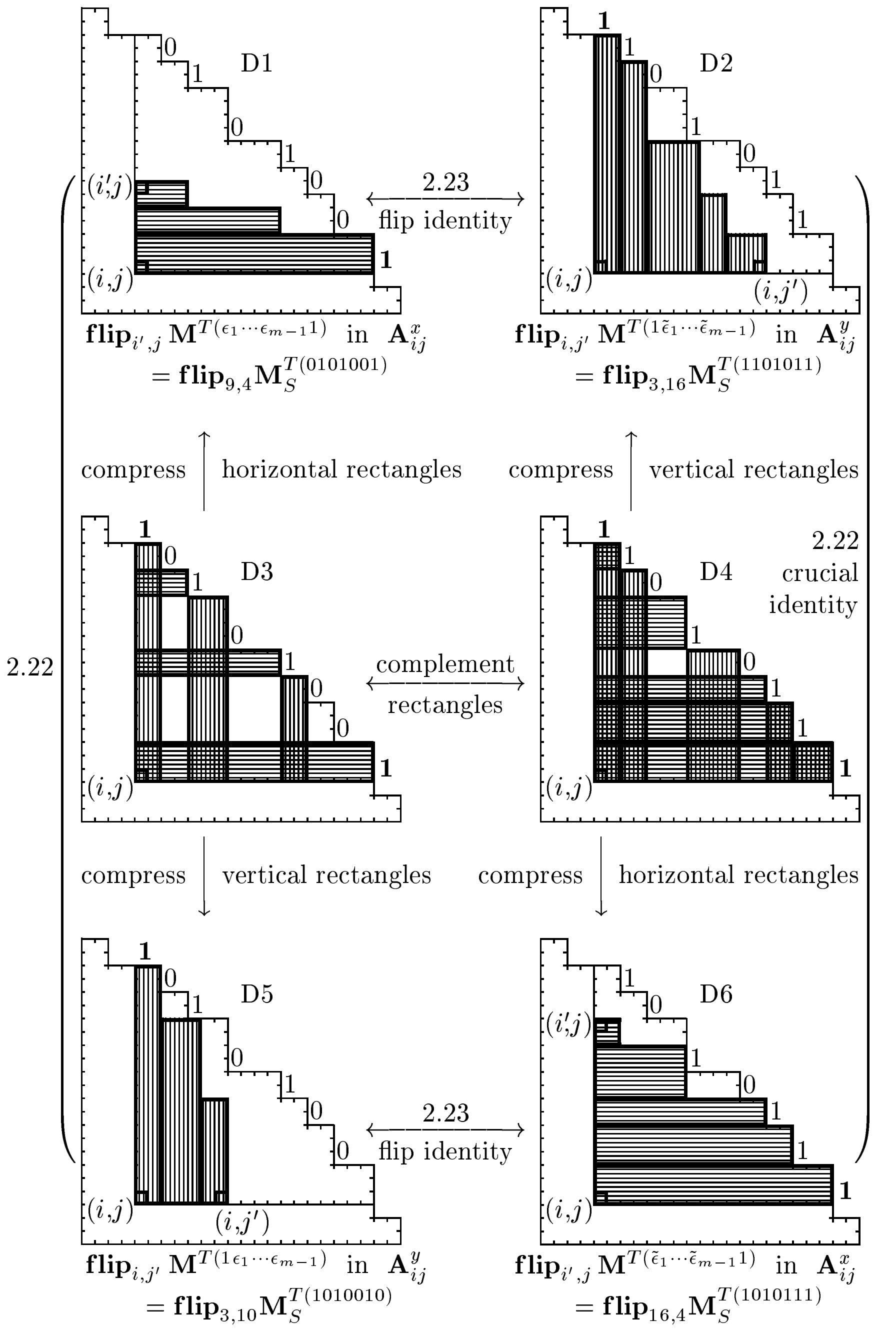
$$\mu = (24^2, 22^4, 19^3, 17^2, 15^2, 11^4, 8^2, 6^2, 2^2) \quad , \quad m = 7 \quad , \quad (i,j) = (3,4) \quad , \quad \epsilon = (0,1,0,1,0,0,\mathbf{1}).$$

Draw 6 copies of the diagram of μ with the shadow of (i,j) marked off. Put three diagrams on the right and three on the left, labelled D1–D6, as shown. In diagram D3, write $\mathbf{1}, \epsilon_1, \ldots, \epsilon_{m-1}, \mathbf{1}$ just northeast of the inner corner cells $u_0^{ij}, \ldots, u_m^{ij}$. Drop vertical lines from each corner to form m vertical rectangles, and shade the rectangles underneath 1's. The $\mathbf{1}$ at the bottom right does not contribute a rectangle since there is nothing beneath it. Slide the shaded rectangles to the left to fill in the gaps, forming the shaded region $D_{ij}(T(\epsilon))$ in D5. The rightmost cell (i,j') on the bottom row of this region is drawn in, and gives a term 2.21 of the direct sum 2.14: $\tilde{\mathbf{A}}^y_{ij}(\eta) = \mathbf{M}^{T(\eta)}_{S_{ij}}(\partial)\Delta_{\mu/i,j'}$, where $\eta = (1, \epsilon_1, \ldots, \epsilon_{m-1})$.

Via the refined crucial identity 2.22, this piece of the y-atom corresponds to a piece of the x-atom that we locate as follows. Extend horizontal lines to the left from each corner in D3, forming m horizontal rectangles. Shade the rectangles that are left of 1's. The $\mathbf{1}$ at the top left does not contribute a rectangle since there is nothing to its left. Slide the shaded rectangles down to fill in the gaps, forming the shaded region in D1. The topmost cell (i',j) in the left column of this region is drawn in, and gives a term 2.20 of the direct sum 2.13: $\tilde{\mathbf{A}}^x_{ij}(\epsilon) = \mathbf{M}^{T(\epsilon)}_{S_{ij}}(\partial)\Delta_{\mu/i',j}$. This term is related to the term from D5 via 2.22.

The three diagrams of μ on the right side of the figure illustrate what happens when we apply $\mathbf{flip}_{\mu/ij}$ to the modules constructed on the left side of the figure. Let $\tilde{\epsilon}_i = 1 - \epsilon_i$. In D4, write

$\mathbf{1}, \tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_{m-1}, \mathbf{1}$ just northeast of the inner corner cells, and then shade vertical and horizontal rectangles according to whether they have a 1 along the edge at their end. This has the effect of complementing which rectangles are shaded in or not shaded in, except that the vertical rectangle on the left and the horizontal rectangle on the bottom are fixed. Now slide all vertical rectangles left to fill in the gaps, and place the result in D2. Its bottom rightmost cell gives a term $\tilde{\mathbf{A}}^{y}_{ij}(\eta)$ of 2.14 for which the refined flip identity 2.23 holds with $\eta = (1, \tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_{m-1})$. Finally, slide down the horizontal rectangles in D4 to form D6. Its top left cell gives a term of 2.13 corresponding to the one in 2.14 from D2 via the crucial identity 2.22 and to the one in D5 via the flip identity 2.23.



D1
(i',j)
(i,j)
2.23 flip identity
$\mathbf{flip}_{i',j}\ \mathbf{M}^{T(\epsilon_1\cdots\epsilon_{m-1}1)}$ in $\mathbf{A}^x_{ij}$
$= \mathbf{flip}_{9,4}\mathbf{M}_S^{T(0101001)}$
D2
(i,j)
(i,j')
$\mathbf{flip}_{i,j'}\ \mathbf{M}^{T(1\tilde\epsilon_1\cdots\tilde\epsilon_{m-1})}$ in $\mathbf{A}^y_{ij}$
$= \mathbf{flip}_{3,16}\mathbf{M}_S^{T(1101011)}$
compress
horizontal rectangles
compress
vertical rectangles
D3
complement rectangles
D4
2.22 crucial identity
2.22
compress
vertical rectangles
compress
horizontal rectangles
D5
2.23 flip identity
D6
(i',j)
(i,j)
(i,j')
$\mathbf{flip}_{i,j'}\ \mathbf{M}^{T(1\epsilon_1\cdots\epsilon_{m-1})}$ in $\mathbf{A}^y_{ij}$
$= \mathbf{flip}_{3,10}\mathbf{M}_S^{T(1010010)}$
$\mathbf{flip}_{i',j}\ \mathbf{M}^{T(\tilde\epsilon_1\cdots\tilde\epsilon_{m-1}1)}$ in $\mathbf{A}^x_{ij}$
$= \mathbf{flip}_{16,4}\mathbf{M}_S^{T(1010111)}$

Remark 2.1

We should point out that since 2.22 is equivalent to the four term recursion and the latter in turn implies the expansion in I.16, it follows from 2.6 that the Frobenius characteristic $C_{\mu/ij}$ of $\mathbf{M}_{\mu/ij}$ is given by the formula

$$C_{\mu/ij} \;=\; \frac{1}{M}\frac{T_{\mu/ij}}{\nabla}\;\Big(\prod_{s=0}^{m}\Big(1-\nabla\frac{u_s^{ij}}{T_{\mu/ij}}\Big)\Big)\,\phi_S^{(m)}\;.$$

The reader may find it challenging to derive this identity directly from 2.6 making only use of the fact that when $S_{ij}=\{\,\alpha^{(1)},\alpha^{(2)},\ldots,\alpha^{(m)},\,\}$ and $\sum_{i=1}^m \epsilon_i = k$ we have

$$\mathcal{F}\,\mathrm{ch}\,\mathbf{M}_{S_{ij}}^{T(\epsilon)} \;=\; \frac{\phi_{S_{ij}}^{(k)}}{\prod_{s=1}^m T_{\alpha^{(s)}}^{1-\epsilon_s}} \;=\; \Big(\prod_{s=1}^{m}\Big(-\frac{\nabla}{T_{\alpha^{(s)}}}\Big)^{1-\epsilon_s}\Big)\phi_{S_{ij}}^{(m)}\;.$$

We terminate this section with a proof that the bases for $\mathbf{M}_\mu$ constructed in [1] by the recursive algorithm of Bergeron-Haiman, may directly be obtained by the same module assignment process we used in 2.6. We begin with a compact summary of this algorithm; then we give a direct formula for the final result of the recursion. As before we set

$$\mathcal{P}red(\mu) \;=\; \{\nu^{(1)},\nu^{(2)},\ldots,\nu^{(d)}\},$$

with the corner cells $\mu/\nu^{(1)},\mu/\nu^{(2)},\ldots,\mu/\nu^{(d)}$ ordered from left to right and respective weights

$$x_1 = t^{l'_1}q^{a'_1}\;,\; x_2 = t^{l'_2}q^{a'_2}\;,\;\ldots\;,\; x_d = t^{l'_d}q^{a'_d}\;.$$

The Algorithm is conjectured to produce a basis for $\mathbf{M}_\mu$ from bases of $\mathbf{M}_{\nu^{(1)}},\ldots,\mathbf{M}_{\nu^{(d)}}$. We abbreviate $\mathbf{M}_{\nu^{(r)}}$ as $\mathbf{M}_r$, and we work with the "Science Fiction Conjecture" that $\mathbf{M}_1,\ldots,\mathbf{M}_d$ generate a distributive lattice under span and intersection.

In [1] the algorithm assigns a module $\mathbf{B}_{ij}$ to each cell (i,j) of μ by a process that starts from the top row then proceeds down one row at the time ending at first row. For notational convenience we shall also assign modules here to cells left or right of μ in the strip $0\le i<l(\mu),\ -\infty<j<\infty$, according to the following recipe:

$$\mathbf{B}_{ij} \;=\; \begin{cases} \sum_{s=1}^d \mathbf{M}_s & \text{if } j<0\,, \\ \{0\} & \text{if } j\ge \mu_{i+1}\,. \end{cases} \tag{2.32}$$

The algorithm starts with setting

$$\mathbf{B}_{ij} \;=\; \mathbf{M}_1 \qquad \forall \quad (i,j)\quad \text{in the top row of } \mu\;, \tag{2.33}$$

this given, for all lower rows, the assignment is

$$\mathbf{B}_{ij} \;=\; \begin{cases} \mathbf{B}_{i+1,j} + (\mathbf{B}_{i+1,j-w}\cap \mathbf{M}_r) & \text{if row } i+1 \text{ contains the } r^{th} \text{ corner of } \mu \\ & \text{and } \mu_{i+1}-\mu_{i+2}=w\;, \\ \mathbf{B}_{i+1,j} & \text{if } \mu_{i+1}=\mu_{i+2}\;. \end{cases} \tag{2.34}$$

It is conjectured in [1] that, for any $\mu \vdash n$, the module $\mathbf{M}_\mu = \mathcal{L}_\partial[\Delta_\mu]$, decomposes as the direct sum

$$\mathbf{M}_\mu \;=\; \bigoplus_{(i,j)\in\mu} \mathbf{B}_{ij}(\partial)\partial_{x_n}^i\partial_{y_n}^j\Delta_\mu\,. \qquad 2.35$$

If $\mathcal{B}_{ij}$ is a basis of $\mathbf{B}_{ij}$, then a basis for $\mathbf{M}_\mu$ should be given by the collection

$$\mathcal{B}_\mu \;=\; \bigcup_{(i,j)\in\mu} \mathcal{B}_{ij}(\partial)\partial_{x_n}^i\partial_{y_n}^j\Delta_\mu\;. \qquad 2.36$$

Since the distributivity conjecture assures that each $\mathbf{B}_{ij}$ decomposes into a direct sum of various components $\mathbf{M}^{\epsilon_1\cdots\epsilon_d} = \mathbf{M}_S^{T(\epsilon)}$ where $S = \mathcal{P}red(\mu)$ and $T(\epsilon) = \{\,\nu^{(r)} \,:\, \epsilon_r = 1\,\}$, we must have direct sum decompositions of the form

$$\mathbf{B}_{ij} \;=\; \bigoplus_{\epsilon\in\mathcal{E}_{ij}} \mathbf{M}^{\epsilon_1\cdots\epsilon_d}$$

for suitable subsets $\mathcal{E}_{ij}$. It develops that these subsets can be given explicitly by a formula which is essentially 2.6 for $\mathbf{M}_{\mu/00}$. In point of fact we have put together this formula by simply discovering how to place the components $\mathbf{M}^{\epsilon_1\cdots\epsilon_d}$ directly into the Young diagram of μ , bypassing the recursive process defined by 2.32, 2.33 and 2.34. To be precise we have

Proposition 2.2

Let $w_1 = a'_1+1$ and $w_s = a'_s - a'_{s-1}$ for $s = 2,\ldots,d$. Assuming the Science Fiction Conjecture, the Bergeron-Haiman recursion is equivalent to placing $\mathbf{M}^{\epsilon_1\cdots\epsilon_d}$ in cells (i,j) with $j < \epsilon_1 w_1 + \cdots + \epsilon_r w_r$, where r is the number of corners of μ that are above row $i+1$. In symbols,

$$\mathbf{B}_{ij} \;=\; \bigoplus_{\substack{\epsilon_1\cdots\epsilon_d\\ j\,<\,\epsilon_1 w_1+\cdots+\epsilon_r w_r}} \mathbf{M}^{\epsilon_1\cdots\epsilon_d} \qquad 2.37$$

where $\epsilon_1,\cdots,\epsilon_d$ independently run over $\{0,1\}$ in all ways with at least one of them being nonzero.

Proof

For convenience, we define

$$W_r(\epsilon) \;=\; \epsilon_1 w_1 + \epsilon_2 w_2 + \cdots + \epsilon_r w_r. \qquad 2.38$$

We shall work our way from the top row of the partition down, to establish that the $\mathbf{B}_{ij}$ as given by 2.37 satisfy the Bergeron-Haiman recursion. We start by checking the definition of $\mathbf{B}_{ij}$ for cells external to μ. Noting that $0 \le W_r(\epsilon) \le w_1 + \cdots + w_r = \mu_{i+1}$, 2.37 states that $\mathbf{B}_{ij}$ is the span of all $\mathbf{M}^{\epsilon_1\cdots\epsilon_d}$'s when $j<0$ and is $\{0\}$ when $j \ge \mu_{i+1}$, in agreement with 2.32.

On the top row of the partition, we have $r = 1$, $w_1 = \mu_{i+1}$, and

$$W_r(\epsilon) \;=\; \begin{cases} \mu_{i+1} & \text{if } \epsilon_1 = 1 \\ 0 & \text{if } \epsilon_1 = 0, \end{cases}$$

so that in 2.37, $\mathbf{B}_{ij}$ is the span of all $\mathbf{M}^{\epsilon_1\cdots\epsilon_d}$'s for which $\epsilon_1 = 1$; and this is just $\mathbf{M}_1$, agreeing with 2.33.

On any subsequent row that does not contain a corner, we have $\mu_{i+1} = \mu_{i+2}$, and 2.34 gives $\mathbf{B}_{ij} = \mathbf{B}_{i+1,j}$; at the same time in this case we must use the same r in 2.37 for rows i and $i+1$, and this gives $\mathbf{B}_{ij} = \mathbf{B}_{i+1,j}$, as desired.

Finally, consider the row containing the r^{th} corner, that is, $i < l(\mu) - 1$ with $a'_r + 1 = \mu_{i+1} > \mu_{i+2} = a'_{r-1} + 1$ and thus $\mu_{i+1} - \mu_{i+2} = w_r$. In other words we must take $w = w_r$ in the first case of 2.34. Now according to 2.37 we have $\mathbf{M}^{\epsilon_1 \cdots \epsilon_d} \subseteq \mathbf{B}_{ij}$ if and only if $j < W_r(\epsilon) = W_{r-1}(\epsilon) + \epsilon_r w_r$. On the other hand if we assume inductively, that both $\mathbf{B}_{i+1,j}$ and $\mathbf{B}_{i+1,j-w}$ are given by 2.37, then we have

$$\mathbf{M}^{\epsilon_1 \cdots \epsilon_d} \subseteq \mathbf{B}_{i+1,j} + (\mathbf{B}_{i+1,j-w} \cap \mathbf{M}_r)$$

if and only if either
(a) $j < W_{r-1}(\epsilon)$, or
(b) $j - w_r < W_{r-1}(\epsilon)$ and $\epsilon_r = 1$.
When $\epsilon_r = 0$, (b) is false, while (a) is equivalent to $j < W_r(\epsilon)$ because $W_r(\epsilon) = W_{r-1}(\epsilon) + \epsilon_r w_r = W_{r-1}(\epsilon) + 0$. When $\epsilon_r = 1$, (a) is equivalent to $j < W_{r-1}(\epsilon)$, and (b) to $j < W_{r-1}(\epsilon) + w_r = W_r(\epsilon)$, so when (a) holds, so does (b). In total, (a) or (b) holds when $j < W_r(\epsilon)$. This assures the equality

$$\mathbf{B}_{ij} \;\; = \;\; \mathbf{B}_{i+1,j} + (\mathbf{B}_{i+1,j-w} \cap \mathbf{M}_r)$$

in this case and completes our proof that the assignment in 2.37 satisfies the Bergeron-Haiman recursion.

3. Some examples.

In this section we shall illustrate the theory we have developed by applying it to the "hook" case $\mu=(n-k,1^k)$. We shall see that in this case all our predictions go through in the finest detail. More significantly, this example gives us a glimpse of the additional ingredients that are needed to carry out our program in the general case.

We shall begin with the special case when μ reduces to a column ($\mu=(1^n)$) or a row ($\mu=(n)$). In either case, bases for $\mathbf{M}_\mu$ are well known (see [1], [6]). For instance in the case $\mu=(1^n)$, Δ_μ reduces to the Vandermonde determinant in $x_1,\ldots,x_n$

$$\Delta_{1^n} \;=\; \det \| \, x_j^{i-1} \, \|_{i,j=1}^{n} \;=\; \Delta_n(x_1,x_2,\ldots,x_n) \ .$$

The basic observation here is that we have

$$\Delta_n(x_1,x_2,\ldots,x_n) \;=\; x_1^0x_2^1x_3^2\cdots x_n^{n-1} \;+\; < \cdots$$

where the symbol "$<\cdots$" is to mean that the monomials in the omitted terms are all greater than the preceding one in the lexicographic order. This given, it is easily seen that, when $\epsilon_i\leq i-1$, we also have

$$\partial_{x_1}^{\epsilon_1}\partial_{x_2}^{\epsilon_2}\cdots\partial_{x_n}^{\epsilon_n}\Delta_n(x_1,x_2,\ldots,x_n) \;=\; c(\epsilon)\, x_1^{0-\epsilon_1}x_2^{1-\epsilon_2}x_3^{2-\epsilon_3}\cdots x_n^{n-1-\epsilon_n} \;+\; < \cdots$$

with $c(\epsilon)$ a nonvanishing constant. This shows that the Vandermonde $\Delta_n(x_1,\ldots,x_n)$ has at least $n!$ independent derivatives. Since we know ([7], [10]) that $\dim \mathbf{M}_\mu\leq n!$ for $\mu\vdash n$, it follows that the collection

$$\mathcal{B}_n(x_1,x_2,\ldots,x_n) \;=\; \Big\{\, \partial_{x_1}^{\epsilon_1}\partial_{x_2}^{\epsilon_2}\cdots\partial_{x_n}^{\epsilon_n}\Delta_n(x_1,x_2,\ldots,x_n) \;:\; 0\leq\epsilon_i\leq i-1 \,\Big\} \qquad 3.1$$

is a basis for $\mathbf{M}_{1^n}$.

Of course, we have an analogous result in the "row" case $\mu=(n)$. In fact, then we have

$$\Delta_\mu \;=\; \Delta_n(y_1,y_2,\ldots,y_n)$$

and thus a basis for $\mathbf{M}_n$ is given by the collection

$$\mathcal{B}_n(y_1,y_2,\ldots,y_n,) \;=\; \Big\{\, \partial_{y_1}^{\epsilon_1}\partial_{y_2}^{\epsilon_2}\cdots\partial_{y_n}^{\epsilon_n}\Delta_n(y_1,y_2,\ldots,y_n) \;:\; 0\leq\epsilon_i\leq i-1 \,\Big\} \ . \qquad 3.2$$

These classical results translate into the following basic facts concerning the modules $\mathbf{M}_{1^{n+1}/i,0}$ and $\mathbf{M}_{n+1/0,j}$:

Theorem 3.1

For each $1\leq i,j\leq n$ we have the following direct sum decompositions:

$$\mathbf{M}_{1^{n+1}/i,0} \;=\; \mathbf{M}_{1^n}(\partial)\Delta_{1^{n+1}/i,0} \,\oplus\, \mathbf{M}_{1^n}(\partial)\Delta_{1^{n+1}/i+1,0} \,\oplus\, \cdots \,\oplus\, \mathbf{M}_{1^n}(\partial)\Delta_{1^{n+1}/n+1,0} \qquad 3.3$$

$$\mathbf{M}_{n+1/0,j} \;=\; \mathbf{M}_n(\partial)\Delta_{n+1/0,j} \,\oplus\, \mathbf{M}_n(\partial)\Delta_{n+1/0,j+1} \,\oplus\, \cdots \,\oplus\, \mathbf{M}_n(\partial)\Delta_{n+1/0,n+1} \ . \qquad 3.4$$

Moreover, we can represent their respective atoms by the following modules.

(1) For $\mu = (1^{n+1})$,

$$a)\ \mathbf{A}^x_{i,0} \;=\; \mathbf{M}_{1^n} \quad , \qquad b)\ \mathbf{A}^y_{i,0} \;=\; \mathbf{M}_{1^n}(\partial)\Delta_{1^{n+1}/i,0} \ . \tag{3.5}$$

(2) For $\mu = (n+1)$,

$$a)\ \mathbf{A}^y_{i,0} \;=\; \mathbf{M}_{n} \quad , \qquad a)\ \mathbf{A}^x_{0,j} \;=\; \mathbf{M}_{n}(\partial)\Delta_{1^{n+1}/0,j} \ . \tag{3.6}$$

Proof

By Proposition I.1 we have

$$\sum_{i=1}^{n+1} \partial_{x_i}\, \Delta_{1^{n+1}} \;=\; 0 \ .$$

It thus follows that if we set $D_x = \sum_{i=1}^n \partial_{x_i}$, $D_y = \sum_{i=1}^n \partial_{y_i}$ then

$$\partial_{x_{n+1}}\, \Delta_{1^{n+1}} \;=\; -D_x\, \Delta_{1^{n+1}} \quad , \quad D_y\, \Delta_{1^{n+1}} \;=\; 0 \ . \tag{3.7}$$

Thus the fact that $\mathcal{B}^x_{n+1}$, as given by 3.1, is a basis for $\mathbf{M}_{1^{n+1}}$ yields that the collection

$$\bigcup_{k=0}^{n} \left\{ \partial_{x_1}^{\epsilon_1}\partial_{x_2}^{\epsilon_2}\cdots\partial_{x_n}^{\epsilon_n} D_x^k\, \Delta_{1^{n+1}} \ :\ 0 \le \epsilon_i \le i-1 \right\}$$

is also a basis. Since

$$\partial_{x_n}^{\epsilon_n} D_x^k\, \Delta_{1^{n+1}}\, \Big|_{x_{n+1}=0} \;=\; \Delta_{1^{n+1}/k,0} \ ,$$

Proposition 1.1 yields that the collection

$$\mathcal{B}_{1^{n+1}/00} \;=\; \bigcup_{k=0}^{n} \left\{ \partial_{x_1}^{\epsilon_1}\partial_{x_2}^{\epsilon_2}\cdots\partial_{x_n}^{\epsilon_n} \Delta_{1^{n+1}/k,0} \ :\ 0 \le \epsilon_i \le i-1 \right\} \ .$$

is also a basis for $\mathbf{M}_{1^{n+1}}$. Taking account of 3.1 we may also write this in the form

$$\mathcal{B}_{1^{n+1}/00} \;=\; \bigcup_{k=0}^{n} \mathcal{B}_n(\partial)\, \Delta_{1^{n+1}/k,0} \ .$$

From this we immediately derive the direct sum decomposition

$$\mathbf{M}_{1^{n+1}/00} \;=\; \bigoplus_{k=0}^{n} \mathcal{L}\big[\, \mathcal{B}_n(\partial)\, \Delta_{1^{n+1}/k,0} \,\big] \ . \tag{3.8}$$

Moreover, since $\mathbf{M}_{1^{n+1}/i,0} = D_x^i\, \mathbf{M}_{1^{n+1}/00}$ and $D_x^i\, \Delta_{1^{n+1}/k,0} = 0$ for $i+k>n$, by applying D_x^i to both sides of 3.8 we also get that

$$\mathbf{M}_{1^{n+1}/i,0} \;=\; \bigoplus_{k=i}^{n} \mathcal{L}\big[\, \mathcal{B}_n(\partial)\, \Delta_{1^{n+1}/k,0} \,\big] \ . \tag{3.9}$$

In particular we deduce that

$$\dim \mathbf{M}_{1^{n+1}/i,0} \;=\; (n+1-i)\times n! \ .$$

This given, since each of the summands in 3.3 has dimension $n!$ and there are $n+1-i$ of them, to show 3.3 we need only verify that they are independent. To this end, assume that for some elements $a_i, a_{i+1}, \ldots, a_{n+1} \in \mathbf{M}_{1^n}$ we have

$$a_i(\partial)\Delta_{1^{n+1}/i,0} \;+\; a_{i+1}(\partial)\Delta_{1^{n+1}/i+1,0} \;+\; \cdots \;+\; a_{n+1}(\partial)\Delta_{1^{n+1}/n+1,0} \;=\; 0 \ . \qquad 3.10$$

Note first that for $i=n+1$, this equation reduces to

$$a_{n+1}(\partial)\Delta_{1^n} \;=\; 0$$

and since by choice $a_{n+1}\in \mathcal{L}_\partial[\Delta_{1^n}]$ this forces $a_{n+1}=0$. So to show that $a_i,\ldots,a_{n+1}=0$ we can proceed by descent induction on i. That is, we can assume that 3.10 for $i{+}1$ forces $a_{i+1},\ldots,a_{n+1}=0$. This given, note that applying D_x^{n+1-i} to 3.10 reduces it to

$$a_i(\partial)\;\Delta_{1^{n+1}/n+1,0} \;=\; 0$$

or, equivalently,

$$a_i(\partial)\;\Delta_{1^n} \;=\; 0 \ .$$

But this, as we have seen, forces $a_i=0$, yielding that we must have

$$a_{i+1}(\partial)\Delta_{1^{n+1}/1+1,0} \;+\; \cdots \;+\; a_{n+1}(\partial)\Delta_{1^{n+1}/n+1,0} \;=\; 0 \ ,$$

and the induction hypothesis yields $a_{i+1},\ldots,a_{n+1}=0$, completing the induction and proving 3.3. It goes without saying that 3.4 may be proved in exactly the same way.

To show 3.5 a) we simply note that

$$D_x\, \mathbf{M}_{1^n}(\partial)\Delta_{1^{n+1}/k,0} \;=\; \begin{cases} \mathbf{M}_{1^n}(\partial)\Delta_{1^{n+1}/k+1,0} & \text{for } k\le n \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Thus from 3.3 it follows that

$$\mathbf{K}^x_{i,0} \;=\; \mathbf{M}_{1^n}(\partial)\Delta_{1^n} = \;\mathbf{M}_{1^n}$$

and since in this case $\mathbf{K}^x_{i,1}=\{0\}$ we must have $\mathbf{K}^x_{i,0}=\mathbf{A}^x_{i,0}$.

On the other hand, 3.5 b) follows from the fact that in this case for all i we have

$$\mathbf{K}^y_{i,0} \;=\; \mathbf{M}_{1^{n+1}/i,0}$$

Thus, using 3.3 again we get

$$\mathbf{A}^y_{i,0} \;=\; \mathbf{K}^y_{i,0}/\mathbf{K}^y_{i+1,0} \;=\; \mathbf{M}_{1^{n+1}/i,0}/\mathbf{M}_{1^{n+1}/i+1,0} \;=\; \mathbf{M}_{1^n}(\partial)\;\Delta_{1^{n+1}/i,0} \ .$$

This completes our proof since 3.6 a) and b) can be derived from 3.4 in precisely the same way.

Remark 3.1

We should note that the direct sum expansions in 3.3 and 3.4 bring to evidence that the modules $\mathbf{M}_{1^{n+1}/i,0}$ and $\mathbf{M}_{1^{n+1}/0,i}$ afford exactly $n+1-i$ copies of the left regular representation of S_n in complete agreement with our Conjecture I.2.

Before we treat the general hook case $\mu = (n+1-k, 1^k)$, it will be good to start by working with $\mu = (5,1,1,1)$. In this case we set

$$\begin{aligned} \mathbf{M}^{11} &= \mathbf{M}_{511} \cap \mathbf{M}_{4111} \quad , \\ \mathbf{M}^{10} &= \mathbf{M}_{511} \cap \big(\mathbf{M}_{511} \cap \mathbf{M}_{4111}\big)^{\perp} , \\ \mathbf{M}^{01} &= \mathbf{M}_{4111} \cap \big(\mathbf{M}_{511} \cap \mathbf{M}_{4111}\big)^{\perp} . \end{aligned} \tag{3.11}$$

and obtain the decompositions:

$$\begin{aligned} \mathbf{M}_{511} &= \mathbf{M}^{11} \oplus \mathbf{M}^{10} \\ \mathbf{M}_{4111} &= \mathbf{M}^{11} \oplus \mathbf{M}^{01} \end{aligned}$$

Using the convention that placing a module $\mathbf{M}$ or a basis $\mathcal{B}$ in cell (i,j) of the diagram of μ represents applying $\mathbf{M}(\partial)$ or $\mathcal{B}(\partial)$ to $\Delta_{\mu/ij}$, formula 2.6 asserts that we must have

$$\mathbf{M}_{5111/00} = \begin{array}{|c|c|c|c|c|} \cline{1-1} \mathbf{M}^{11} \\ \cline{1-1} \mathbf{M}^{11} \\ \cline{1-1} \mathbf{M}^{11} \\ \hline \mathbf{M}^{11} & \mathbf{M}^{11} & \mathbf{M}^{11} & \mathbf{M}^{11} & \mathbf{M}^{11} \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|c|} \cline{1-1} \mathbf{M}^{10} \\ \cline{1-1} \mathbf{M}^{10} \\ \cline{1-1} \mathbf{M}^{10} \\ \hline \mathbf{M}^{10} & \emptyset & \emptyset & \emptyset & \emptyset \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|c|} \cline{1-1} \emptyset \\ \cline{1-1} \emptyset \\ \cline{1-1} \emptyset \\ \hline \mathbf{M}^{01} & \mathbf{M}^{01} & \mathbf{M}^{01} & \mathbf{M}^{01} & \emptyset \\ \hline \end{array}$$

Taking account of 3.11 and setting $\mathbf{A} = \mathbf{M_{511}}$ and $\mathbf{B} = \mathbf{M_{4111}}$, this identity may be compressed to

$$\mathbf{M}_{5111/00} = \begin{array}{|c|c|c|c|c|} \cline{1-1} \mathbf{A} \\ \cline{1-1} \mathbf{A} \\ \cline{1-1} \mathbf{A} \\ \hline \mathbf{A}+\mathbf{B} & \mathbf{B} & \mathbf{B} & \mathbf{B} & \mathbf{A}\cap\mathbf{B} \\ \hline \end{array}$$

Letting $\mathcal{B}_a$, $\mathcal{B}_b$, $\mathcal{B}_{a+b}$, $\mathcal{B}_{a\cap b}$ denote bases for $\mathbf{A}$, $\mathbf{B}$, $\mathbf{A}+\mathbf{B}$, $\mathbf{A}\cap\mathbf{B}$ respectively, and writing Δ_{ij} for $\Delta_{\mu/ij}$, this formula asserts that a basis for the module $\mathbf{M}_{5111/00}$ is given by the collection

$$\begin{aligned} \mathcal{B}_{5111/00} = \ & \mathcal{B}_a(\partial)\Delta_{30} \ \cup \ \mathcal{B}_a(\partial)\Delta_{20} \ \cup \ \mathcal{B}_a(\partial)\Delta_{10} \\ & \cup \ \mathcal{B}_{a+b}(\partial)\Delta_{00} \ \cup \ \mathcal{B}_b(\partial)\Delta_{01} \ \cup \ \mathcal{B}_b(\partial)\Delta_{02} \ \cup \ \mathcal{B}_b(\partial)\Delta_{03} \ \cup \ \mathcal{B}_{a\cap b}(\partial)\Delta_{04} \ . \end{aligned} \tag{3.12}$$

In particular we get that this set has cardinality

$$4 \times \dim \mathbf{A} \ + \ 4 \times \dim \mathbf{B} \ .$$

Thus, assuming that $\dim \mathbf{A} = \dim \mathbf{B} = 7!$, we deduce that $\mathcal{B}_{5111/00}$ has precisely 8! elements. Since it was shown in [7] that for $\mu \vdash n$ we have $\dim \mathbf{M}_\mu \leq n!$, from Proposition I.5 we derive that $\dim \mathbf{M}_{5111/00} \leq 8!$. Thus, to show that $\mathcal{B}_{5111/00}$ is a basis we need only verify that it is an independent set. To this end let (for $i = 1, \ldots, 3$)

$$a_i \in \mathcal{L}[\mathcal{B}_a] \ \ , \ \ b_i \in \mathcal{L}[\mathcal{B}_b] \ \ , \ \ u \in \mathcal{L}[\mathcal{B}_{a+b}] \, , \ \ v \in \mathcal{L}[\mathcal{B}_{a\cap b}]$$

and suppose if possible that

$$\begin{aligned} & a_3(\partial)\Delta_{30} \ + \ a_2(\partial)\Delta_{20} \ + \ a_1(\partial)\Delta_{10} \ + \\ & \qquad + \ u(\partial)\Delta_{00} \ + \ b_1(\partial)\Delta_{01} \ + \ b_2(\partial)\Delta_{02} \ + \ b_3(\partial)\Delta_{03} \ + \ v(\partial)\Delta_{04} \ = \ 0 \ . \end{aligned} \qquad 3.13$$

Let the symbol " $\doteq$ " represent equality up to a constant factor. Proposition I.2 gives

$$\begin{aligned} & D_x^3\Delta_{00} \doteq D_x^2\Delta_{10} \doteq D_x\Delta_{20} \doteq \Delta_{30} \\ & D_y^4\Delta_{00} \doteq D_y^3\Delta_{01} \doteq D_y^2\Delta_{02} \doteq D_y\Delta_{03} \doteq \Delta_{04} \\ & D_x\Delta_{0,i} = 0 \quad \& \quad D_y\Delta_{i,0} = 0 \quad \text{for} \quad i > 0 \\ & D_x\Delta_{30} = 0 \quad \& \quad D_y\Delta_{04} = 0 \quad , \end{aligned} \qquad 3.14$$

where the last of these equations results from the fact that

$$\Delta_{30} \ = \ \Delta_{511} \quad \text{and} \quad \Delta_{04} \ = \ \Delta_{4111} \ .$$

Thus applying D_x^3 to 3.13 reduces it to

$$u(\partial)\, \Delta_{511} \ = \ 0 \ . \qquad 3.15$$

Similarly applying D_y^4 to 3.13 gives

$$u(\partial)\, \Delta_{4111} \ = \ 0 \ . \qquad 3.16$$

Since by assumption $u \in \mathcal{L}_\partial[\Delta_{511}] + \mathcal{L}_\partial[\Delta_{4111}]$, equations 3.15 and 3.16 force u to be orthogonal to itself and therefore identically zero. So 3.3 becomes

$$\begin{aligned} & a_3(\partial)\Delta_{30} \ + \ a_2(\partial)\Delta_{20} \ + \ a_1(\partial)\Delta_{10} \\ & \qquad + \ b_1(\partial)\Delta_{01} \ + \ b_2(\partial)\Delta_{02} \ + \ b_3(\partial)\Delta_{03} \ + \ v(\partial)\Delta_{04} \ = \ 0 \ . \end{aligned} \qquad 3.17$$

Now, the relations in 3.14 yield that applying D_x^2 to 3.17 reduces it to

$$a_1(\partial)\Delta_{511} \ = \ 0$$

and since $a_1 \in \mathcal{L}_\partial[\Delta_{511}]$, we derive that $a_1 = 0$ as well, reducing 3.17 to

$$\begin{aligned} & a_3(\partial)\Delta_{30} \ + \ a_2(\partial)\Delta_{20} \\ & \qquad + \ b_1(\partial)\Delta_{01} \ + \ b_2(\partial)\Delta_{02} \ + \ b_3(\partial)\Delta_{03} \ + \ v(\partial)\Delta_{04} \ = \ 0 \ . \end{aligned} \qquad 3.18$$

Now an application of D_x yields

$$a_2(\partial)\Delta_{511} \;=\; 0$$

and $a_2 \in \mathcal{L}_\partial[\Delta_{511}]$ yields again $a_2 = 0$, reducing 3.18 to

$$\begin{aligned} &a_3(\partial)\Delta_{30} + \\ &\qquad\qquad + \; b_1(\partial)\Delta_{01} \;+\; b_2(\partial)\Delta_{02} \;+\; b_3(\partial)\Delta_{03} \;+\; v(\partial)\Delta_{04} \;=\; 0 \;. \end{aligned} \tag{3.19}$$

Since $D_y\Delta_{30} \;=\; 0$, we can now apply D_y^3, D_y^2, D_y^1 in succession to 3.9 and, by a similar process, successively derive that

$$b_1 \;,\; b_2 \;,\; b_3 \;=\; 0 \;,$$

reducing 3.19 to

$$a_3(\partial)\Delta_{511} \;+\; v(\partial)\Delta_{4111} \;=\; 0 \;. \tag{3.20}$$

As we let a_3 vary in $\mathcal{L}_\partial[\mathcal{B}_a]$ without restriction, the term $a_3(\partial)\Delta_{511}$ will necessarily describe all of $\mathbf{A}$. On the other hand, as v varies in $\mathcal{L}_\partial[\mathcal{B}_{a\cap b}]$ the term $v(\partial)\Delta_{4111}$ will describe $\mathbf{flip}_{4111}\big(\mathbf{A}\cap\mathbf{B}\big)$.

This given, to conclude from for 3.10 that a_3 and v must vanish we need to know that $\mathbf{A}$ and $\mathbf{flip}_{4111}\big(\mathbf{A}\cap\mathbf{B}\big)$ have no element in common other than 0. It is at this point that the SF hypothesis plays a role. In fact, in the particular case of a 2-corner partition μ, with two predecessors α_1, α_2, condition (iii) of SF asserts (see [1] I.29) that

$$1) \quad \mathbf{flip}_{\alpha_1}\mathbf{M}^{11} \;=\; \mathbf{M}^{10} \qquad \text{and} \qquad 2) \quad \mathbf{flip}_{\alpha_2}\mathbf{M}^{11} \;=\; \mathbf{M}^{01} \;. \tag{3.21}$$

This of course guarantees that the two terms in 3.20 must separately vanish, completing the proof that the collection $\mathcal{B}_{5111/00}$ defined in 3.13 is a basis for $\mathbf{M}_{5111/00}$.

Remark 3.2

We should note that although they can be verified by computer in several special cases, the identities in 3.21 may be too strong to be true in general. A weaker form, which does not affect the final conclusion, is obtained by changing the definitions of $\mathbf{M}^{10}$ and $\mathbf{M}^{01}$ by dropping the condition that they be orthogonal complements of $\mathbf{M}^{11}$ in $\mathbf{M}_{511}$ and $\mathbf{M}_{4111}$ respectively and just require that they be simply "complements" constructed so that the relations in 3.21 are satisfied. Another way to get around requiring 3.21 is to observe that the desired implication

$$a_3(\partial)\Delta_{511} \;+\; v(\partial)\Delta_{4111} \;=\; 0 \qquad \Longrightarrow \qquad a_3(\partial)\Delta_{511} \;\&\; v(\partial)\Delta_{4111} \;=\; 0$$

immediately follows, if the collection $\mathcal{B}_{a\cap b}$ is replaced by any basis of $\mathbf{flip}_{4111}^{-1}\;\mathbf{M}^{01}$. This choice guarantees that $\mathcal{B}_{5111/00}$ is an independent set. However, to conclude that $\mathcal{B}_{5111/00}$ is a basis we need

$$\dim \mathbf{M}^{11} \;=\; \dim \mathbf{M}^{01} \;,$$

or equivalently

$$\dim \mathbf{M}^{11} \;=\; \frac{\dim \mathbf{M}_{\alpha_1}}{2} \;. \tag{3.22}$$

Unfortunately, this equality, which has come to be referred to as the $n!/2$ conjecture, has to this date remained unproved in full generality (even in the "hook" case). As a result, this modified construction of $\mathcal{B}_{5111/00}$ only generalizes to a proof that

$$\dim \mathbf{M}^{11} \geq \dim \mathbf{M}^{01} \quad .$$

These observations are essentially all contained in the SF paper [1]. What is new here is that the introduction of the "atoms" $\mathbf{A}_{ij}^{x}$ and $\mathbf{A}_{ij}^{y}$ leads to a very elegant construction of a basis of $\mathbf{M}_\mu$ when μ is a hook without any need of unproved auxiliary conjectures. We shall illustrate it here again in the case $\mu = (5,1,1,1,1)$.

Since the construction is inductive on the size of μ we shall again assume that both $\mathbf{M}_{511}$ and $\mathbf{M}_{4111}$ have dimension 7! and that we have chosen $\mathcal{B}_{511}$ and $\mathcal{B}_{4111}$ as their respective bases. This given, we may represent our alternate construction of a basis $\tilde{\mathcal{B}}_{5111/00}$ for $\mathbf{M}_{5111/00}$ by the diagram

$$\tilde{\mathcal{B}}_{5111/00} = \begin{array}{|c|ccccc} \hline \emptyset & & & & \\ \hline \mathcal{B}_{511} & & & & \\ \hline \mathcal{B}_{511} & & & & \\ \hline \mathcal{B}_{511} \cup \mathcal{X} & \multicolumn{1}{|c|}{\mathcal{B}_{4111}} & \multicolumn{1}{c|}{\mathcal{B}_{4111}} & \multicolumn{1}{c|}{\mathcal{B}_{4111}} & \multicolumn{1}{c|}{\mathcal{B}_{4111}} \\ \hline \end{array} \qquad 3.23$$

where $\mathcal{X}$ is a suitable collection of monomials. Before we exhibit our choice of $\mathcal{X}$, it will be instructive to see that 3.23 gives a basis for $\mathbf{M}_{5111/00}$ as soon as $\mathcal{X}$ satisfies the following three conditions:

(i) $\mathcal{X}(\partial)\Delta_{00}$ is an independent set of cardinality 7!,

(ii) $D_x\, m(\partial)\Delta_{00} = 0 \quad \forall \quad m \in \mathcal{X}$, 3.24

(iii) For any $0 \neq \xi \in \mathcal{L}[\mathcal{X}]$ the element $\xi(\partial)\Delta_{00}$ is never in
$$\mathcal{L}\big[\, \mathcal{B}_{4111}(\partial)\Delta_{01} \ \cup\ \mathcal{B}_{4111}(\partial)\Delta_{02} \ \cup\ \mathcal{B}_{4111}(\partial)\Delta_{03} \ \cup\ \mathcal{B}_{4111}(\partial)\Delta_{04} \,\big] \ .$$

In fact, suppose that for some $\quad a_0, a_1, a_2 \in \mathcal{L}[\mathcal{B}_{511}]\,, \quad b_1, b_2, b_3, b_4 \in \mathcal{L}[\mathcal{B}_{4111}] \quad$ and $\quad \xi \in \mathcal{L}[\mathcal{X}]$ we have

$$\begin{aligned} &a_2(\partial)\Delta_{20} \ + \ a_1(\partial)\Delta_{10} \ + \ a_0(\partial)\Delta_{00} \\ &\qquad + \ \xi(\partial)\Delta_{00} \ + \ b_1(\partial)\Delta_{01} \ + \ b_2(\partial)\Delta_{02} \ + \ b_3(\partial)\Delta_{03} \ + \ b_4(\partial)\Delta_{04} \ = \ 0 \ . \end{aligned} \qquad 3.25$$

To show that this forces $a_1, a_2, a_3, \xi, b_1, b_2, b_3, b_4 = 0$ we apply D_x to both sides and, using the relations in 3.14 and condition (ii) of 3.24, immediately derive that

$$a_2(\partial)\Delta_{30} \ + \ a_1(\partial)\Delta_{20} \ + \ a_0(\partial)\Delta_{10} \ = \ 0 \ . \qquad 3.26$$

This given, an application of D_x^2 reduces this to

$$a_0(\partial)\Delta_{30} \ = \ 0 \ ,$$

which as we have seen forces $a_0 = 0$ and 3.26 becomes

$$a_2(\partial)\Delta_{30} \ + \ a_1(\partial)\Delta_{20} \ = \ 0 \ . \qquad 3.27$$

Applying D_x , we now get

$$a_1(\partial)\Delta_{30} \;=\; 0 \;,$$

which forces $a_1 = 0$, reducing 3.17 to

$$a_2(\partial)\Delta_{30} \;=\; 0 \;,$$

and this in turn yields

$$a_2 \;=\; 0 \;.$$

So 3.15 becomes

$$\xi(\partial)\Delta_{00} \;+\; b_1(\partial)\Delta_{01} \;+\; b_2(\partial)\Delta_{02} \;+\; b_3(\partial)\Delta_{03} \;+\; b_4(\partial)\Delta_{04} \;=\; 0 \;.$$

But then condition (iii) of 3.14 assures that we must separately have

$$\begin{aligned} \xi(\partial)\Delta_{00} \;&=\; 0 \\ b_1(\partial)\Delta_{01} \;+\; b_2(\partial)\Delta_{02} \;+\; b_3(\partial)\Delta_{03} \;+\; b_4(\partial)\Delta_{04} \;&=\; 0 \end{aligned}$$

Now, the first equation (using 3.24(i)) yields $\xi = 0$, while the second yields $b_1, b_2, b_3 = 0$ by successive applications of D_y^3, D_y^2, D_y, as we have seen before. We are finally left with

$$b_4(\partial)\Delta_{04} \;=\; 0 \;,$$

which forces $b_4 = 0$ and completes the proof of independence of $\tilde{\mathcal{B}}_{5111/00}$. Since by virtue of (i) in 3.14 the cardinality of $\tilde{\mathcal{B}}_{5111/00}$ evaluates to 8! , we must conclude that $\tilde{\mathcal{B}}_{5111/00}$ must also be a basis.

It develops that a collection of monomials that satisfies all of the condition in 3.14 is obtained by setting

$$\mathcal{X} = \bigcup_{\substack{1\le i_1<i_2<i_3\le 7 \\ 1\le j_1<j_2<j_3<j_4\le 7 \\ \{i_1,i_2,i_3,j_1,j_2,j_3,j_4\}=\{1,2,3,4,5,6,7\}}} \{\, x_{i_1}^{1+\epsilon_1}\, x_{i_2}^{1+\epsilon_2}\, x_{i_3}^{1+\epsilon_1}\; y_{j_1}^{\eta_1}\, y_{j_2}^{\eta_2}\, y_{j_3}^{\eta_3}\, y_{j_4}^{\eta_4} \;:\; 0\le \epsilon_i \le i-1\,;\; 0\le \eta_j \le j-1 \;\} \tag{3.28}$$

To see this, note first that since

$$\Delta_{5111/00} \;=\; \det \begin{pmatrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5v & y_6^2 & y_7^2 \\ y_1^3 & y_2^3 & y_3^3 & y_4^3 & y_5^3 & y_6^3 & y_7^3 \\ y_1^4 & y_2^4 & y_3^4 & y_4^4 & y_5^4 & y_6^4 & y_7^4 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 & x_7^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 & x_5^3 & x_6^3 & x_7^3 \end{pmatrix}$$

we have

$$\begin{aligned} \partial_{x_1}\partial_{x_2}\partial_{x_3}\Delta_{5111/00} \;&=\; 6\times\det\begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix} \;\times\; \det\begin{pmatrix} y_4 & y_5 & y_6 & y_7 \\ y_4^2 & y_5 & y_6^2 & y_7^2 \\ y_4^3 & y_5^3 & y_6^3 & y_7^3 \\ y_4^4 & y_5^4 & y_6^4 & y_7^4 \end{pmatrix} \\ &=\; 6\times y_4\, y_5\, y_6\, y_7 \;\times\; \det\begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix} \;\times\; \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ y_4 & y_5 & y_6 & y_7 \\ y_4^2 & y_5 & y_6^2 & y_7^2 \\ y_4^3 & y_5^3 & y_6^3 & y_7^3 \end{pmatrix} \;. \end{aligned}$$

Thus using the notation we introduced at the beginning of the section (see 3.1 and 3.2) we may write

$$\Big\{\, \partial_{x_1}^{1+\epsilon_1}\partial_{x_2}^{1+\epsilon_2}\partial_{x_3}^{1+\epsilon_3}\partial_{y_1}^{\eta_4}\partial_{y_5}^{\eta_2}\partial_{y_6}^{\eta_3}\partial_{y_7}^{\eta_3}\Delta_{5111/00} \ : 0\le \epsilon_i \le i-1\ ;\ 0\le \eta_j\le j-1 \,\Big\}$$
$$\doteq\ y_4\, y_5\, y_6\, y_7\ \times\ \mathcal{B}_3(x_1,x_2,x_3)\ \times\ \mathcal{B}_3(y_4,y_5,y_6,y_7)$$

So we see that the module $\mathcal{L}[\mathcal{X}(\partial)\Delta_{5111/00}]$ may be expressed as the direct sum

$$\mathcal{L}\big[\mathcal{X}(\partial)\Delta_{5111/00}\big] \;=\; \bigoplus_{\substack{1\le i_1<i_2<i_3\le 7\\ 1\le j_1<j_2<j_3<j_4\le 7\\ \{i_1,i_2,i_3;j_1,j_2,j_3,j_4\}=\{1,2,3,4,5,6,7\}}} y_{j_1}\,y_{j_2}\,y_{j_3}\,y_{j_4}\times \mathbf{M}_{1^3}[x_{i_1},x_{i_2},x_{i_3}]\times \mathbf{M}_4[y_{j_1},y_{j_2},y_{j_3},y_{j_4}]\,, \tag{3.29}$$

where the symbols $\mathbf{M}_{1^3}[x_{i_1},x_{i_2},x_{i_3}]$ and $\mathbf{M}_4[y_{j_1},y_{j_2},y_{j_3},y_{j_4}]$ denote $\mathbf{M}_{1^3}$ and $\mathbf{M}_4$ with x_s replaced by x_{i_s} and y_r replaced by y_{j_r}. Now we immediately derive from this that

$$\dim \mathcal{L}\big[\mathcal{X}(\partial)\Delta_{5111/00}\big] \;=\; \binom{7}{3}\,3!\,4! \;=\; 7!$$

yielding 3.14 (i). Now 3.14 (ii) is immediate since for any choice of $x_{i_1}x_{i_2}x_{i_3}$ we have

$$D_x\ \Delta_3(x_{i_1},x_{i_2},x_{i_3}) \;=\; 0\ .$$

Finally we note that every one of the determinants $\Delta_{i,0}$ is a sum of monomials only containing three different $y_i\,'\,s$ and thus none of their derivatives can contain monomials with four different $y_i\,'\,s$. Since each element of $\mathcal{L}\big[\mathcal{X}(\partial)\Delta_{5111/00}\big]$ has $y_{j_1}\,y_{j_2}\,y_{j_3}\,y_{j_4}$ as a factor we see that 3.24 (iii) must necessarily hold true precisely as required.

Now the fact that 3.13, with $\mathcal{X}$ given by 3.28, gives a basis for $\mathbf{M}_{5111/00}$ yields that $\mathbf{M}_{5111/00}$ has a direct sum decomposition

$$\begin{aligned}\mathbf{M}_{5111/00} \;=\; &\mathbf{M}_{511}(\partial)\Delta_{5111/20}\ \oplus\ \mathbf{M}_{511}(\partial)\Delta_{5111/10}\\ &\oplus\ \mathcal{L}[\mathcal{X}(\partial)\Delta_{00}]\ \oplus\ \mathbf{M}_{511}(\partial)\Delta_{00}\\ &\oplus\ \mathbf{M}_{4111}(\partial)\Delta_{5111/01}\ \oplus\ \mathbf{M}_{4111}(\partial)\Delta_{5111/02}\\ &\oplus\ \mathbf{M}_{4111}(\partial)\Delta_{5111/03}\ \oplus\ \mathbf{M}_{4111}(\partial)\Delta_{5111/04}\ .\end{aligned} \tag{3.30}$$

Since $\mathbf{M}_{5111/i,0} = D_x^i\,\mathbf{M}_{5111/00}$ for $i=1,2,3$ and D_x kills $\mathcal{L}[\mathcal{X}(\partial)\Delta_{00}]$ as well as each $\Delta_{5111/0,j}$, we immediately derive, by applying D_x, D_x^2, D_x^3 to both sides of 3.30, that

$$\begin{aligned}\mathbf{M}_{5111/10} &= \mathbf{M}_{511}(\partial)\Delta_{5111/10}\ \oplus\ \mathbf{M}_{511}(\partial)\Delta_{5111/20}\ \oplus\ \mathbf{M}_{511}\\ \mathbf{M}_{5111/20} &= \mathbf{M}_{511}(\partial)\Delta_{5111/20}\ \oplus\ \mathbf{M}_{511}\\ \mathbf{M}_{5111/30} &= \mathbf{M}_{511}\ ,\end{aligned} \tag{3.31}$$

where we have used the fact that $\Delta_{5111/30}=\Delta_{511}$. Similarly by inverting the roles played by the x and y variables we derive the direct sum decompositions

$$\begin{aligned}\mathbf{M}_{5111/01} &= \mathbf{M}_{4111}(\partial)\Delta_{5111/01}\ \oplus\ \mathbf{M}_{4111}(\partial)\Delta_{5111/02}\ \oplus\ \mathbf{M}_{4111}(\partial)\Delta_{5111/03}\ \oplus\ \mathbf{M}_{4111}\\ \mathbf{M}_{5111/02} &= \mathbf{M}_{4111}(\partial)\Delta_{5111/02}\ \oplus\ \mathbf{M}_{4111}(\partial)\Delta_{5111/03}\ \oplus\ \mathbf{M}_{4111}\\ \mathbf{M}_{5111/03} &= \mathbf{M}_{4111}(\partial)\Delta_{5111/03}\ \oplus\ \mathbf{M}_{4111}\\ \mathbf{M}_{5111/04} &= \mathbf{M}_{4111}\ .\end{aligned} \tag{3.32}$$

Note that 3.30 gives as also that

$$\mathbf{K}^x_{00} \;=\; \mathcal{L}[\mathcal{X}(\partial)\Delta_{00}] \;\oplus\; \mathbf{M}_{4111}(\partial)\Delta_{5111/01} \;\oplus\; \mathbf{M}_{4111}(\partial)\Delta_{5111/02} \;\oplus\; \mathbf{M}_{4111}(\partial)\Delta_{5111/03} \;\oplus\; \mathbf{M}_{4111}(\partial)\Delta_{5111/04} \;.$$

In other words

$$\mathbf{K}^x_{00} \;=\; \mathcal{L}[\mathcal{X}(\partial)\Delta_{00}] \;\oplus\; \mathbf{M}_{5111/01} \;.$$

Since D_x kills all of $\mathbf{M}_{5111/01}$. This may be rewritten as

$$\mathbf{K}^x_{00} \;=\; \mathcal{L}[\mathcal{X}(\partial)\Delta_{00}] \;\oplus\; \mathbf{K}^x_{01} \;,$$

yielding that in this case we have

$$\mathbf{A}^x_{00} \;=\; \mathcal{L}[\mathcal{X}(\partial)\Delta_{00}] \;. \tag{3.33}$$

By "equality," we mean that $\mathcal{L}[\mathcal{X}(\partial)\Delta_{00}]$ is a complement of $\mathbf{K}^x_{01}$ within $\mathbf{K}^x_{00}$, thus forming a system of representatives of the quotient $\mathbf{A}^x_{00} = \mathbf{K}^x_{00}/\mathbf{K}^x_{01}$. Similarly we can derive from 3.31 and 3.32 that

$$\mathbf{A}^x_{10} = \mathbf{A}^x_{20} = \mathbf{A}^x_{30} = \mathbf{M}_{511} \tag{3.34}$$

and

$$\mathbf{A}^x_{0,i} = \mathbf{M}_{4111}(\partial)\Delta_{5111/0,i} \qquad (\text{ for } i = 1,2,3,4\ ) \;. \tag{3.35}$$

We should point out that analogous results concerning the atoms $\mathbf{A}^y_{ij}$ can be obtained if construct the basis $\tilde{\mathcal{B}}_{5111/00}$ according to the "transposed" diagram

$$\tilde{\mathcal{B}}_{5111/00} = \begin{array}{|c|c|c|c|c|} \cline{1-1} \mathcal{B}_{511} \\ \cline{1-1} \mathcal{B}_{511} \\ \cline{1-1} \mathcal{B}_{511} \\ \hline \mathcal{B}_{4111}\cup\mathcal{Y} & \mathcal{B}_{4111} & \mathcal{B}_{4111} & \mathcal{B}_{4111} & \emptyset \\ \hline \end{array} \tag{3.36}$$

with

$$\mathcal{Y} = \bigcup_{\substack{1\le i_1<i_2<i_3\le 7\\ 1\le j_1<j_2<j_3<j_4\le 7\\ \{i_1,i_2,i_3;j_1,j_2,j_3,j_4\}=\{1,2,3,4,5,6,7\}}} \left\{\ x_{i_1}^{\epsilon_1}\, x_{i_2}^{\epsilon_2}\, x_{i_3}^{\epsilon_3}\, y_{j_1}^{1+\eta_1}\, y_{j_2}^{1+\eta_2}\, y_{j_3}^{1+\eta_3}\, y_{j_4}^{1+\eta_4}\ :\ 0\le\epsilon_i\le i-1\ ;\ 0\le\eta_j\le j-1\ \right\} \;.$$

It should also be clear that the argument we have illustrated in the case $\mu = 5111$ can be carried out for all hook partitions. In fact, in this case all our conjectures can be proved in full including the $C = \tilde{H}$ conjecture and the four term recursion.

For a given subset $S = \{i_1 < i_2 < \cdots < i_k\}$ let $|S| = k$ and set

$$X(S) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \quad , \quad Y(S) = \{y_{i_1}, y_{i_2}, \ldots, y_{i_k}\} \;.$$

Moreover, if $\mathbf{M}$ is a space of polynomials in the variables $x_1, x_2, \ldots, x_k$, let $\mathbf{M}[X(S)]$ denote the space obtained by replacing x_s by x_{i_s} in all elements of $\mathbf{M}$. Let $\mathbf{M}[Y(S)]$ be analogously defined with the $y's$ replacing the $x's$. Recall that according to the definitions made in the introduction, $\mathbf{M}_{1^n}$ and $\mathbf{M}_n$ denote the linear spans of derivatives of the Vandermonde determinants in $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \ldots, y_n$ respectively. With these conventions, our general result for hooks may be stated as follows.

Theorem 3.2

For $\mu = (n+1-k, 1^k)$, set $\alpha = (n+1-k, 1^{k-1})$ and $\beta = (n-k, 1^k)$. Let

$$\mathbf{X} \;=\; \bigoplus_{\substack{|S|=k \\ |T|=n-k \\ S+T=\{1,2,\ldots,n\}}} \Big(\prod_{j\in T} y_j\Big) \times \mathbf{M}_{1^k}\big[X(S)\big] \times \mathbf{M}_{n-k}\big[Y(T)\big] \tag{3.37}$$

and

$$\mathbf{Y} \;=\; \bigoplus_{\substack{|S|=k \\ |T|=n-k \\ S+T=\{1,2,\ldots,n\}}} \Big(\prod_{i\in S} x_i\Big) \times \mathbf{M}_{1^k}\big[X(S)\big] \times \mathbf{M}_{n-k}\big[Y(T)\big] \ . \tag{3.38}$$

This given, we have the following direct sum decompositions:

$$\begin{aligned} a)\quad \mathbf{M}_{\mu/00} \;&=\; \bigoplus_{i=0}^{k-1} \mathbf{M}_\alpha(\partial)\Delta_{\mu/i,0} \;\oplus\; \mathbf{X} \;\oplus\; \bigoplus_{j=1}^{n-k} \mathbf{M}_\beta(\partial)\Delta_{\mu/0,j} \\ b)\quad \mathbf{M}_{\mu/00} \;&=\; \bigoplus_{i=1}^{k} \mathbf{M}_\alpha(\partial)\Delta_{\mu/i,0} \;\oplus\; \mathbf{Y} \;\oplus\; \bigoplus_{j=0}^{n-k-1} \mathbf{M}_\beta(\partial)\Delta_{\mu/0,j} \end{aligned} \tag{3.39}$$

$$a)\quad \mathbf{M}_{\mu/i,0} \;=\; \bigoplus_{r=i}^{k} \mathbf{M}_\alpha(\partial)\Delta_{\mu/r,0} \quad , \qquad b)\quad \mathbf{M}_{\mu/0,j} \;=\; \bigoplus_{s=j}^{n-k} \mathbf{M}_\beta(\partial)\Delta_{\mu/0,s} \tag{3.40}$$

with

$$\begin{aligned} a)\quad & \mathbf{A}^x_{00} \;=\; \mathbf{X} \;, \qquad \mathbf{A}^x_{i,0} \;=\; \mathbf{M}_\alpha \quad , \qquad \mathbf{A}^y_{0,j} \;=\; \mathbf{M}_\beta(\partial)\Delta_{\mu/0,j} \;, \\ b)\quad & \mathbf{A}^y_{00} \;=\; \mathbf{Y} \;, \qquad \mathbf{A}^y_{i,0} \;=\; \mathbf{M}_\alpha(\partial)\Delta_{\mu/i,0} \;, \qquad \mathbf{A}^y_{0,j} \;=\; \mathbf{M}_\beta \ . \end{aligned} \tag{3.41}$$

Moreover, the Frobenius characteristics of these modules may be expressed in terms of the Macdonald polynomials as follows:

$$\begin{aligned} a)\quad & \mathcal{F}\,\mathrm{ch}\,\mathbf{A}^x_{00} \;=\; q^{n-k}\,\tilde H_{1^k}\,\tilde H_{n-k} \\ b)\quad & \mathcal{F}\,\mathrm{ch}\,\mathbf{A}^y_{00} \;=\; t^k\,\tilde H_{1^k}\,\tilde H_{n-k} \end{aligned} \qquad c)\quad \mathcal{F}\,\mathrm{ch}\,\mathbf{M}_{(n+1-k,1^k)} = \tilde H_{(n+1-k,1^k)} \tag{3.42}$$

Proof

Formulas 3.39 a) and b) may be obtained by generalizing the argument that yielded 3.30. Similarly 3.40 a) and b) can be easily established by the process that gives 3.31 and 3.32. This given, since $D_x\mathbf{X} = \{0\}$ and $D_x\Delta_{\mu/0,j} = 0$, it follows from 3.39 a) and 3.40 a) that

$$D_x\,\mathbf{M}_{\mu/00} = \bigoplus_{i=1}^{k} \mathbf{M}_\alpha(\partial)\Delta_{\mu/i,0} = \mathbf{M}_{\mu/10} \quad , \qquad D_x\,\mathbf{M}_{\mu/01} = 0 \ .$$

Thus

$$\mathbf{K}^x_{01} = \mathbf{M}_{\mu/10} \quad , \quad \mathbf{K}^x_{00} \;=\; \mathbf{X} \;\oplus\; \bigoplus_{j=1}^{n-k} \mathbf{M}_\beta(\partial)\Delta_{\mu/0,j} \;=\; \mathbf{X} \;\oplus\; \mathbf{M}_{\mu/10} \;,$$

yielding

$$\mathbf{K}^x_{00} \;=\; \mathbf{X} \;\oplus\; \mathbf{K}^x_{01}$$

and 3.41 then follows from the definition I.18. Formula 3.39 b) is established in a similar manner. The remaining identities in 3.41 follow from 3.39 and the stated properties of D_x and D_y.

Thus it only remains to prove the Macdonality of the Frobenius characteristics as stated in 3.42. To begin with we note that it is well known (see [2], [10]) that the linear span of the derivatives of the Vandermonde determinant $\Delta_n(x_1, x_2, \ldots, x_n)$ yields a graded version of the left regular representation of S_n with Frobenius characteristic given by the symmetric polynomial

$$(1-t)(1-t^2)\cdots(1-t^n)\, h_n\big[\tfrac{X}{1-t}\big] \;=\; \sum_{\lambda\vdash n} S_\lambda[X]\, S_\lambda[1,t,t^2,\ldots](1-t)(1-t^2)\cdots(1-t^n) \;.$$

Now we have shown in [10] that

$$\tilde H_{1^n} \;=\; (1-t)(1-t^2)\cdots(1-t^n)\, h_n\big[\tfrac{X}{1-t}\big]$$

and

$$\tilde H_n \;=\; (1-q)(1-q^2)\cdots(1-q^n)\, h_n\big[\tfrac{X}{1-q}\big] \;.$$

Thus formula 3.37 defines $\mathbf{X}$ as the bigraded module obtained by inducing from $S_k \times S_{n-k}$ to S_n the tensor product of a representation with Frobenius characteristic $\tilde H_{1^k}$ by one of Frobenius characteristic $q^{n-k}\tilde H_{n-k}$. A known result of representation theory (see [20]) then yields that

$$\mathcal{F}\,\mathrm{ch}\,\mathbf{X} \;=\; q^{n-k}\, \tilde H_{1^k}\, \tilde H_{n-k}$$

and 3.42 a) then follows 3.41 a). Similarly we derive 3.42 b) from 3.38 and 3.41 b).

We should note at this point that the identities we have established so far already yield an inductive mechanism for proving the $n!$ conjecture for hooks. Indeed, making use of I.11 we immediately derive from 3.39 a) and b) that

$$\begin{aligned} &a)\;\; \partial_{p_1} C_\mu = C_{\mu/00} = (t+t^2+\cdots+t^k)\, C_\alpha + q^{n-k}\, \tilde H_{1^k}\, \tilde H_{n-k} + (1+q+\cdots+q^{n-k-1})\, C_\beta \\ &b)\;\; \partial_{p_1} C_\mu = C_{\mu/00} = (1+t+\cdots+t^{k-1})\, C_\alpha + t^k\, \tilde H_{1^k}\, \tilde H_{n-k} + (q+q^2+\cdots+q^{n-k})\, C_\beta \end{aligned} \qquad 3.43$$

Now either of these two equalities yields the implication

$$\dim \mathbf{M}_\alpha = \dim\; \mathbf{M}_\beta = n! \quad\Longrightarrow\quad \dim \mathbf{M}_\mu = (n+1)! \;. \qquad 3.44$$

In fact, applying $\partial_{p_1}^n$ to both sides of 3.43 a) gives (using the notation in I.4)

$$F_{(n+1-k,1^k)} \;=\; t\,[k]_t\, F_{(n+1-k,1^{k-1})} \;+\; q^{n-k}\binom{n}{k}\, [k]_t!\, [n-k]_q! \;+\; [n-k]_q\, F_{(n-k,1^k)} \qquad 3.45$$

with $[k]_t = 1+\cdots+t^{k-1}$, $[k]_t! = [1]_t[2]_t\cdots[k]_t$ and $[n-k]_q$, $[n-k]_q!$ analogously defined. Thus 3.44 follows from 3.45 by setting $t=q=1$.

To prove 3.42 c) we need a few auxiliary identities. To begin with note that subtracting 3.43 b) from 3.43 a) we obtain that

$$\tilde H_{1^k}\,\tilde H_{n-k} \;=\; \frac{t^k-1}{t^k-q^{n-k}}\;C_\alpha \;+\; \frac{1-q^{n-k}}{t^k-q^{n-k}}\;C_\beta \;. \qquad 3.46$$

On the other hand, from suitably modified Macdonald Pieri rules (see [6] or [8]) we derive that

$$\tilde H_{1^k}\,\tilde H_{n-k} \;=\; \frac{t^k-1}{t^k-q^{n-k}}\;\tilde H_{(n+1-k,1^{k-1})} \;+\; \frac{1-q^{n-k}}{t^k-q^{n-k}}\;\tilde H_{(n-k,1^k)} \;. \qquad 3.47$$

Finally, subtracting 3.47 from 3.46 and recalling that $\alpha=(n+1-k,1^{k-1})$ and $\beta=(n-k,1^k)$ we are led to the recursion

$$\begin{aligned}\frac{1-q^{n-k}}{t^k-q^{n-k}}\,C_{(n-k,1^k)} \;&=\; \frac{1-q^{n-k}}{t^k-q^{n-k}}\,\tilde H_{(n-k,1^k)}\\ &\quad+\;\frac{t^k-1}{t^k-q^{n-k}}\,\tilde H_{(n+1-k,1^{k-1})} \;-\; \frac{t^k-1}{t^k-q^{n-k}}\;C_{(n+1-k,1^{k-1})}\;.\end{aligned} \qquad 3.48$$

This enables us to prove 3.42 c) for each n by induction on k . Indeed, since $\mathbf{M}_{(n+1)}$, by definition, is the linear span of derivatives of the Vandermonde determinant in $(y_1,y_2,\cdots,y_n)$ we necessarily have

$$\mathcal{F}\,\mathrm{ch}\,\mathbf{M}_{(n+1)} \;=\; \tilde H_{(n+1)} \;.$$

This gives 3.42 c) for $k=0$. However, if by induction, we assume 3.42 c) for $k-1$, which is

$$C_{(n+1-k,1^{k-1})} \;=\; \mathcal{F}\,\mathrm{ch}\,\mathbf{M}_{(n+1-k,1^{k-1})} \;=\; \tilde H_{(n+1-k,1^{k-1})} \;,$$

from 3.48 we immediately obtain that

$$C_{(n-k,1^k)} \;=\; \tilde H_{(n-k,1^k)} \;.$$

Thus 3.43 c) must hold true for all k and our proof is complete.

Remark 3.3

We should point out the remarkable agreement that our conjectures have with the theory of Macdonald polynomials. To begin with note that substituting 3.46 in 3.43 a) or b) and carrying out the simplifications yields that

$$\partial_{p_1}\,\tilde H_{(n+1-k,1^k)} \;=\; \frac{q^{n-k}-t^{k+1}}{q^{n-k}-t^k}\;\frac{1-t^k}{1-t}\;\tilde H_{(n+1-k,1^{k-1})} \;+\; \frac{t^k-q^{n+1-k}}{t^k-q^{n-k}}\;\frac{1-q^{n-k}}{1-q}\;\tilde H_{(n-k,1^k)}$$

and this is precisely what may be obtained from I.13 and I.14. In the same vein, we can show that 3.46 itself, which is an instance of higher order Pieri rules, is in fact a consequence of Conjecture I.16 or the four term recursion (which are the same because of Theorem I.1). This can be seen from the following formula which expresses Frobenius characteristics of atoms directly in terms of Macdonald Polynomials.

Theorem 3.3

Let l and a be the leg and arm of (i,j), let τ be the partition in the shadow of (i,j). As in the proof of Proposition I.8, let $x_0^{ij},\ldots,x_m^{ij}$; $u_0^{ij},\ldots,u_m^{ij}$ be the corner weights of τ, $\rho^{(1)},\rho^{(2)},\ldots,\rho^{(m)}$ be the predecessors of τ ordered from left to right so that $x_1^{ij},\ldots,x_m^{ij}$ are the respective weights of the cells $\tau/\rho^{(1)},\ldots,\tau/\rho^{(m)}$. Set $\alpha^{(s)}=\mu-\tau+\rho^{(s)}$. Then on the $C=\tilde H$ conjecture, we have

$$\frac{1}{q^a}A_{ij}^x \;=\; \frac{1}{t^l}A_{ij}^y \;=\; \sum_{s=1}^m \frac{\prod_{r=1}^{m-1}(x_s^{ij}-u_r^{ij})}{\prod_{r=1\,;\,r\neq s}^m(x_s^{ij}-x_r^{ij})}\;\tilde H_{\alpha^{(s)}}\;. \qquad 3.49$$

Proof

Our point of departure is the definition

$$A_{ij}^x \;=\; C_{\mu/ij}-t\,C_{\mu/i+1,j}-C_{\mu/i,j+1}+t\,C_{\mu/i+1,j+1} \qquad 3.50$$

with the $C\,'s$ computed by means of formula 1.20, that is

$$C_{\mu/ij}(x;q,t) \;=\; \frac{1}{M}\sum_{s=1}^m \frac{1}{x_s^{ij}}\,\frac{\prod_{r=0}^m\big(x_s^{ij}-u_r^{ij}\big)}{\prod_{r=1\,;\,r\neq s}^m\big(x_s^{ij}-x_r^{ij}\big)}\;\tilde H_{\alpha^{(s)}}\;, \qquad 3.51$$

where $M=(1-1/t)(1-1/q)$. For simplicity we shall assume that the shadows of $(i,j),(i+1,j),(i,j+1)$ and $(i+1,j+1)$ contain the same corners of μ. This given, note that for $s\neq 0$ we have the relations

$$x_s^{ij} \;=\; t\,q\,x_s^{i+1,j+1}\;,\qquad x_s^{i+1,j} \;=\; q\,x_s^{i+1,j+1}\;,\qquad x_s^{i,j+1} \;=\; t\,q\,x_s^{i+1,j+1}\;.$$

Moreover, we recall that

$$\frac{1}{t}\,u_0^{ij} \;=\; \frac{1}{t}\,u_0^{i,j+1} \;=\; u_0^{i+1,j} \;=\; u_0^{i+1,j+1}\;,$$

and

$$\frac{1}{q}\,u_m^{ij} \;=\; \frac{1}{q}\,u_m^{i+1,j} \;=\; u_m^{i,j+1} \;=\; u_0^{i+1,j+1}\;.$$

Using these relations in 3.51 written for $(i,j),(i+1,j),(i,j+1)$ and $(i+1,j+1)$, we obtain from 3.50 that the coefficient of $\tilde H_{\alpha^{(s)}}$ in A_{ij}^x is

$$\begin{aligned} A_{ij}^x\,\big|_{\tilde H_{\alpha^{(s)}}} = \frac{CF}{M}\Big(x_s\,t\,q\Big(1-\frac{tu_0}{tqx_s}\Big)\Big(1-\frac{qu_m}{tqx_s}\Big) & \\ -\;x_s\,t\,q\Big(1-\frac{u_0}{qx_s}\Big)\Big(1-\frac{qu_m}{qx_s}\Big) & \\ -\;x_s\,t\Big(1-\frac{tu_0}{tx_s}\Big)\Big(1-\frac{u_m}{tx_s}\Big) & \\ -\;x_s\,t\Big(1-\frac{u_0}{x_s}\Big)\Big(1-\frac{u_m}{x_s}\Big)\Big) & \end{aligned} \qquad 3.52$$

where for convenience we have set

$$x_s^{i+1,j+1} = x_s \quad , \qquad u_0^{i+1,j+1} = u_0 \quad , \qquad u_m^{i+1,j+1} = u_m$$

and

$$CF \;=\; \frac{\prod_{r=1}^{m-1}(x_s^{ij} - u_r^{ij})}{\prod_{r=1\,;\,r\neq s}^{m}(x_s^{ij} - x_r^{ij})} \quad .$$

Now a little manipulation simplifies 3.52 to

$$A_{ij}^x \;\Big|_{\tilde H_{\alpha^{(s)}}} \;=\; CF \;\frac{x_s\, t\,(\frac{1}{t} - 1)(1-q)}{M}\;\frac{u_m}{x_s} \;=\; CF\, q\, t\, u_m$$

and this is 3.49 since

$$t\,q\,u_m \;=\; t\,q\,u_m^{i+1,j+1} \;=\; q^a \quad .$$

this completes our proof.

Note that for $\mu = (1^k, n+1-k)$, and $i = j = 0$, formula 3.49 gives

$$\frac{1}{q^{n-k}}\;A_{00}^x \;=\; \frac{x_1^{00} - u_1^{00}}{x_1^{00} - x_2^{00}}\;\tilde H_{(1^{k-1},n+1-k)} \;+\; \frac{x_2^{00} - u_1^{00}}{x_2^{00} - x_1^{00}}\;\tilde H_{(1^k,n-k)} \quad . \tag{3.53}$$

Since in this case

$$x_1^{00} = t^k \quad , \quad x_2^{00} = q^{n-k} \quad \text{and} \quad u_1^{00} = 1 \quad ,$$

substituting this in 3.53 we get that

$$\frac{1}{q^{n-k}}\;A_{00}^x \;=\; \frac{t^k - 1}{t^k - q^{n-k}}\;\tilde H_{(n+1-k,1^{k-1})} \;+\; \frac{q^{n-k} - 1}{q^{n-k} - t^k}\;\tilde H_{(n-k,1^k)} \quad ,$$

which is in complete agreement with what we obtain by combining 3.42 a) with the Macdonald Pieri rule given in 3.47.

4. Dimension bounds.

In this section, we derive a dimension bound for the spaces $\mathbf{M}_{\mu/ij}$. We begin by reviewing the construction that yields the dimension bound of $n!$ for $\mathbf{M}_\mu$. The reader is referred to [10] for proofs and further details.

Given a finite subset S of n-dimensional Cartesian space, we let J_S denote the ideal of polynomials $P(x_1,x_2,\ldots,x_n)$ which vanish on S. The quotient ring $\mathbf{R}_S=\mathbf{Q}[x_1,x_2,\ldots,x_n]/J_S$ may be viewed as the coordinate ring of the algebraic variety consisting of the elements of S. This given, it is clear that

$$\dim \mathbf{R}_S \;=\; |S| \ . \tag{4.1}$$

Although $\mathbf{R}_S$ is not graded it has a filtration given by the subspaces $\mathcal{H}_{\leq k}(\mathbf{R}_S)$ spanned by the monomials $x^p = x_1^{p_1}x_2^{p_2}\cdots x_n^{p_n}$ which are of degree $\leq k$. A graded version of $\mathbf{R}_S$ is obtained by setting

$$\mathrm{gr}\,\mathbf{R}_S \;=\; \mathbf{Q}[x_1,x_2,\ldots,x_n]/\,\mathrm{gr}\,J_S \tag{4.2}$$

with

$$\mathrm{gr}\,J_S \;=\; \big(\, h(P)\ :\ P\in J_S\,\big)$$

where for a polynomial P we let $h(P)$ denote the homogeneous component of P that is of highest degree. It is also convenient to introduce the space $\mathbf{H}_S=\big(\,\mathrm{gr}\,J_S\,\big)^{\perp}$, the orthogonal complement of $\mathrm{gr}\,J_S$ with respect to the scalar product

$$\langle P\ ,\ Q\rangle \;=\; P(\partial)Q(x)\,\big|_{x=0}\ .$$

We may also define $\mathbf{H}_S$ as the space of polynomials that are killed by elements of gr J_S as differential operators. In symbols

$$\mathbf{H}_S \;=\; \big\{Q(x)\ :\ P(\partial)\,Q=0\quad\forall\quad P\in\mathrm{gr}\,J_S\,\big\}\ . \tag{4.3}$$

It is easy to show (see [6]) that any homogeneous basis $\mathcal{B}_S$ for $\mathbf{H}_S$ is also a basis of $\mathrm{gr}\,\mathbf{R}_S$ and $\mathbf{R}_S$. In particular, the dimensions of these three spaces must be the same and thus equal to $|S|$. In fact, we also have for all $k\geq 0$

$$\dim\mathcal{H}_{\leq k}(\mathbf{R}_S) \;=\; \sum_{s=0}^{k}\dim\mathcal{H}_{=s}(\mathbf{H}_S) \;=\; \sum_{s=0}^{k}\dim\mathcal{H}_{=s}(\mathrm{gr}\,\mathbf{R}_S)\ , \tag{4.4}$$

where $\mathcal{H}_{=s}(\mathbf{H}_S)$ and $\mathcal{H}_{=s}(\mathrm{gr}\,\mathbf{R}_S)$ denote the subspaces of $\mathbf{H}_S$ and $\mathrm{gr}\,\mathbf{R}_S$ consisting of their homogeneous elements of degree s.

The natural action of GL_n on polynomials $P(x_1,x_2,\ldots,x_n)$ is defined by setting for an $n\times n$ matrix $A=\|a_{ij}\|_{i,j=1}^n$

$$T_A\,P(x) \;=\; P(xA) \tag{4.5}$$

where xA denotes matrix multiplication of the row vector $x=(x_1,x_2,\ldots,x_n)$ by A. It is not difficult to show that if A is an orthogonal matrix, then for all $P,Q\in\mathbf{Q}[x_1,x_2,\ldots,x_n]$ we have

$$\big\langle T_AP\ ,\ T_AQ\big\rangle \;=\; \big\langle P\ ,\ Q\big\rangle \tag{4.6}$$

If G is a group of $n\times n$ matrices that leave S invariant then both J_S and $\mathrm{gr}\, J_S$ remain invariant under T_A for every $A\in G$ and we can define an action of G on the two quotient spaces $\mathbf{R}_S$ and $\mathrm{gr}\,\mathbf{R}_S$. It develops that the resulting G-modules are easily shown to be equivalent. If in addition G consists of orthogonal matrices, then from 4.6 it follows that $\mathbf{H}_S=(\mathrm{gr}\, J_S)^{\perp}$ is also G-invariant and equivalent to $\mathrm{gr}\,\mathbf{R}_S$ as a graded G-module. Moreover we have the following character identity for all $k\geq 0$:

$$\mathrm{ch}\,\mathcal{H}_{\leq k}(\mathbf{R}_S) \;=\; \sum_{s=0}^{k}\mathrm{ch}\,\mathcal{H}_{=s}(\mathbf{H}_S) \;=\; \sum_{s=0}^{k}\mathrm{ch}\,\mathcal{H}_{=s}(\mathrm{gr}\,\mathbf{R}_S)\ . \qquad 4.7$$

Given a group G, the simplest G-invariant subsets are its "*orbits*." More precisely, for any point $\rho=(\rho_1,\rho_2,\ldots,\rho_n)$, we set

$$[\,\rho\,]_G \;=\; \{\ \rho A\ :\ A\in G\ \}\ . \qquad 4.8$$

Clearly, G acts on the orbit $[\,\rho\,]_G$ as it does on the left cosets of the subgroup that leaves ρ invariant. It follows from this that both $\mathbf{R}_{[\,\rho\,]_G}$ and $\mathrm{gr}\,\mathbf{R}_{[\,\rho\,]_G}$ afford this left coset action; in particular, if ρ is a regular point (that is, ρ has a trivial stabilizer), then $\mathbf{R}_{[\,\rho\,]_G}$ and $\mathrm{gr}\,\mathbf{R}_{[\,\rho\,]_G}$ are versions of the left regular representation of G. Moreover, if G is a group of orthogonal matrices, then $\mathbf{H}_{[\,\rho\,]_G}$ affords a graded version of the left regular representation of G and consists of polynomials that are killed by all G-invariant differential operators (see [6]). In particular, all elements of $\mathbf{H}_{[\,\rho\,]_G}$ are harmonic.

To get our dimension bounds we need to suitably specialize G and the point ρ. To this end, given $\mu=(\mu_1\geq\mu_2\geq\ldots\geq\mu_k>0)\vdash n$ let $h=\mu_1$ be the number of parts of the conjugate of μ and let $(\alpha_1,\alpha_2,\ldots,\alpha_k;\beta_1,\beta_2,\ldots,\beta_h)$ be distinct rational numbers. If preferred, the latter may be taken to be two additional sets of indeterminates. Recall that an injective tableau T of shape $\mu\vdash n$ is a labeling of the cells of μ by the numbers $\{1,2,\ldots,n\}$. The collection of all such tableaux is denoted by $\mathcal{IT}(\mu)$. Given a tableau $T\in\mathcal{IT}(\mu)$, for each $i=1,2,\ldots,n$ we set

$$a_i(T)=\alpha_r\quad,\qquad b_i(T)=\beta_c \qquad 4.9$$

if the label i is at the intersection of row r with column c. The resulting point of $2n$-dimensional space will be denoted by $\rho(T)$. In other words we set

$$\rho(T)=\big(a_1(T),a_2(T),\ldots,a_n(T);b_1(T),b_2(T),\ldots,b_n(T)\big)\ .$$

For instance, for $\mu=(3,2)$ and

$$T \;=\; \begin{array}{|c|c|c}\hline 5&3&\multicolumn{1}{c}{}\\\hline 2&1&\multicolumn{1}{c|}{4}\\\hline\end{array}$$

we set

$$\rho(T) \;=\; (\alpha_1,\alpha_1,\alpha_2,\alpha_1,\alpha_2\,;\,\beta_2,\beta_1,\beta_2,\beta_3,\beta_1\ )\ .$$

Note that the collection

$$\{\ \rho(T)\ :\ T\in\mathcal{IT}(\mu)\ \} \qquad 4.10$$

consists of $n!$ distinct points. Indeed, since the $\alpha\,'s$ and the $\beta\,'s$ are assumed to be distinct, we can reconstruct the position of any label i in T by simply looking at the i^{th} and the $(n+i)^{th}$

coordinates of $\rho(T)$. Note that the collection in 4.10 may also be viewed as an S_n-orbit under the diagonal action. More precisely, we see that for any $T \in \mathcal{IT}(\mu)$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in S_n$, we have

$$\sigma\rho(T) \;=\; \Big(a_{\sigma_1}(T), a_{\sigma_2}(T), \ldots, a_{\sigma_n}(T)\,;\, b_{\sigma_1}(T), b_{\sigma_2}(T), \ldots, b_{\sigma_n}(T)\Big) \;=\; \rho(\sigma^{-1}\,T) \;,$$

where $\sigma^{-1}\,T$ is the tableau obtained by replacing the label i in T by the label σ_i^{-1}. This given, we can consider the collection in 4.10 as the S_n-orbit of a point ρ_μ corresponding some specially chosen injective tableau of shape μ. To be specific we may let T_0 be the "superstandard tableau"; this is the tableau obtained by labeling the cells of $\mu \vdash n$ successively from $1, \ldots, n$ starting from the bottom row and proceeding on up, from left to right in each row. Set

$$\rho_\mu \;=\; \rho(T_0) \;. \tag{4.11}$$

We can thus apply the theory we have outlined at the beginning of the section with G specialized to the group of matrices yielding the diagonal action of S_n and construct the three spaces

$$\mathbf{R}_{[\rho_\mu]} \quad , \quad \mathrm{gr}\,\mathbf{R}_{[\rho_\mu]} \quad \text{and} \quad \mathbf{H}_{[\rho_\mu]}$$

where $[\rho_\mu]$ denotes the orbit of $\rho(T_0)$ or, equivalently, the subset of $2n$-dimensional space defined by 4.10. We thus obtain three left regular representations of S_n and in particular we have

$$\dim \mathbf{R}_{[\rho_\mu]} = \dim \mathrm{gr}\,\mathbf{R}_{[\rho_\mu]} = \dim \mathbf{H}_{[\rho_\mu]} = n! \;. \tag{4.12}$$

The definition of these spaces suggests that they may depend on our choice of the $\alpha_i' s$ and $\beta_j' s$. This is clearly the case for the coordinate ring $\mathbf{R}_{[\rho_\mu]}$. Nevertheless, there is strong evidence that the space of harmonics $\mathbf{H}_{[\rho_\mu]}$ as well as the ideal $\mathrm{gr}\,J_{[\rho_\mu]}$ and the quotient ring $\mathrm{gr}\,\mathbf{R}_{[\rho_\mu]}$ only depend on the choice of the partition μ. The reason for this stems from the following result:

Proposition 4.1

If (i,j) is an outer corner cell of μ then for any $s = 1, 2, \ldots, n$ the monomial $x_s^i y_s^j$ belongs to the ideal $\mathrm{gr}\,J_{[\rho_\mu]}$. In particular, if a monomial $x^p y^q = x_1^{p_1} \cdots x_n^{p_n} y_1^{q_1} \cdots y_n^{q_n}$ does not vanish in $\mathrm{gr}\,R_{[\rho_\mu]}$ then all the pairs (p_s, q_s) must give cells of μ. For the same reason, every polynomial in $\mathbf{H}_{[\rho_\mu]}$ must be a linear combination of monomials satisfying the same condition.

Proof

This result was first proved in [10] (see Proposition 1.2 there). Since the argument is quite simple and illuminating, we will include a proof here as well. To this end note that the polynomial

$$P_{(i,j)}(x,y) = \prod_{i'=1}^{i} (x_s - \alpha_{i'}) \prod_{j'=1}^{j} (y_s - \beta_{j'})$$

must necessarily vanish throughout $[\rho_\mu]$. Indeed, for any point

$$\rho(T) = (a_1, \ldots, a_n; b_1, \ldots, b_n) \in [\rho_\mu]$$

our definition gives that $a_s = \alpha_{i'}$ for some $i' \le i$ if s is south of (i,j) in T and $b_s = \beta_{j'}$ for some $j' \le j$ if s is west of (i,j). Since every cell of μ satisfies at least one of these conditions we see that at least one of the factors of $P_{(i,j)}$ must necessarily vanish for $(x;y) = (a_1,\ldots,a_n; b_1,\ldots,b_n)$. This places $P_{(i,j)}$ in $J_{[\rho_\mu]}$ and its highest homogeneous component $x_s^i y_s^j$ in $\mathrm{gr}\, J_{[\rho_\mu]}$. Thus every monomial which contains $x_s^i y_s^j$ as a factor must necessarily vanish in $\mathrm{gr}\,\mathbf{R}_{[\rho_\mu]}$ and every polynomial in $\mathbf{H}_{[\rho_\mu]}$ must be killed by $\partial_{x_s}^i \partial_{y_s}^j$. Since this must hold true for any $s = 1,\ldots,n$, we deduce that every element of $\mathrm{gr}\,\mathbf{R}_{[\rho_\mu]}$ or $\mathbf{H}_{[\rho_\mu]}$ must be a linear combination of monomials $x_1^{p_1}\cdots x_n^{p_n} y_1^{q_1}\cdots y_n^{q_n}$ where each pair (p_s,q_s) must be a cell of μ.

This result has the following immediate corollary

Theorem 4.1

For any choice of the α_i and β_j we have the containment

$$\mathbf{M}_\mu \subseteq \mathbf{H}_{[\rho_\mu]} \ . \tag{4.13}$$

In particular,

$$\dim \mathbf{M}_\mu \le n! \ . \tag{4.14}$$

Thus on the $n!$ conjecture we have

$$\mathbf{M}_\mu \;=\; \mathbf{H}_{[\rho_\mu]} \qquad \textit{and} \qquad \mathrm{gr}\, J_{[\rho_\mu]} \;=\; I_{\Delta_\mu} \ , \tag{4.15}$$

where I_{Δ_μ} denotes the ideal of polynomials that kill Δ_μ.

Proof

These results were first proved in [10] (see Theorems 1.1 and 1.2 there). We sketch the idea of the argument here. Since $\mathbf{H}_{[\rho_\mu]}$ affords a version of the left regular representation of S_n, it must contain a polynomial $\Delta(x;y)$, unique up to a scalar factor, which alternates under the diagonal action. Clearly all the monomials appearing in $\Delta(x;y)$ must be of the form

$$x^p y^q = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} y_1^{q_1} y_2^{q_2} \cdots y_n^{q_n}$$

with $(p_1,q_1),(p_2,q_2),\ldots,(p_n,q_n)$ all distinct. On the other hand, Proposition 4.1 guarantees that each of these pairs must give a cell of μ. Combining these two facts yields that the sequence

$$\big\{(p_1,q_1),(p_2,q_2),\ldots,(p_n,q_n)\big\}$$

must be a permutation of the cells of μ. Thus $\Delta(x;y)$ can only be a multiple of $\Delta_\mu(x;y)$ and we must have

$$\Delta_\mu(x;y) \in \mathbf{H}_{[\rho_\mu]} \ . \tag{4.16}$$

However, since $\mathbf{H}_{[\rho_\mu]}$ is derivative closed, we must also have

$$\mathbf{M}_\mu \;=\; \mathcal{L}_\partial[\Delta_\mu] \subseteq \mathbf{H}_{[\rho_\mu]} \ ,$$

proving 4.13. This completes our proof since 4.14 and 4.15 are immediate consequences of 4.13.

Now let $\mu \vdash n+1$ and $[\rho_\mu]_{ij}$ denote the subset of the orbit $[\rho_\mu]$ consisting of the points $\rho(T)$ corresponding to tableaux T where $n+1$ lies in the shadow of the cell (i,j). Clearly the cardinality of this set is

$$\big|\, [\rho_\mu]_{ij} \,\big| \;=\; \#\text{shadow}(i,j) \;\times\; n! \qquad 4.17$$

where "$\#\text{shadow}(i,j)$" denotes the number of cells that are in the shadow of (i,j). Moreover, it is easy to see that under the diagonal action of S_n, the set $[\rho_\mu]_{ij}$ splits into as many as $\#\text{shadow}(i,j)$ distinct regular orbits. It follows then that each of the three spaces

$$\mathbf{R}_{[\rho_\mu]_{ij}}\;, \qquad \text{gr}\,\mathbf{R}_{[\rho_\mu]_{ij}}\;, \qquad \text{and} \qquad \mathbf{H}_{[\rho_\mu]_{ij}}\;,$$

breaks up into a direct sum of $\#\text{shadow}(i,j)$ regular representations of S_n. These observations yield the following extension of Theorem 4.1.

Theorem 4.2

For any choice of the α_i and β_j and any cell $(i,j)\in\mu$, we have the containment

$$\mathcal{L}_\partial[\,\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu(x;y)\,] \;\subseteq\; \mathbf{H}_{[\rho_\mu]_{ij}}\;. \qquad 4.18$$

In particular,

$$\dim\mathcal{L}_\partial[\,\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu(x;y)\,] \le \#\text{shadow}(i,j)\;\times\; n!\;. \qquad 4.19$$

Moreover, equality here forces the equalities

$$\mathcal{L}_\partial[\,\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu(x;y)\,] = \mathbf{H}_{[\rho_\mu]_{ij}}\quad,\qquad \text{gr}\,J_{[\rho_\mu]_{ij}} \;=\; I_{\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu}\;, \qquad 4.20$$

where $I_{\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu}$ denotes the ideal of polynomials that kill $\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu$. But then $\mathcal{L}_\partial[\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu(x;y)\,]$ must necessarily break up into a direct sum of $\#\text{shadow}(i,j)$ regular representations of S_n.

Proof

Note that if $P(x;y)$ is an element of the ideal $J_{[\rho_\mu]_{ij}}$ then the polynomial

$$Q(x;y) \;=\; P(x;y)\prod_{i'=1}^{i}\big(x_{n+1}-\alpha_{i'}\big)\prod_{j'=1}^{j}\big(y_{n+1}-\beta_{j'}\big)$$

must necessarily vanish throughout the orbit $[\rho_\mu]$. In fact, $P(x;y)$ vanishes in $[\rho_\mu]_{ij}$ and the product of the two remaining factors vanishes in the rest of $[\rho_\mu]$. This places $Q(x;y)$ in $J_{[\rho_\mu]}$. Denoting as before by $h(P)$ and $h(Q)$ the highest homogeneous components of P and Q, we derive that

$$h(Q) = x^i_{n+1}y^j_{n+1}\;\,h(P)\in \text{gr}\,J_{[\rho_\mu]}\;,$$

and therefore $h(Q)$ must kill all the elements of $\mathbf{H}_{[\rho_\mu]}$. In particular, in view of 4.16 we must also have

$$h(P)(\partial)\,\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu \;=\; 0\;.$$

Since this holds true for any $P \in J_{[\rho_\mu]_{ij}}$ we are brought to the conclusion that

$$\mathrm{gr}\, J_{[\rho_\mu]_{ij}} \;\subseteq\; I_{\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu} \;. \tag{4.21}$$

Now it is easy to show that

$$I_{\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu} \;=\; \mathcal{L}_\partial[\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu]^\perp \;.$$

This gives

$$\big(\, I_{\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu}\,\big)^\perp \;=\; \mathcal{L}_\partial[\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu]$$

and thus 4.18 follows from 4.21 by taking orthogonal complements. This given, 4.19 follows from 4.17 and 4.18 since

$$\dim \mathbf{H}_{[\rho_\mu]} \;=\; \big|\,[\rho_\mu]\,\big| \;.$$

Finally, equality in 4.19 forces equality in 4.18 which in turn can only hold true if equality holds in 4.21. This completes our proof since the last assertion is a consequence of our preliminary observations.

We are now in a position to derive the main result of this section.

Theorem 4.3

For any $\mu \vdash n+1$ and any cell $(i,j) \in \mu$ we have

$$\dim \mathbf{M}_{\mu/ij} \;\leq\; \#\mathrm{shadow}(i,j) \;\times\; n! \;. \tag{4.22}$$

Moreover, if equality holds here, then $\mathbf{M}_{\mu/ij}$, breaks up into a direct sum of $\#\mathrm{shadow}(i,j)$ regular representations of S_n.

Proof

In view of Theorem 4.2 we only need to show that $\mathbf{M}_{\mu/ij}$ and $\mathcal{L}_\partial[\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu(x;y)\,]$ are equivalent as S_n-modules under the diagonal action. To this end note that from 1.16, we derive that:

$$\begin{aligned}\frac{1}{i!}\frac{1}{j!}\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu(x;y) \;&=\; \epsilon_{i,j}\;\Delta_{\mu/ij}(x_1,\ldots,x_n;y_1,\ldots,y_n)\\ &\quad+ \sum_{\substack{(i',j')\,\in\,\mu\\ i'>i \;\text{ or }\; j'>j}} x_{n+1}^{i'-i}\,y_{n+1}^{j'-j}\;c_{i',j'}\;\Delta_{\mu/i'j'}(x_1,\ldots,x_n;y_1,\ldots,y_n)\end{aligned} \tag{4.23}$$

where $\epsilon_{ij} = \pm 1$ and the $c_{i',j'}$ are suitable constants. Thus for any $f \in \mathbf{Q}[x_1,\ldots,x_n;y_1,\ldots,y_n]$ we necessarily have

$$a)\quad f(\partial_x;\partial_y)\,\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu(x;y) \;=\; 0 \qquad\longleftrightarrow\qquad b)\quad f(\partial_x;\partial_y)\,\Delta_{\mu/ij} \;=\; 0 \;.$$

In fact, we see from 4.23 that b) immediately follows from a) by setting $x_{n+1} = y_{n+1} = 0$. Conversely, if b) holds true then applying to it the operator

$$D_{i'-i,j'-j} \;=\; \sum_{s=1}^{n} \partial_{x_s}^{i'-i}\partial_{y_s}^{j'-j}$$

we obtain that

$$f(\partial_x;\partial_y)\,\Delta_{\mu/i',j'} \;\;=\;\; 0$$

must hold true for all $(i',j')\in\mu$ that are in the shadow of (i,j) and this forces a) to hold true as well.

Now from the relations in 1.13 it follows that

$$\begin{aligned}
\partial_{x_{n+1}}\,\big(\,\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu(x;y)\,\big) \;\;&=\;\; -\sum_{s=1}^{n}\partial_{x_s}\,\big(\,\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu(x;y)\,\big)\\
\partial_{y_{n+1}}\,\big(\,\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu(x;y)\,\big) \;\;&=\;\; -\sum_{s=1}^{n}\partial_{y_s}\,\big(\,\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu(x;y)\,\big)\;\;.
\end{aligned}$$

This means that we can construct a basis for

$$\mathcal{L}_\partial[\,\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu(x;y)\,]$$

of the form

$$\mathcal{B}_{ij} \;\;=\;\; \big\{\,b(\partial_x;\partial_y)\,\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu(x;y)\;:\;b\in\mathcal{C}\,\big\}$$

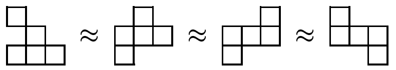
with $\mathcal{C}$ a collection of polynomials in the variables $x_1,\ldots,x_n;y_1,\ldots,y_n$. But then it follows from the observations above that, with the same $\mathcal{C}$, the collection

$$\mathcal{B}^*_{ij} \;\;=\;\; \big\{\,b(\partial_x;\partial_y)\Delta_{\mu/ij}(x;y)\;:\;b\in\mathcal{C}\,\big\}$$

must give a basis for $\mathbf{M}_{\mu/ij}$. This given, if the elements of $\mathcal{C}$ are chosen to be homogeneous, it follows that the action of S_n on the corresponding homogeneous components of $\mathcal{B}_{ij}$ and $\mathcal{B}^*_{ij}$ must be given by exactly the same matrices, proving that $\mathbf{M}_{ij}$ and $\mathcal{L}_\partial[\,\partial^i_{x_{n+1}}\partial^j_{y_{n+1}}\Delta_\mu(x;y)\,]$ must be equivalent also as *graded* S_n-modules. This completes our argument.

5. Atoms and further lattice diagram characteristics.

In [8] Garsia and Haiman call two lattice diagrams D_1 and D_2 "equivalent" and write $D_1 \approx D_2$ if and only D_2 can be obtained from D_1 by a sequence of row and column rearrangements. Diagrams that are equivalent to skew diagrams are briefly referred to there as "gistols." We should note that it is not visually obvious when two diagrams are equivalent. For instance we have



Following standard convention, the "conjugate" of a diagram D, denoted by D' is the diagram obtained by reflecting D across the diagonal line $x = y$. Similarly, the reflection of a lattice square $s = (i, j)$ across $x = y$ is denoted by $s' = (j, i)$. Finally, if D may be decomposed into the union of two diagrams D_1 and D_2 in such a manner that no square of D_2 is in the same row or column of a square of D_1, then we shall say that D is "decomposable" and we write $D = D_1 \times D_2$. This given, Garsia-Haiman postulate the existence of a family of polynomials $\{G_D(x; q, t)\}_D$, and a family of weights $w_{s,D}(q, t)$, with the following basic properties:

$$\begin{cases} (0) \quad G_D(x;q,t) = \tilde{H}_\mu(x;q,t) & \text{if } \ D \text{ is the diagram of } \mu \\ (1) \quad G_{D_1}(x;q,t) = G_{D_2}(x;q,t) & \text{if } \ D_1 \approx D_2 \\ (2) \quad G_{D_1}(x;q,t) = G_{D_2}(x;t,q) & \text{if } \ D_2 \approx D_1' \\ (3) \quad G_D(x;q,t) = G_{D_1}(x;q,t) G_{D_2}(x;q,t) & \text{if } \ D \approx D_1 \times D_2 \\ (4) \quad \partial_{p_1} G_D(x;q,t) = \sum_{s \in D} w_{s,D}(q,t) \; G_{D/s}(x;q,t) \ , & \text{with } \ D/s = D \ \text{ minus } s \ . \end{cases} \qquad 5.1$$

It should be noted at the onset that these properties overdetermine the family $\{G_D(x; q, t)\}_D$, so that existence is by no means guaranteed. Nevertheless, all the experimentations so far indicate that the existence of such a family is consistent with the theory of Macdonald polynomials. In particular it was shown in [8] that for any partition μ we have

$$\tilde{H}_{\mu'}(x;q,t) \ = \ \tilde{H}_\mu(x;t,q)$$

which is in perfect agreement with condition (2) in 5.1.

Experimentation suggests that the weights $w_{s,D}(q, t)$ should be monomials in q, t, but there are no conjectured formulas for general lattice diagrams. Nevertheless, we should point out that if condition (4) holds for the conjugate D' of a diagram D, that is we have

$$\partial_{p_1} G_{D'}(x;q,t) \ = \ \sum_{s' \in D'} w_{s',D'}(q,t) \; G_{D'/s'}(x;q,t) \ , \qquad 5.2$$

then, upon interchanging q and t, from condition (2) we immediately derive that we must also have

$$\partial_{p_1} G_D(x;q,t) \ = \ \sum_{s \in D} w_{s',D'}(t,q) \; G_{D/s}(x;q,t) \ . \qquad 5.3$$

Thus the conditions in 5.1 force the existence of at least one pair of "weights" both yielding the expansion in 5.1 (4). Now, in the case that D is a skew diagram, representation theoretical reasons suggest that we should use either one of the following two choices of weights:

$$a)\quad w[s,D] = t^{l_D(s)} q^{a'_D(s)} \qquad \text{and} \qquad b)\quad w[s,D] = t^{l'_D(s)} q^{a_D(s)} \tag{5.4}$$

where as customary $l_D(s), l'_D(s)$ denote the number of cells strictly north and south, respectively, of s in D, and likewise $a_D(s), a'_D(s)$ give the number of cells strictly east and west, respectively. It is easy to see that is consistent with the relations given in 5.2 and 5.3. Using these weights, we can determine a wide variety of the polynomials G_D, and each via a number of different independent ways all leading to the same final Schur function expansion. Remarkably, all the polynomials thus obtained reduce to $h_1^{|D|}$ when we set $t = q = 1$. In particular, when D is a skew diagram or a diagram obtained by removing a cell from a Ferrers diagram, we invariably obtain an expansion of the form

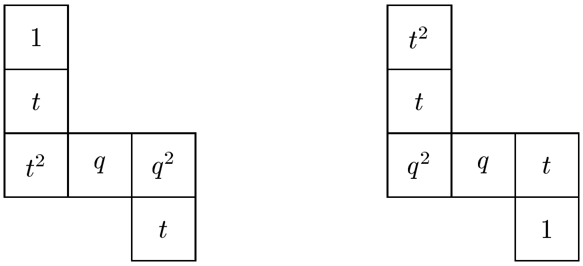
$$G_D(x;q,t) \;=\; \sum_{\lambda \vdash n} S_\lambda(x)\, \tilde{K}_{\lambda,D}(q,t) \tag{5.5}$$

with $\tilde{K}_{\lambda,D}(q,t)$ polynomials with nonnegative integral coefficients satisfying

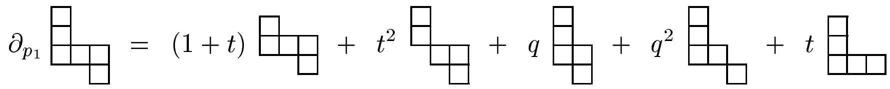
$$\tilde{K}_{\lambda,D}(1,1) \;=\; f_\lambda \;=\; \#\{\text{ standard tableaux of shape } \lambda\ \} \ . \tag{5.6}$$

Even more remarkably, all the identities involving Macdonald polynomials we have been able to derive by means of the rules in 5.1 end up to be computer verifiable and/or theoretically provable.

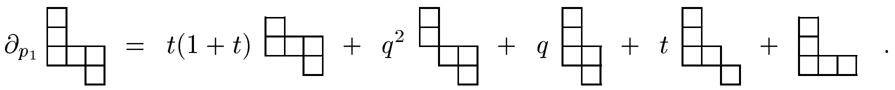
To get our point across, it will be good to review some of these calculations here. As a first example, we shall apply rule (4) to the diagram $D = \{(0,2),(1,0),(1,1),(1,2),(2,0),(3,0)\}$. In the figure below, the first tableau is obtained by filling the cells of D with the weights computed according to formula a) of 5.4 and the second according to formula b).



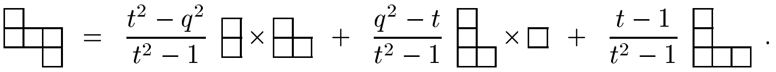
Thus, if we use the diagrams themselves to represent the corresponding polynomial, rule (4) with the first set of weights gives



while the second set gives

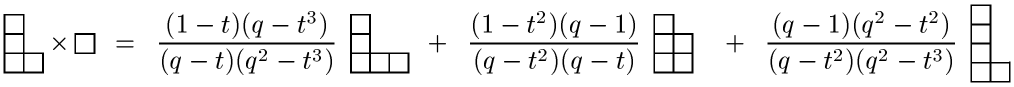


We can thus obtain by subtraction that



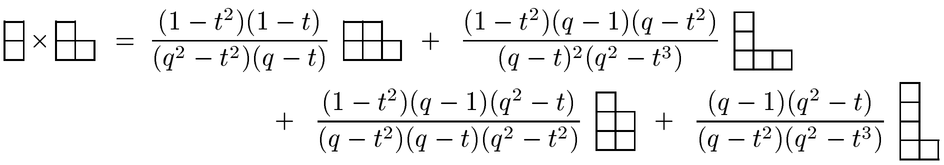
5.7

It easily obtained, either by computer or by means of Macdonald Pieri rules, that



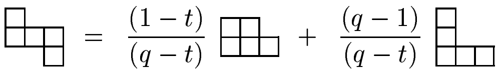
5.8

and



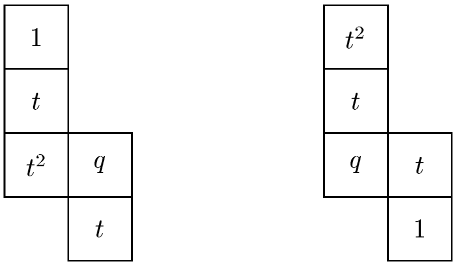
5.9

Using 5.8 and 5.9 in 5.7 yields the surprisingly simple final expression

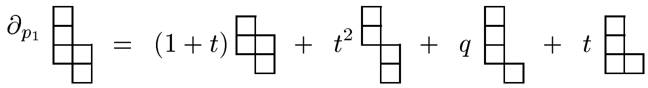


5.10

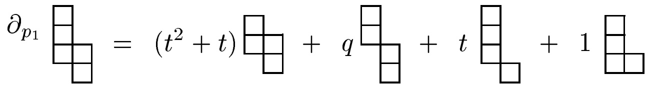
Applying rule (4) to the diagram $\{(0,1),(1,0),(1,1),(2,0),(3,0)\}$ according to the weights



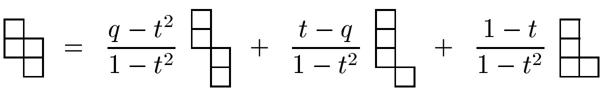
gives



and

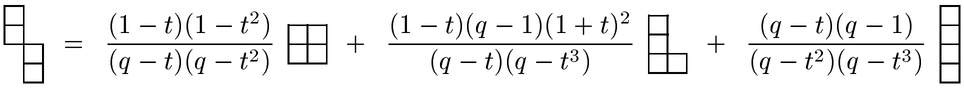


so by subtraction we get



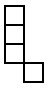
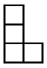
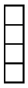
5.11

Using Pieri rules again gives

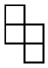
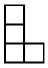


5.12

and

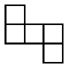
 $= \dfrac{t^3-1}{t^3-q}$  $+ \dfrac{1-q}{t^3-q}$  . 5.13

Substituting 5.12 and 5.13 in 5.11 produces

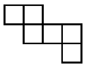
 $= \dfrac{1-t}{q-t}$  $+ \dfrac{q-1}{q-t}$  , 5.14

leaving us with the puzzle of explaining why the coefficients we get here are the same we got in 5.10.

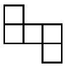
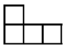
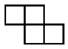
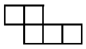
But we have more surprises coming. We have yet another path that yields an expression for the polynomial indexed by the diagram

$D =$  . 5.15

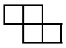
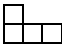
This is based on applying rule (4) to the following diagram.



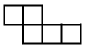
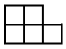
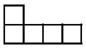
Omitting the details, the resulting expansion turns out to be

 $= \dfrac{t-q}{1-q}$  $\times$  $+ \dfrac{q^2-t}{1-q}$  $\times$  $+ \dfrac{t-q^2}{1-q}$  $\times$  $+ \dfrac{1-t}{1-q}$  5.16

Note next that applying rule (2) we can transform 5.14 into the expansion

 $= \dfrac{(1-q)}{(t-q)}$  $+ \dfrac{(t-1)}{(t-q)}$  . 5.17

Omitting again the details, we can show that

 $= \dfrac{(1-q^2)}{(t-q^2)}$  $+ \dfrac{(t-1)}{(t-q^2)}$  . 5.18

Now miraculously, after we substitute 5.17 and 5.18 into 5.16, apply the required Pieri rules and feed the rather monstrous result into the computer we witness the occurrence of massive simplifications yielding that 5.16 is yet another way of writing 5.10.

The reader may find it amusing to play this game by means of Stembridge's SF Maple package. Seeing is believing that there must be a beautiful explanation for all these miraculous identities. Now it develops that we can use the present theory to remove the mystery out of some of them. To see this we start by writing the identities in I.19 in the form

$$
\begin{array}{llll}
a)_x & C_{\mu/ij} = K^x_{ij} \;+\; t\, C_{\mu/i+1,j} \quad , & a)_y & C_{\mu/ij} = K^y_{ij} \;+\; q\, C_{\mu/i,j+1} \\
b)_x & K^x_{ij} = A^x_{ij} \;+\; K^x_{i,j+1} \quad , & b)_y & K^y_{ij} = A^y_{ij} \;+\; K^x_{i+1,j}
\end{array}
\qquad 5.19
$$

Iterating $a)_x$ and using the fact that $C^x_{\mu/i,\,\mu'_{j+1}} = \{0\}$ we obtain

$$C^x_{\mu/ij} = K^x_{ij} + t\, K^x_{i+1,j} + \cdots + t^{l_{ij}} K^x_{\mu'_{j+1}-1\,,j} \qquad 5.20$$

where $l_{ij} = \mu'_{j+1} - i - 1$ is the leg of the cell (i,j). Similarly from $b)_x$, using $K^x_{i,\mu_{i+1}} = \{0\}$ we derive that

$$K^x_{ij} = A^x_{ij} + A^x_{i,j+1} + \cdots + A^x_{i,\mu_{i+1}-1} \ . \qquad 5.21$$

Taking account of 3.47 let us set for each cell $s = (i,j)$

$$\Xi_{\mu,(i,j)} \;=\; \frac{1}{q^a} A^x_{ij} \;=\; \frac{1}{t^l} A^y_{ij} \;=\; \sum_{s=1}^{m} \frac{\prod_{r=1}^{m-1} (x^{ij}_s - u^{ij}_r)}{\prod_{r=1\ r\neq s}^{m} (x^{ij}_s - x^{ij}_r)}\ \tilde H_{\alpha^{(s)}} \ . \qquad 5.22$$

This given, we may rewrite 5.21 in the form

$$K^x_{ij} = \sum_{(i,j)\to\, s'} q^{a(s')}\, \Xi_{\mu,s'} \ ,$$

where we have used the symbol " $(i,j)\to s'$ " to indicate that we are to sum over cells s' that are directly east of (i,j) including (i,j) itself and $a(s')$ denotes the arm of s' in μ. Using such an expression for each of the characteristics $K^x_{i',j}$ occurring in 5.20, we derive that

$$C_{\mu/ij} \;=\; \sum_{(i,j)\leq\ (i',j')=s'} t^{i'-i}\, q^{a(s')}\, \Xi_{\mu,s'} \qquad 5.23$$

where "$(i,j) \leq\ (i',j')$" is to represent that we are to sum over all cells $(i',j') \in \mu$ that are in the shadow of (i,j). Denoting the partition in the shadow of (i,j) by τ_{ij}, we see that 5.23 may be rewritten as

$$C_{\mu/ij} \;=\; \sum_{(i,j)\leq\ s'} t^{l'_{\tau_{ij}}(s')}\, q^{a_{\tau_{ij}}(s')}\, \Xi_{\mu,s'} \ . \qquad 5.24$$

On the other hand, we can derive from the recurrence in 5.19 $a)_y$ that we also have

$$C_{\mu/ij} = K^y_{ij} + q\, K^y_{i,j+1} + \cdots + q^{a_{ij}} K^y_{i\,,\,\mu_{i+1}-1} \qquad 5.25$$

where $a_{ij} = \mu_{i+1} - j - 1$ is the arm of (i,j) in μ. Moreover, from 5.19 $b)_y$ we derive

$$K^y_{ij} = A^y_{ij} + A^y_{i+1,j} + \cdots + A^y_{\mu'_{j+1}-1\,,\,j} \ . \qquad 5.26$$

Proceeding as we did above, the identities in 5.22, 5.25 and 5.26 now yield that we also have

$$C_{\mu/ij} \;=\; \sum_{(i,j)\leq\ s'} t^{l_{\tau_{ij}}(s')}\, q^{a'_{\tau_{ij}}(s')}\, \Xi_{\mu,s'} \ . \qquad 5.27$$

Since on the $C = \tilde H$ conjecture we have (see I.12)

$$C_{\mu/00} \;=\; \partial_{p_1}\, \tilde H_\mu \ ,$$

we get that the special cases $(i,j)=(0,0)$ of 5.24 and 5.27 yield

$$\partial_{p_1}\ \tilde{H}_\mu \ = \ \sum_{s\in\mu} t^{l'_\mu(s)}\ q^{a_\mu(s)}\ \Xi_{\mu,s} \quad , \tag{5.28}$$

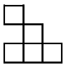
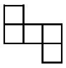
and

$$\partial_{p_1}\ \tilde{H}_\mu \ = \ \sum_{s\in\mu} t^{l_\mu(s)}\ q^{a'_\mu(s)}\ \Xi_{\mu,s} \quad . \tag{5.29}$$

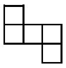
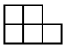
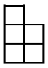
Comparing with 5.1 (4) and 5.3 written for $D=\mu$ and with $w(s,D)$ and $w(s',D')$ respectively given by the weights in 5.4 a) and b) we come to the inescapable conclusion that at least for D the diagram of a partition, these mysterious polynomials $G_{D/s}(x;q,t)$ must be none other than our normalized atom characteristics $\Xi_{\mu,s}$. To be precise, we are thus led to the addition of one further rule to the heuristic apparatus exhibited in 5.1, namely that we must also have

$$(5) \qquad G_{\mu/s} \ = \ \Xi_{\mu,s} \qquad\qquad (\ \forall \quad s\in\mu\ )\ . \tag{5.30}$$

It develops that accepting this hypothesis, we can easily explain a wide variety of formulas that may be derived from the rules in 5.1. This is best seen through a few examples. Let us begin with 5.10 which heretofore could only be obtained through the two intricate paths we illustrated above. Now, we saw at the beginning of the section that we have the equivalence

 $\approx$  .

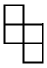
Thus from rule (1) and formula 5.22 for $\mu=(3,2,1)$ and $s=(1,0)$, we obtain the expansion

 $= \dfrac{x_1-u_1}{x_1-x_2}$  $+ \dfrac{x_2-u_1}{x_2-x_1}$  5.31

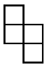
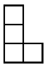
where x_1 and x_2 must be the weights of the two corners of the partition  (which is the shadow of $(1,0)$ in $(3,2,1)$), and u_1 must be the weight of the inner corner. We thus deduce that 5.31 must hold true with

$$x_1=t \quad , \qquad x_2=q \quad , \qquad u_1=1 \ . \tag{5.32}$$

Now we can easily see that making these substitutions in 5.31 immediately yields our formula 5.10. For our next example we take

$D \ =$  .

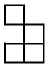
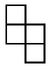
In this case we use 5.30 with $\mu=(2,2,1)$ and $s=(0,0)$, obtaining that we must have

 $= \dfrac{x_1-u_1}{x_1-x_2}$  $+ \dfrac{x_2-u_1}{x_2-x_1}$  . 5.33

Here we must take

$$x_1=t^2 \quad , \qquad x_2=tq \quad , \qquad u_1=t \quad . \tag{5.34}$$

Making these substitutions we see that 5.33 reduces to 5.14. The the fact that the weights in 5.14 are the same as those in 5.10 may be explained from the equivalence

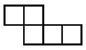
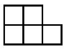
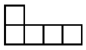
 $\approx$  ,

which shows that we could also use 5.30 with $\mu = (2,2,1)$ and $s = (1,0)$, yielding that we must also have 5.33 with the weights given in 5.32.

Remark 5.1

Incidentally, the reason that the weights in 5.34 yield the same result as those in 5.32 is due to a special instance of Theorem I.3 stated in the introduction. In fact, we should note that we can easily deduce from formula 5.22 itself that the normalized atom characteristics $\Xi_{\mu,s}$ must remain constant within any of the rectangles defined in I.51.

Formula 5.30 can also yields the expansion in 5.18. Indeed if we use it with $\mu = (4,2)$ and $s = (0,0)$ we immediately derive that

 $= \dfrac{x_1 - u_1}{x_1 - x_2}$  $+ \dfrac{x_2 - u_1}{x_2 - x_1}$  5.35

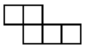
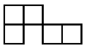
with

$$x_1 = t\,q \quad , \qquad x_2 = q^3 \quad , \qquad u_1 = q \quad ,$$

or with the weights

$$x_1 = t \quad , \qquad x_2 = q^2 \quad , \qquad u_1 = 1 \quad ,$$

because of the equivalence

 $\approx$  .

We should point out that we haven't proved anything here, since a number of the above derivations are based on various yet unproven conjectures. Nevertheless, the variety of identities that may be constructed in this manner should be taken as evidence in support of the conjectures. More importantly, these calculations open up a number of avenues for further investigation. To begin with, it is difficult to believe that we could not find some very natural quotients of subspaces of the modules $\mathbf{M}_\mu$ whose Frobenius characteristics may be identified with the conjectured polynomials $G_D(x;q,t)$ (as we have done for the polynomials A^x_{ij} and A^y_{ij}). Our experience suggests that these subspaces should result from restricting to smaller and smaller Young subgroups of S_n. In this vein, just as the characteristics $\Xi_{\mu,s}$ do extend and simplify the Macdonald (first order) Pieri rules, we would expect that ,using the general polynomials $G_D(x;q,t)$, we should be able to unravel the combinatorics of higher order Pieri rules. From this point of view it appears that we have uncovered what may be the tip of an iceberg of further research. Only time will tell the significance of what may ultimately be found in explaining some of the mysteries that stem from the present developments.

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