Science Fiction
and
Macdonald's Polynomials

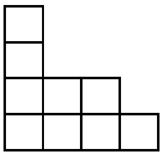
F. Bergeron (†) and A. M. Garsia (††)

Abstract

This work studies the remarkable relationships that hold among certain m-tuples of the Garsia-Haiman modules $\mathbf{M}_\mu$ and corresponding elements of the Macdonald basis. We recall that in [10], $\mathbf{M}_\mu$ is defined for a partition $\mu \vdash n$, as the linear span of derivatives of a certain bihomogeneous polynomial $\Delta_\mu(x,y)$ in the variables $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$. It has been conjectured in [6], [10] that $\mathbf{M}_\mu$ has $n!$ dimensions and that its bigraded Frobenius characteristic is given by the symmetric polynomial $\widetilde{H}_\mu(x;q,t) = \sum_{\lambda \vdash n} S_\lambda(X)\widetilde{K}_{\lambda\mu}(q,t)$ where the $\widetilde{K}_{\lambda\mu}(q,t)$ are related to the Macdonald q,t-Kostka coefficients $K_{\lambda\mu}(q,t)$ by the identity $\widetilde{K}_{\lambda\mu}(q,t) = K_{\lambda\mu}(q,1/t)t^{n(\mu)}$ with $n(\mu)$ the x-degree of $\Delta_\mu(x;y)$. Using this conjectured relation we can translate observed or proved properties of the modules $\mathbf{M}_\mu$ into identities for Macdonald polynomials. Computer data has suggested that as ν varies among the immediate predecessors of a partition μ, the spaces $\mathbf{M}_\nu$ behave like a boolean lattice. The same appears to holds true when ν varies among the immediate successors of μ. Combining this property with a number of observed facts and some "heuristics" we have been led to formulate a number of remarkable conjectures about the Macdonald polynomials. In particular we obtain a representation theoretical interpretation for some of the symmetries that can be found in the computed tables of q,t-Kostka coefficients. The expression "Science Fiction" here refers to a package of "heuristics" that we use to describe relations amongst the modules $\mathbf{M}_\nu$. These heuristics are purely speculative assertions that are used as a convenient guide to the construction of identities relating the corresponding bigraded Frobenius characteristics. Nevertheless computer experimentation reveals that these assertions are "generically" correct. Moreover, the evidence in support of the symmetric function identities we have derived from them are overwhelming. In particular, we show that various independent consequences of our heuristics lead to the same final identities.

Introduction

Throughout this writing μ will be a partition of n (denoted $\mu \vdash n$) and μ' will denote its conjugate. We shall use the French convention here and, given that the parts of μ are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0$, we let the corresponding Ferrer's diagram have μ_i lattice squares in the i^{th} row (counting from the bottom up), hence the diagram of 4311 is:



As in Macdonald [20], for $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0)$, we let

$$n(\mu) := \sum_{i=1}^{k} (i-1)\mu_i, \qquad \text{I.1}$$

(†) With support from NSERC

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We also let the coordinates (i,j) of a cell $c \in \mu$ respectively measure the height of c and the position of c in its row.

We shall also adopt the Macdonald convention of calling the *arm, leg, co-arm* and *co-leg* of a lattice square s the parameters $a(s)$, $l(s)$, $a'(s)$ and $l'(s)$ giving the number of cells of μ that are respectively *strictly* EAST, NORTH, WEST and SOUTH of s in μ. Recall that Macdonald in [20] established the existence of a symmetric function basis $\{P_\mu(x;q,t)\}_\mu$ uniquely characterized by the following conditions

$$\begin{aligned} &\text{a)} \quad P_\lambda = S_\lambda + \sum_{\mu<\lambda} S_\mu \xi_{\mu\lambda}(q,t), \\ &\text{b)} \quad \langle P_\lambda, P_\mu \rangle_{q,t} = 0 \quad \text{for} \quad \lambda \neq \mu, \end{aligned} \tag{I.2}$$

where $\langle , \rangle_{q,t}$ denotes the scalar product of symmetric polynomials defined by setting for the power basis $\{p_\mu\}$

$$\langle p_\mu, p_\lambda \rangle_{q,t} = \begin{cases} z_\mu p_\mu\left[\frac{1-q}{1-t}\right] & \text{if } \mu = \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Here, as customary, z_μ denotes the integer that makes $n!/z_\mu$ the number of permutations with cycle structure μ. We have also used λ-ring notation in I.3.We recall that λ-substitution in a symmetric function is the linear and multiplicative extension of a substitution explicitly defined for the power sums. Where, for any given expression E and any integer $k \geq 1$, we let $p_k[E]$ be the expression obtained from E by replacing all the variables occurring in E by their k^{th}-powers. For example, writing X for $x_1 + x_2 + x_3 + \ldots$, the substitution of $\frac{X}{1-t}$ in a given symmetric function f, henceforth denoted $f[\frac{X}{1-t}]$, corresponds to the linear and multiplicative extention of

$$p_k[X] = x_1^k + x_2^k + x_3^k + \ldots \ \mapsto \ p_k[\tfrac{X}{1-t}] = \frac{1}{1-t^k} \ p_k[X].$$

There are a number of outstanding conjectures concerning the polynomials P_λ (see [21]). We are dealing here with those involving the so called *integral forms* $J_\mu(x;q,t)$ and their associated Macdonald-Kostka coefficients $K_{\lambda\mu}(q,t)$. We shall use the same notation as in [21]. In particular $\{Q_\lambda(x;q,t)\}$ denotes the basis dual to $\{P_\lambda(x;q,t)\}$ with respect to the scalar product $\langle , \rangle_{q,t}$. Clearly, I.2 b) gives

$$Q_\lambda(x;q,t) = d_\lambda(q,t) P_\lambda(x;q,t), \tag{I.3}$$

for a suitable rational function $d_\lambda(q,t)$. However in [21], it is shown that

$$d_\lambda(q,t) = \frac{h_\lambda(q,t)}{h'_\lambda(q,t)} \tag{I.4}$$

with

$$h_\lambda(q,t) := \prod_{s\in\lambda}(1 - q^{a_\lambda(s)} t^{l_\lambda(s)+1}), \qquad h'_\lambda(q,t) := \prod_{s\in\lambda}(1 - q^{a_\lambda(s)+1} t^{l_\lambda(s)}),$$

where s denotes a generic lattice square and $a_\lambda(s)$, $l_\lambda(s)$ respectively denote the *arm* and the *leg* of s in the Ferrers' diagram of λ.

From I.3 and I.4 it follows that we may set, as Macdonald does in [21]:

$$J_\mu(x;q,t) = h_\mu(q,t) P_\mu(x;q,t) = h'_\mu(q,t) Q_\mu(x;q,t)\,. \tag{I.5}$$

The coefficients $K_{\lambda\mu}(q,t)$ can now be defined through an expansion which, in λ-ring notation, may be written as

$$J_\mu(x;q,t) = \sum_\lambda S_\lambda[X(1-t)] K_{\lambda\mu}(q,t). \tag{I.6}$$

Macdonald conjectured that these coefficients are polynomials in q and t with non-negative integer coefficients. There are now a number of proofs of integrality ([12], [13], [16], [17], [23]) but the positivity has still to be demonstrated.

We define

$$H_\mu(x;q,t) := \sum_\lambda S_\lambda K_{\lambda\mu}(q,t) = J_\mu[X/(1-t);q,t],$$

and set

$$\widetilde{H}_\mu(x;q,t) := H_\mu(x;q,1/t)t^{n(\mu)}. \tag{I.7}$$

Note that we also have

$$\widetilde{H}_\mu(x;q,t) := \sum_\lambda S_\lambda \widetilde{K}_{\lambda\mu}(q,t), \tag{I.8}$$

with

$$\widetilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)} K_{\lambda\mu}(q,1/t).$$

We shall set

$$B_\mu(q,t) := \sum_{(i,j)\in\mu} t^{i-1}q^{j-1}, \qquad T_\mu(q,t) := \prod_{(i,j)\in\mu} t^{i-1}q^{j-1} = q^{n(\mu')}t^{n(\mu)}. \tag{I.9}$$

The pairs $(i-1,j-1)$ occurring in the above sum are briefly referred to as the *biexponents* of μ and $B_\mu(q,t)$ itself will be called the *biexponent generator* of μ.

Now let $(p_1,q_1),\ldots,(p_n,q_n)$ denote the set of biexponents arranged in lexicographic order and set

$$\Delta_\mu(x,y) := \Delta_\mu(x_1,\ldots,x_n;y_1,\ldots,y_n) = \det \| x_i^{p_j} y_i^{q_j} \|_{i,j=1..n}. \tag{I.10}$$

For example

$$\Delta_{(3,2,1)} = \det \begin{pmatrix} 1 & x_1 & {x_1}^2 & y_1 & x_1\,y_1 & {y_1}^2 \\ 1 & x_2 & {x_2}^2 & y_2 & x_2\,y_2 & {y_2}^2 \\ 1 & x_3 & {x_3}^2 & y_3 & x_3\,y_3 & {y_3}^2 \\ 1 & x_4 & {x_4}^2 & y_4 & x_4\,y_4 & {y_4}^2 \\ 1 & x_5 & {x_5}^2 & y_5 & x_5\,y_5 & {y_5}^2 \\ 1 & x_6 & {x_6}^2 & y_6 & x_6\,y_6 & {y_6}^2 \end{pmatrix} \tag{I.11}$$

This given we let $\mathbf{M}_\mu$ be the collection of polynomials in the variables $x_1,\ldots,x_n;y_1,\ldots,y_n$ obtained by taking the linear span of all the partial derivatives of Δ_μ. In symbols we may write

$$\mathbf{M}_\mu = \mathcal{L}[\partial_x^p \partial_y^q \Delta_\mu(x;y)] \tag{I.12}$$

where $\partial_x^p = \partial_{x_1}^{p_1}\cdots\partial_{x_n}^{p_n}$, $\partial_y^q = \partial_{y_1}^{q_1}\cdots\partial_{y_n}^{q_n}$.

The natural action of a permutation $\sigma = (\sigma_1,\ldots,\sigma_n)$ on a polynomial $P(x_1,\ldots,x_n;y_1,\ldots,y_n)$ is the so called *diagonal action* which is defined by setting

$$\sigma P(x_1,\ldots,x_n;y_1,\ldots,y_n) := P(x_{\sigma_1},\ldots,x_{\sigma_n};y_{\sigma_1},\ldots,y_{\sigma_n}).$$

Since $\sigma\Delta_\mu = \pm\Delta_\mu$ according to the sign of σ, the space $\mathbf{M}_\mu$ necessarily remains invariant under this action.

Note that, since Δ_μ is bihomogeneous of degree $n(\mu)$ in x and $n(\mu')$ in y, we also have the direct sum decomposition

$$\mathbf{M}_\mu = \bigoplus_{h=0}^{n(\mu)} \bigoplus_{k=0}^{n(\mu')} \mathcal{H}_{h,k}(\mathbf{M}_\mu),$$

where $\mathcal{H}_{h,k}(\mathbf{M}_\mu)$ denotes the subspace of $\mathbf{M}_\mu$ spanned by its bihomogeneous elements of degree h in x and degree k in y. Since the diagonal action clearly preserves bidegree, each of the subspaces $\mathcal{H}_{h,k}(\mathbf{M}_\mu)$ is also S_n-invariant. Thus we see that $\mathbf{M}_\mu$ has the structure of a bigraded module. The generating function of the characters of its bihomogeneous components, which we shall refer to as the *bigraded character* of $\mathbf{M}_\mu$, may be written in the form

$$\chi^\mu(q,t) = \sum_{h=0}^{n(\mu)} \sum_{k=0}^{n(\mu')} t^h q^k \operatorname{char} \mathcal{H}_{h,k}(\mathbf{M}_\mu).$$

The bivariate Frobenius characteristic of $\mathbf{M}_\mu$ is

$$\mathcal{F}(\mathbf{M}_\mu) := \frac{1}{n!} \sum_{\sigma \in S_n} \chi^\mu(\sigma; q,t) p_{\lambda(\sigma)}(x). \tag{I.13}$$

where $\chi^\mu(\sigma;q,t)$ denotes the value of $\chi^\mu(q,t)$ at σ and $p_{\lambda(\sigma)}(x)$ is the power symmetric function indexed by the shape of σ. It will be convenient to set

$$\mathcal{F}(\mathbf{M}_\mu) \;=\; C_\mu(x;q,t) \;=\; \sum_{\lambda \vdash n} S_\lambda(x) C_{\lambda\mu}(q,t) \;. \tag{I.14}$$

Where, $C_{\lambda\mu}(q,t)$ is the polynomial which gives the occurrences of the irreducible character χ^λ in the various bihomogeneus components of $\mathbf{M}_\mu$. More precisely the coefficient of $t^h q^k$ in $C_{\lambda\mu}(q,t)$ gives the multiplicity of χ^λ in the submodule $\mathcal{H}_{h,k}(\mathbf{M}_\mu)$. It has been conjectured in [6] and supported in [8], [10] and [22] (by verification of special cases and several representation theoretical considerations), that

$$C_\mu(x;q,t) \;=\; \widetilde{H}_\mu(x;q,t) \tag{I.15}$$

This forces the equality

$$C_{\lambda\mu}(q,t) = \widetilde{K}_{\lambda\mu}(q,t) \tag{I.16}$$

which in particular implies the Macdonald conjecture that the $K_{\lambda\mu}(q,t)$ are polynomials with non-negative integer coefficients. We shall briefly refer to I.15 as the $C = \widetilde{H}$ conjecture.

Macdonald in [21] derives a number of properties of the $K_{\lambda\mu}(q,t)$, in particular he shows that for any partition μ

$$K_{\lambda\mu}(1,1) = f_\lambda,$$

where as customary f_λ denotes the number of standard tableaux of shape λ. Thus the validity of the $C = \widetilde{H}$ conjecture requires that $\mathbf{M}_\mu$ should yield a bigraded version of the left regular representation. Then a fortiori we should have

$$\dim \mathbf{M}_\mu = n! \tag{I.17}$$

This has come to be referred to as the *n-factorial Conjecture* (see [10]). In the nearly six years since this deceptively simple conjecture was formulated, an overwhelming amount of theoretical and experimental evidence in its support has been gathered. But perhaps the most surprising development in this context is a recent work of M. Haiman [15] where it is shown that the n-factorial conjecture for a given μ is all that is needed to establish the identity $C_\mu(x;q,t) = \widetilde{H}_\mu(x;q,t)$ for that same μ. It is easy to see that, for $\mu = 1^n$ and $\mu = (n)$, Δ_μ reduces to the Vandermonde determinant in x and y respectively. In these cases (I.15) is a classical result (see [7]).

Experimental evidence has revealed that the modules $\mathbf{M}_\nu$ corresponding to partitions ν lying immediately below (resp. above) a given partition μ have remarkable intersection properties. Our purpose here is to explore various results and conjectures that can be obtained by combining these findings with the $C = \widetilde{H}$ conjecture.

Let us begin by observing that a space $\mathbf{M}$ spanned by the partial derivatives of a single bihomogeneous polynomial $\Delta(x,y) = \Delta(x_1,\ldots,x_n;y_1,\ldots,y_n)$ of bidegree (a,b), in symbols

$$\mathbf{M} = \mathcal{L}[\partial_x^p \partial_y^q \, \Delta(x,y)],$$

has a bidegree complementing automorphism, called **flip**, defined by setting, for every $P \in \mathbf{M}$,

$$\mathbf{flip}\, P := P(\partial_x, \partial_y)\, \Delta(x, y) = P(\partial)\, \Delta(x, y),$$

where here and after, if P is a polynomial in $x_1, \ldots, x_n; y_1, \ldots, y_n$, $P(\partial)$ is to represent the differential operator obtained by replacing in P, x_i by ∂_{x_i} and y_i by ∂_{y_i}.

Note that, if Δ is alternating then $\mathbf{M}$ is invariant under the diagonal action of S_n, and moreover the effect of flip will be to sign-twist each of the irreducible submodules. Hence, if $\Phi(x; q, t)$ is the bivariate Frobenius characteristic of $\mathbf{M}$, then we necessarily have that

$$\Phi(x; q, t) = t^a q^b \omega\, \Phi(x; q^{-1}, t^{-1}). \tag{I.18}$$

Here, as usual, ω denotes the involution that sends S_λ to $S_{\lambda'}$. More generally, for any bigraded submodule $\mathbf{M_1}$ of $\mathbf{M}$, the subspace

$$\mathbf{flip}\, \mathbf{M_1} := \mathcal{L}[\mathbf{flip}\, P \mid P \in \mathbf{M_1}], \tag{I.19}$$

is also a bigraded submodule of $\mathbf{M}$. Moreover, if $\Psi(x; q, t)$ is the bivariate Frobenius characteristic of $\mathbf{M}$, then the bivariate Frobenius characteristic of $\mathbf{flip}\, \mathbf{M_1}$ is given by the formula

$$\mathcal{F}(\mathbf{flip}\, \mathbf{M_1}) = t^a q^b \ \downarrow \Psi(x; q, t), \tag{I.20}$$

where we set

$$\downarrow \Psi(x; q, t) := \omega \Psi(x; q^{-1}, t^{-1}). \tag{I.21}$$

Here and after, we shall assume that our partition μ has m corners, and that the set of predecessors of μ is

$$\pi(\mu) = \{\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(m)}\},$$

with $\alpha^{(i)}$ the partition obtained by removing the i^{th}-corner (as we encounter it from left to right). Our computer experimentations indicate that the space $\mathbf{V}_\mu$, sum of the $\mathbf{M}_{\alpha^{(i)}}$'s, in symbols

$$\mathbf{V}_\mu := \bigvee_{i=1}^{m} \mathbf{M}_{\alpha^{(i)}}, \tag{I.22}$$

has a basis $\mathcal{B} = \mathcal{B}(\mu)$, of bihomogeneous polynomials, with the properties (i)–(v) given below.

(i) *For each $1 \leq i \leq m$, we have a subset $\mathcal{B}_i \subset \mathcal{B}$ such that*

$$\mathbf{M}_{\alpha^{(i)}} = \mathcal{L}[\mathcal{B}_i], \tag{I.23}$$

For a word $\epsilon = \epsilon_1 \epsilon_2 \cdots \epsilon_m$, with $\epsilon_i = 0, 1$, set

$$\mathcal{B}^\epsilon := \bigcap_{i=1}^{m} \mathcal{B}_i^{\epsilon_i},$$

where, for a given subset $S \subset \mathcal{B}$, we let

$$S^1 := S\,, \qquad S^0 := \mathcal{B} \setminus S\,.$$

This given,

(ii) *For each ϵ the space*

$$\mathbf{M}_\mu^\epsilon := \mathcal{L}[\mathcal{B}_\mu^\epsilon], \tag{I.24}$$

is S_n-invariant, and therefore has the structure of a bigraded S_n-module, whose bivariate Frobenius characteristic we denote $\Phi^\epsilon = \Phi^\epsilon_\mu$, in symbols

$$\Phi^\epsilon_\mu(x;q,t) := \mathcal{F}(\mathbf{M}^\epsilon_\mu). \qquad \text{I.25}$$

Note that if we set

$$\mathbf{M}^1_{\alpha^{(i)}} := \mathcal{L}[\mathcal{B}_i] = \mathbf{M}_{\alpha^{(i)}}\,, \qquad \mathbf{M}^0_{\alpha^{(i)}} := \mathcal{L}[\mathcal{B}\setminus\mathcal{B}_i]\,,$$

then we necessarily have

$$\mathbf{M}^\epsilon_\mu = \bigcap_{i=1}^m \mathbf{M}^{\epsilon_i}_{\alpha^{(i)}}. \qquad \text{I.26}$$

Let now, $\mathbf{flip}_i$ denote the flip operation within the space $\mathbf{M}_{\alpha^{(i)}}$, and let $\downarrow_i$ denote its effect on the bivariate Frobenius characteristics of the bigraded submodules of $\mathbf{M}_{\alpha^{(i)}}$. According to (I.20) and (I.21), if $\Psi(x;q,t)$ is such a characteristic, then we have

$$\downarrow_i \Psi(x;q,t) = T_i \downarrow \Psi(x;q,t), \qquad \text{I.27}$$

where hereafter we simply use T_i for $T_{\alpha_i}(q,t)$.

Let τ_i denote the operator on words in $0,1$ which complements all but the i^{th} letter. That is, setting $\tilde\epsilon_j := 1-\epsilon_j$, we have

$$\tau_i\big(\epsilon_1\cdots\epsilon_{i-1}\epsilon_i\epsilon_{i+1}\cdots\epsilon_m\big) := \tilde\epsilon_1\cdots\tilde\epsilon_{i-1}\epsilon_i\tilde\epsilon_{i+1}\cdots\tilde\epsilon_m. \qquad \text{I.28}$$

This given, the most remarkable property of our flip operations can be stated as follows

(iii) *For all words $\epsilon = \epsilon_1\epsilon_2\cdots\epsilon_m$ such that $\epsilon_i = 1$, we have*

$$(a) \quad \mathbf{flip}_i\,\mathbf{M}^\epsilon_\mu \cong \mathbf{M}^{\tau_i\,\epsilon}_\mu. \qquad \text{(b)} \qquad \mathbf{M}^{\tau_i\,\epsilon}_\mu \cap \mathbf{flip}_i\,\mathbf{M}^\epsilon_\mu \;=\; \{0\} \qquad \text{I.29}$$

Note that I.29 (a) can be written more explicitly as

$$\begin{aligned}\mathbf{flip}_i\big(\mathbf{M}^{\epsilon_1}_{\alpha^{(1)}}\cap\cdots\cap\mathbf{M}^{\epsilon_{i-1}}_{\alpha^{(1)}}\cap\mathbf{M}_{\alpha^{(i)}}\cap\mathbf{M}^{\epsilon_{i+1}}_{\alpha^{(1)}}\cap\cdots\cap\mathbf{M}^{\epsilon_m}_{\alpha_m}\big) \cong\\ \cong\; \mathbf{M}^{\tilde\epsilon_1}_{\alpha^{(1)}}\cap\cdots\cap\mathbf{M}^{\tilde\epsilon_{i-1}}_{\alpha^{(1)}}\cap\mathbf{M}_{\alpha^{(i)}}\cap\mathbf{M}^{\tilde\epsilon_{i+1}}_{\alpha^{(1)}}\cap\cdots\cap\mathbf{M}^{\tilde\epsilon_m}_{\alpha_m}\;.\end{aligned}$$

Furthermore from (I.27) we derive that the bivariate Frobenius characteristics $\Phi^\epsilon_\mu(x;q,t)$ are related by the following identity

$$\downarrow_i \Phi^\epsilon_\mu = \Phi^{\tau_i\,\epsilon}_\mu. \qquad \text{I.30}$$

There is a natural scalar product $\langle\ ,\ \rangle$ in the space of polynomials in $x_1,x_2,\ldots,x_n;y_1,\ldots,y_n$ which is defined by setting

$$\langle P\ ,\ Q\rangle \;=\; P(\partial_x,\partial_y)Q(x,y)\,|_{x=y=0}\;. \qquad \text{I.31}$$

It can be shown that the orthogonal complement of the space $\mathbf{M}_\mu$ consists of the ideal

$$\mathcal{J}_\mu \;=\; \{\,P(x,y)\;:\;P(\partial_x,\partial_y)\,\Delta_\mu(x,y) = 0\,\}\;. \qquad \text{I.32}$$

Our examples suggest thatwe should be able to construct the basis $\mathcal{B}_\mu$ so that

(iv) *If ϵ and η are such that $\epsilon_i = 1$ and $\eta_i = 0$ then*

$$\mathbf{M}^\eta_\mu \;\subseteq\; \big(\,\mathbf{M}^\epsilon_\mu\,\big)^\perp \qquad \text{I.33}$$

where the symbol "$\perp$" is to represent the operation of taking orthogonal complements with respect to the scalar product in I.31.

Finally, one further property that is suggested by our computer experimentations is that the intersection of any k of the modules $\mathbf{M}_{\alpha^{(i)}}$ always turns out to have dimension $1/k^{\text{th}}$ of the common dimension of the $\mathbf{M}_{\alpha^{(i)}}$'s. More precisely:

(v) *For any $\mu \vdash n+1$, with m parts, and for any k-subset S of $\{1, 2, \ldots, m\}$*

$$\dim \bigcap_{i \in S} \mathbf{M}_{\alpha^{(i)}} = \frac{n!}{k}. \qquad \text{I.34}$$

This given, here and after we shall work under the following basic

Heuristic: *For every partition μ, a basis $\mathcal{B}_\mu$ exists with properties* (i)–(v) above.

We shall refer to this assertion as the "Science fiction heuristic" or briefly "SF". We do not venture calling this a *conjecture* since it is only generically true and was constructed from data obtained from experimentations with relatively small partitions.Nevertheless we shall see that, by combining SF with the $C = \widetilde{H}$ conjecture, we can predict several surprising identities for the polynomials $\widetilde{H}_\mu(x; q, t)$ and the coefficients $\widetilde{K}_{\lambda\mu}(q, t)$. Remarkably, these identities can actually be established outright in some cases and in others they can be verified numerically for significantly large examples. We should note that tables of the $\widetilde{K}_{\lambda\mu}(q, t)$ are now available up to $n = 12$.

For example, note that the definition (I.25) gives that Φ^ϵ_μ is a bivariate Frobenius characteristic. Therefore it has a Schur function expansion

$$\Phi^\epsilon_\mu(x; q, t) = \sum_\lambda S_\lambda \, \varphi^\epsilon_{\lambda\mu}(q, t), \qquad \text{I.35}$$

with the $\varphi_{\lambda\mu}$ polynomials in q, t with positive integer coefficients, a property of symmetric polynomials which here and after will by briefly referred to as "Schur positivity".

It develops further that the symmetric functions Φ^ϵ_μ depend on ϵ in a very simple manner. In fact, using our heuristic we will show that, for each $\mu \vdash n$ with m corners, we can construct m Schur positive symmetric functions

$$\Phi^{(1)}(x; q, t)\ ,\ \Phi^{(2)}(x; q, t)\ ,\ \ldots\ ,\ \Phi^{(m)}(x; q, t) \qquad \text{I.36}$$

yielding (for $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n = k$)

$$\Phi^\epsilon_\mu \;=\; \frac{\Phi^{(k)}(x; q, t)}{\prod_{\epsilon_i = 0} T_i} \qquad \text{I.37}$$

One of the most remarkable consequences of I.37 is that, combined with the $C = \widetilde{H}$ conjecture, it yields that the Macdonald polynomial $\widetilde{H}_{\alpha^{(i)}}$ has the expansion

$$\widetilde{H}_{\alpha^{(i)}}(x; q, t) \;=\; \sum_{k=1}^m \Phi^{(k)} \; e_{m-k}\left[\frac{1}{T_1} + \frac{1}{T_2} + \cdots + \frac{1}{T_m} \;-\; \frac{1}{T_i}\right]. \qquad \text{I.38}$$

Equating coefficients of S_λ we can then further derive that

$$\widetilde{K}_{\lambda, \alpha^{(i)}}(q, t) \;=\; \sum_{k=1}^m \Phi^{(k)}_{\lambda, k}(q, t) \; e_{m-k}\left[\frac{1}{T_1} + \frac{1}{T_2} + \cdots + \frac{1}{T_m} \;-\; \frac{1}{T_i}\right]. \qquad \text{I.39}$$

where, (assuming SF), the $\Phi_{\lambda,k}(q, t)$ must be polynomials with positive integer coefficients. In this manner, we obtain a representation theoretical explanation for many properties of the Kostka-Macdonald tables which have been previously observed by examination of data. In particular, this brings to light that, when λ remains fixed, the dependence of $\widetilde{K}_{\lambda\mu}(q, t)$ on μ is governed by a yet finer mechanism than the one that was noticed by L. Butler (†).

(†) Personal communication.

It is appropriate to review Butler's observations here. Let μ and ν be partitions of the same number. Following A. Young it is customary to say that μ is obtained from ν by a "Raising Operator" if μ can be obtained by lifting a number of cells of ν from lower to upper rows (††). A raising operator will be called minimal if it lifts a single cell up a single row or to the right a single column. It can be shown (see [17]) that the dominance partial order is the transitive closure of the relation $\mu = R\nu$ with R a minimal raising operator. L. Butler [1] observed that the Macdonald coefficients $\widetilde{K}_{\lambda\mu}(q,t)$ change in a remarkably simple way when μ is obtained from ν by a minimal raising operator. Her observation which was originally made from tables computed by hand by Macdonald [21] (up to $n \leq 6$) is now confirmed by more extensive tables obtained by computer (up to $n \leq 12$). Butler noticed that for any such pair of partitions μ, ν and any λ there are two polynomials Φ_λ and Θ_λ with non-negative integer coefficients such that

$$\begin{aligned} \widetilde{K}_{\lambda\mu}(q,t) &= \Phi_\lambda + \Theta_\lambda \\ \widetilde{K}_{\lambda\nu}(q,t) &= \Phi_\lambda + \Theta_\lambda\, T_\nu/T_\mu, \end{aligned} \qquad \text{I.40}$$

This has come to be referred to as the *Butler conjecture*. To explore the full implications of this conjecture let us set for a moment $\psi_\lambda = \Theta_\lambda/T_\mu$ and write these two identities in the more symmetric form

$$\begin{aligned} \widetilde{K}_{\lambda\mu}(q,t) &= \Phi_\lambda + \Psi_\lambda\, T_\mu \\ \widetilde{K}_{\lambda\nu}(q,t) &= \Phi_\lambda + \Psi_\lambda\, T_\nu, \end{aligned}$$

Equivalently we may write

$$\Phi_\lambda = \frac{T_\nu\, \widetilde{K}_{\lambda\mu} - T_\mu\, \widetilde{K}_{\lambda\nu}}{T_\nu - T_\mu} \quad , \qquad \Psi_\lambda = \frac{\widetilde{K}_{\lambda\mu} - \widetilde{K}_{\lambda\nu}}{T_\mu - T_\nu} \ .$$

Now it may be shown (see [7]) that for any $\lambda, \mu \vdash n$ we have

$$\widetilde{K}_{\lambda\mu}(1/q, 1/t)\ T_\mu = \widetilde{K}_{\lambda'\mu}(q,t) \ , \qquad \widetilde{K}_{\lambda\nu}(1/q,1/t)\ T_\nu = \widetilde{K}_{\lambda'\nu}(q,t) \ .$$

These identities imply that Θ_λ is superfluous. In fact, combinining them with I.40 we can easily derive that

$$\Psi_\lambda(q,t) = \Phi_{\lambda'}(1/q, 1/t) \ .$$

Setting $\Phi(x;q,t) = \sum_\lambda \Phi_\lambda(q,t)\, S_\lambda(x)$ the identity in I.8 written for μ and ν yields that

$$\widetilde{H}_\mu(x;q,t) = \Phi(x;q,t) + T_\mu \downarrow \Phi(x;q,t) \ , \qquad \widetilde{H}_\nu(x;q,t) = \Phi(x;q,t) + T_\nu \downarrow \Phi(x;q,t) \ . \qquad \text{I.41}$$

Our findings in the case of the pair $\mu = 31$, $\nu = 22$ (which is treated in the next section) reveals that the phenomenon observed by Butler should be the result of a beautiful mechanism which we venture to state as follows.

Conjecture 1.1 *If $\mu = R\nu$ with R a minimal raising operator then the identities in I.41 hold true with $\Phi(x;q,t)$ the bigraded Frobenius characteristic of $\mathbf{M}_\mu \wedge \mathbf{M}_\nu$*

On the validity of our $C = \widetilde{H}$ conjecture, this property of the Macdonald polynomials would be an immediate consequence of

(††) This definition requires that diagrams of partitions be drawn by the english convention

Conjecture 1.2 *If $\mu = R\nu$ with R a minimal raising operator then*

$$\begin{aligned} \mathbf{M}_\mu &= \mathbf{M}_\mu \wedge \mathbf{M}_\nu \oplus \mathbf{flip}_\mu \mathbf{M}_\mu \wedge \mathbf{M}_\nu \ , \\ \mathbf{M}_\nu &= \mathbf{M}_\mu \wedge \mathbf{M}_\nu \oplus \mathbf{flip}_\nu \mathbf{M}_\nu \wedge \mathbf{M}_\nu \ . \end{aligned} \qquad \text{I.42}$$

It not difficult to see that both these conjectures may be derived from our heuristic even in the more general case that μ and ν are any two predecessors of the same partition.

Finally, we should mention that in recent joint work M. Haiman and C. Chang have been able to give an Algebraic Geometrical setting to some of the identities we derive here. It develops that in their setting, some of the ingredients we use here in a purely heuristic way take a remarkable natural place within the theory of Hilbert schemes. This permitted them to establish the validity of our assumption for any two part partition and to construct a general mechanism for proving its validity in full generality.

The contents of this paper are divided into four sections. In the first section we work out some examples in full detail. We do this to show that our SF heuristic stems right out of a very natural approach to the proof the n-factorial conjecture. In section 2, we work out the case $\mu = 321$ since it is the first case which clearly displays all the facets of the SF heuristic.

In section 3, using $C = \widetilde{H}$, we derive formulas for all the characteristics Φ_μ^ϵ in terms of the polynomials $\widetilde{H}_\mu(x; q, t)$ and derive from them some additional positivity properties of the Macdonald basis. We also show in section 2 that (I.15) combined with (I.34) yields precise dimension counts for all the submodules $\mathbf{M}_\mu^\epsilon$.

In section 4 we present applications. In particular we derive there new and refined versions of Pieri formulas for the polynomials $\widetilde{H}_\mu(x; q, t)$. The fact that these formulas are completely consistent with those originally obtained by Macdonald should provide support for the validity of at least some weakened version of the SF heuristic.

1. A recursive approach to the $n!$-conjecture

There is an elementary approach to proving the $n!$ conjecture which can be successfully used in various special cases. To get accross the basic idea we shall systematically work out in this manner the proof of the $n!$-conjecture for all $n \leq 5$. Of course, in doing this we shall have to take for granted some of the results obtained in previous work. However, all the auxiliary material we need here can be found in [7], [9] or [10].

Our point of departure is a result (proved in [10]) that for any partition μ we have

$$\dim \mathbf{M}_\mu \ \leq \ n! \qquad 1.1$$

This means that to establish the $n!$-factorial conjecture for a given μ we need only exhibit $n!$ independent elements in $\mathbf{M}_\mu$.

Note next that from the definition I.10 we immediately derive that

$$\Delta_{\mu'}(x, y) \ = \ \Delta_\mu(y, x)$$

In particular the Frobenius characteristics $C_\mu(x; q, t)$ are related by the identity

$$C_{\mu'}(x; q, t) \ = \ C_\mu(x; t, q) \ .$$

This is consistent with the $C = \widetilde{H}$-conjecture since the duality result of Macdonald implies that

$$\widetilde{H}_{\mu'}(x; q, t) \ = \ \widetilde{H}_\mu(x; t, q) \ .$$

Thus we need establish the $n!$-conjecture only for one member of each pair of conjugate partitions.

Let us recall that the portion of 0 y-degree in any of our modules $\mathbf{M}_\mu$ has dimension $\binom{n}{\mu}$. This may be easily derived from the basic result in [11] and the fact that this bihomogeneous subspace of $\mathbf{M}_\mu$ is simply the linear span of derivatives of the Garnir elements of shape μ. Details of this derivation can be found in [5]. Thus in the particular case that $\mu = 21^{n-2}$ this dimension is $n!/2$. Since **flip** operation in any $\mathbf{M}_\mu$ yields a sign-twisted S_n-module isomorphism of the 0 y-degree portion into the $n(\mu')$ y-degree portion, we see that for all these partitions (and their conjugates) the $n!$-conjecture must necessarily hold true. Moreover the case of $\mu = 1^n$ (single columns) or $\mu = (n)$ (single rows) is classical since Δ_μ then reduces to the Vandermonde determinant in the $x's$ and the $y's$ respectively. In particular, for $n = 4$ we are only left with $\mu = 22$. We shall start by dealing with this case.

Expanding with respect to the last column we get

$$\Delta_{22} \;=\; \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \end{pmatrix} \;=\; \Phi_{00} + x_4\,\Phi_{10} + y_4\,\Phi_{01} + x_4y_4\Phi_{11}\ . \tag{1.2}$$

So we may write

$$\partial_{x_4}\Delta_{22} = \Phi_{10} + y_4\Phi_{11}\ ,\quad \partial_{y_4}\Delta_{22} = \Phi_{01} + x_4\Phi_{11}\ ,\quad \partial_{x_4}\partial_{y_4}\Delta_{22} = \Phi_{11}\ . \tag{1.3}$$

Note that

$$\Phi_{11} \;=\; \Delta_{21} \;=\; \det\begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \tag{1.4}$$

Recall that if P is a polynomial in $x_1, .., x_n; y_1, .., y_n$, then $P(\partial) = P(\partial_{x_1}, .., \partial_{x_n}; \partial_{y_1}, .., \partial_{y_n})$. This given, let $\mathcal{B}$ be a collection of monomials in $x_1, x_2, x_3; y_1, y_2, y_3$ chosen so that $\{b(\partial)\Delta_{21}\}_{b\in\mathcal{B}}$ is a basis for $\mathbf{M}_{21}$ (†). We claim that the polynomials in the union

$$\mathcal{B}_{22} = \{b(\partial)\Delta_{22}\}_{b\in\mathcal{B}} + \{b(\partial)\partial_{x_4}\Delta_{22}\}_{b\in\mathcal{B}} + \{b(\partial)\partial_{y_4}\Delta_{22}\}_{b\in\mathcal{B}} + \{b(\partial)\partial_{x_4}\partial_{y_4}\Delta_{22}\}_{b\in\mathcal{B}} \tag{1.5}$$

are linearly independent. Note that since from 1.1 we get that $\dim \mathbf{M}_{22} \leq 24$ we can then conclude that $\mathcal{B}_{22}$ is a basis for $\mathbf{M}_{22}$ and that the conjecture is true for $\mu = 22$. To this end, let there be $b_{00}, b_{10}, b_{01}, b_{11}$ in the linear span of $\mathcal{B}$ giving

$$b_{00}(\partial)\Delta_{22} \;+\; b_{10}(\partial)\partial_{x_4}\Delta_{22} \;+\; b_{01}(\partial)\partial_{y_4}\Delta_{22} \;+\; b_{11}(\partial)\partial_{x_4}\partial_{y_4}\Delta_{22} \;=\; 0\ .$$

Using 1.2, 1.3 and 1.4 and extracting the coefficients of $1, x_4, y_4, x_4y_4$ we derive that we must have the simultaneous equations

$$\begin{array}{ccccccccc} b_{00}(\partial)\Phi_{00} & + & b_{10}(\partial)\Phi_{10} & + & b_{01}(\partial)\Phi_{01} & + & b_{11}(\partial)\Delta_{21} & = & 0 \\ & & b_{00}(\partial)\Phi_{10} & & & + & b_{01}(\partial)\Delta_{21} & = & 0 \\ & & & & b_{00}(\partial)\Phi_{01} & + & b_{10}(\partial)\Delta_{21} & = & 0 \\ & & & & & & b_{00}(\partial)\Delta_{21} & = & 0 \end{array}$$

Note now that if $b_{00} = \sum_{b\in\mathcal{B}} c_b\, b$ then the last equation may be written as

$$\sum_{b\in\mathcal{B}} c_b\; b(\partial)\Delta_{21} \;=\; 0\ ,$$

but this contraddicts the choice of $\mathcal{B}$ unless the coefficients c_b are all equal to zero. Substituting $b_{00} = 0$ in the third equation reduces it to $b_{10}(\partial)\Delta_{21} = 0$ which again implies that the coefficients of b_{10} must all vanish as well. Substituting $b_{00} = 0$ in the second equation yields the same for the coefficients of b_{01}. Finally, setting $b_{00} = b_{10} = b_{01} = 0$ in the first equation forces the vanishing of the coefficients of b_{11}. This proves that $\mathcal{B}_{22}$ is an independent set as asserted.

(†) We may take $\mathcal{B} = \{1, x_1, x_2, y_1, y_2, x_1y_2\}$

We should note that our argument here establishes a bit more than the validity of the $n!$ conjecture for $\mu = 22$. Let us recall that if Θ is the Frobenius image of an S_n character χ then the partial derivative [(†)] $\partial_{p_1}\Theta$ yields the Frobenius image of the restriction of χ to S_{n-1}. In particular, the polynomial

$$G_\mu(q,t) = \partial_{p_1}^n\, C_\mu(x;q,t) \;=\; \sum_{\lambda\vdash n} f_\lambda C_{\lambda\mu}(q,t)$$

must give the bigraded Hilbert series of $\mathcal{B}_\mu$. Of course under the $C = \widetilde{H}$ conjecture this Hilbert series is given by the polynomial

$$F_\mu(q,t) \;=\; \partial_{p_1}^n\, \widetilde{H}_\mu(x;q,t) \;=\; \sum_{\lambda\vdash n} f_\lambda \widetilde{K}_{\lambda\mu}(q,t)\ . \qquad 1.6$$

We should also keep in mind (see [7]) that the "duality" result for Macdonald polynomials in our notation becomes

$$\omega \widetilde{H}_\mu(x;1/q,1/t)t^{n(\mu)}q^{n(\mu')} \;=\; \widetilde{H}_\mu(x;q,t) \qquad 1.7$$

where ω is the involution which sends S_λ into $S_{\lambda'}$. Thus 1.7 implies that we must also have

$$F_\mu(1/q,1/t)t^{n(\mu)}q^{n(\mu')} \;=\; F_\mu(q,t)\ . \qquad 1.8$$

The analogous identities

$$\omega C_\mu(x;1/q,1/t)t^{n(\mu)}q^{n(\mu')} \;=\; C_\mu(x;q,t) \qquad 1.9$$

$$G_\mu(1/q,1/t)t^{n(\mu)}q^{n(\mu')} \;=\; G_\mu(q,t)\ . \qquad 1.10$$

follow from I.18.

It is easy to see that if $\beta(q,t)$ gives the bidegree distribution of the monomials in $\mathcal{B}$ then $qt\beta(1/q,1/t)$ must give the Hilbert series of $\mathbf{M}_{21}$. Thus from 1.10 we get that $\beta = G_{21}(q,t)$. From our construction 1.4 of the basis $\mathcal{B}_{22}$ we can then immediately derive that the Hilbert series of $\mathbf{M}_{22}$ must be given by formula

$$G_{22}(q,t) = \beta(1/q,1/t)(1+1/t+1/q+1/tq)q^2t^2 = (1+t+q+tq)G_{21}(q,t)\ .$$

A slightly more refined argument which takes account of the action of S_3 on the basis $\mathcal{B}_{22}$, yields that we must also have

$$\partial_{p_1} C_{22}(x;q,t) \;=\; (1+t+q+tq)C_{21}(x;q,t)\ . \qquad 1.11$$

It is not difficult to see that the argument we have given here can be generalized to the case of arbitrary rectangular partitions $\mu = r^s$ and obtain that

Theorem 1.1 *If the $n!$ conjecture holds for the partition $r-1, r^{s-1}$ then it holds for r^s, Moreover,*

$$\partial_{p_1} C_{r^s}(x;q,t) \;=\; B_{r^s}(q,t)\, C_{r-1,r^{s-1}}(x;q,t) \qquad 1.12$$

$$G_{r^s}(q,t) \;=\; B_{r^s}(q,t)\, G_{r-1,r^{s-1}}(q,t) \qquad 1.13$$

Continuing with our examples we see that for $n = 5$ we only need to deal with the partitions 32 and 311. We start with 32. Expanding again with respect to the last column we get

$$\Delta_{32} \;=\; \det\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_5y_5 \end{pmatrix} \;=\; \Phi_{00} + y_5\,\Phi_{01} + y_5^2\,\Phi_{02} + x_5\Phi_{10} + x_5y_5\Phi_{11}\ . \qquad 1.14$$

(†) Here p_1 denotes the first power symmetric polynomial and the symbol $\partial_{p_1}\Theta$ is to mean the partial derivative of Θ as a polynomial in the power symmetric functions.

So we may write

$$
\begin{array}{rcrcrcrcrcr}
\Delta_{32} & = & b_{00}(\partial)\Phi_{00} & + & y_5\Phi_{01} & + & y_5^2\Delta_{22} & + & x_5\Phi_{10} & + & x_5y_5\,\Delta_{211} \\
\partial_{y_5}\Delta_{32} & = & & & \Phi_{01} & + & 2y_5\Delta_{22} & & & + & x_5\Delta_{211} \\
\partial_{y_5^2}\Delta_{32} & = & & & & & 2\Delta_{22} & & & & \\
\partial_{x_5}\Delta_{32} & = & & & & & & & \Phi_{10} & + & y_5\Delta_{211} \\
\partial_{x_5}\partial_{y_5}\Delta_{32} & = & & & & & & & & & \Delta_{211}
\end{array}
\tag{1.15}
$$

Proceeding as we did for $\mu = 22$ we are to construct 5 collections $\mathcal{B}_{00}, \mathcal{B}_{01}, \mathcal{B}_{02}, \mathcal{B}_{10}, \mathcal{B}_{11}$ of polynomials in $x_1, x_2, x_3, x_4; y_1, y_2, y_3, y_4$ such that the polynomials in the union

$$
\begin{aligned}
\mathcal{B}_{32} = \left\{b(\partial)\Delta_{32}\right\}_{b\in\mathcal{B}_{00}} + \left\{b(\partial)\partial_{y_5}\Delta_{32}\right\}_{b\in\mathcal{B}_{01}} + \left\{b(\partial)\partial_{y_5}^2\Delta_{32}\right\}_{b\in\mathcal{B}_{02}} + \\
+ \left\{b(\partial)\partial_{x_5}\Delta_{32}\right\}_{b\in\mathcal{B}_{10}} + \left\{b(\partial)\partial_{x_5}\partial_{y_5}\Delta_{32}\right\}_{b\in\mathcal{B}_{11}}
\end{aligned}
$$

are independent. If we succeed in choosing them so that $\mathcal{B}_{32}$ has altogether 120 elements we will have proved the $n!$ conjecture for $\mu = 32$. In this case we shall not venture a guess and determine the $\mathcal{B}_{ij}$ from the equations that would have to hold if there was a dependence between the elements of $\mathcal{B}_{32}$. So let b_{ij} be in the linear span of $\mathcal{B}_{ij}$ and assume that we have

$$
b_{00}(\partial)\Delta_{32} + b_{01}(\partial)\partial_{y_5}\Delta_{32} + b_{02}(\partial)\partial_{y_5}^2\Delta_{32} + b_{10}(\partial)\partial_{x_5}\Delta_{32} + b_{11}(\partial)\partial_{x_5}\partial_{y_5}\Delta_{32} \;=\; 0
$$

Using the expansions in 1.15 and equating to zero the coefficients of $1, y_5, y_5^2, x_5, x_5y_5$ we get the system of equations

$$
\begin{array}{rcrcrcrcrl}
b_{00}(\partial)\Phi_{00} & + & b_{01}(\partial)\Phi_{01} & + & 2b_{02}(\partial)\Delta_{22} & + & b_{10}(\partial)\Phi_{10} & + & b_{11}(\partial)\Delta_{31} & = 0 \\
 & & b_{00}(\partial)\Phi_{01} & + & 2b_{01}(\partial)\Delta_{22} & & & + & b_{10}(\partial)\Delta_{31} & = 0 \\
 & & & & b_{00}(\partial)\Delta_{22} & & & & & = 0 \\
 & & & & & & b_{00}(\partial)\Phi_{10} & + & b_{01}(\partial)\Delta_{31} & = 0 \\
 & & & & & & & & b_{00}(\partial)\Delta_{31} & = 0
\end{array}
\tag{1.16}
$$

Note that in I.32 we have defined $\mathcal{J}_\mu$ as the ideal of polynomials which kill Δ_μ. With this notation the third and fifth equations yield us that $b_{00} \in \mathcal{J}_{31} \wedge \mathcal{J}_{22}$ and since $(\mathcal{J}_{31} \wedge \mathcal{J}_{22})^\perp = \mathbf{M}_{31} \vee \mathbf{M}_{22}$, we see that we can choose $\mathcal{B}_{00}$ to be any bihomogenous basis of $\mathbf{M}_{31} \vee \mathbf{M}_{22}$. Indeed, if we do so any linear combination b_{00} of elements of $\mathcal{B}_{00}$ which satisfies the third and fifth equations in 1.16 would have to be orthogonal to itself and therefore must have all its coefficients equal to zero. We shall visualize this choice by writing

$$
\mathcal{B}_{00} \;=\; \ydiagram{1,3} \vee \ydiagram{2,2}
\tag{1.17}
$$

This done, setting $b_{00} = 0$, the fourth equation reduces to $b_{01}(\partial)\Delta_{31} = 0$ which gives $b_{01} \in \mathcal{J}_{31}$. Thus using the same imagery we can set

$$
\mathcal{B}_{01} \;=\; \ydiagram{1,3}
\tag{1.18}
$$

Substituting $b_{00} = b_{01} = 0$ the second equation reduces to $b_{10}(\partial)\Delta_{31} = 0$. Thus we can again set

$$
\mathcal{B}_{10} \;=\; \ydiagram{1,3}
\tag{1.19}
$$

Substituting $b_{00} = b_{01} = b_{10} = 0$ the first equation reduces to $b_{02}(\partial)\Delta_{22} + b_{11}(\partial)\Delta_{31} = 0$. Here we have two possible choices. We can let b_{11} vary freely in $\mathbf{M}_{31}$ and force $b_{02}(\partial)\Delta_{22}$ out of $\mathbf{M}_{31}/\{0\}$. Alternatively we can let b_{02} vary freely in $\mathbf{M}_{22}$ and force $b_{11}(\partial)\Delta_{31}$ out of $\mathbf{M}_{22}/\{0\}$. Let us take

$$
\mathcal{B}_{11} \;=\; \ydiagram{1,3}
\tag{1.20}
$$

With this choice, we want $b_{02}(\partial)\Delta_{22} \in \mathbf{M}_{31}$ to imply $b_{02} = 0$. We can assure this if $b_{02}(\partial)\Delta_{22}$ lies in the orthogonal complement of $\mathbf{M}_{31}\wedge\mathbf{M}_{22}$ in $\mathbf{M}_{22}$. In other words we want $\mathbf{flip}_{22} b_{02}$ † to lie in $\big(\mathbf{M}_{31}\wedge\mathbf{M}_{22}\big)^\perp\wedge\mathbf{M}_{22}$.

† Here and after $\mathbf{flip}_\mu$ will denote flipping with respect to Δ_μ.

To do this we must take $\mathcal{B}_{02}$ to be a basis of $\mathbf{flip}_{22}^{-1}\Big[\big(\mathbf{M}_{31}\wedge\mathbf{M}_{22}\big)^{\perp}\wedge\mathbf{M}_{22}\Big]$. We represent this final choice by writing

$$\mathcal{B}_{02} \;=\; \mathbf{flip}_{22}^{-1}\left[\left(\boxplus\!\square\wedge\boxplus\right)^{\perp}\wedge\boxplus\right]$$

We have now assured that $\mathcal{B}_{32}$ is an independent set. Let us count how many elements it has. We see that 1.18, 1.19 and 1.20 yield 3×24 elements, moreover, 1.17 yields $2\times 24-\dim\mathbf{M}_{31}\wedge\mathbf{M}_{22}$ elements. Since $\dim\mathbf{M}_{32}\le 120$ the independence of $\mathcal{B}_{32}$ yields the inequality

$$5\times 24 \;-\; \dim\begin{smallmatrix}\square\\ \square\\ \square\square\end{smallmatrix}\wedge\boxplus \;+\; \dim\left(\boxplus\!\square\wedge\boxplus\right)^{\perp}\wedge\boxplus \;\le 120$$

or better

$$\dim\left(\boxplus\!\square\wedge\boxplus\right)^{\perp}\wedge\boxplus \;\;\le\;\; \dim\boxplus\!\square\wedge\boxplus \qquad 1.21$$

In particular, $\mathcal{B}_{32}$ would be a basis for $\mathbf{M}_{32}$ and the $n!$ conjecture would then be established for $\mu=32$ if we prove that equality holds in 1.21. Amazingly it develops that this equality holds true in the strongest possible sense. More precisely, we can prove that

$$\mathbf{flip}_{22}\left[\mathbf{M}_{31}\wedge\mathbf{M}_{22}\right] \;=\; \left(\mathbf{M}_{31}\wedge\mathbf{M}_{22}\right)^{\perp}\wedge\mathbf{M}_{22}\;. \qquad 1.22$$

Let us postpone the proof of this identity and proceed to examine all its implications. To begin with it shows that we can take $\mathcal{B}_{02}$ to be any bihomogeneous basis of $\mathbf{M}_{31}\wedge\mathbf{M}_{22}$. This done we can visualise the Hilbert series of $\mathbf{M}_{32}$ by writing

$$G_{32} \;=\; \boxplus\!\square \;+\; \boxplus \;-\; \boxplus\!\square\wedge\boxplus \;+\; q\,\boxplus\!\square \;+\; t\,\boxplus\!\square \;+\; tq\,\boxplus\!\square \;+\; q^2\,\boxplus\!\square\vee\boxplus\;.$$

To be precise, the left hand side of this relation is $G_{32}(1/q,1/t)t^2q^4$, however we can write it that way because of 1.10. It will be convenient to extend the definition of $\mathbf{flip}_{\mu}$ to act on a Frobenius characteristic $H(x;q,t)$ and on a Hilbert series $F(q,t)$ by setting

$$\mathbf{flip}_{\mu}\,H(x;q,t)=\omega H(x;1/q,1/t)\,t^{n(\mu)}q^{n(\mu')}\;,\qquad \mathbf{flip}_{\mu}\,F(q,t)=F(1/q,1/t)\,t^{n(\mu)}q^{n(\mu')} \qquad 1.23$$

This given, the more refined argument which takes account of the action of S_4 on the basis $\mathcal{B}_{32}$ yields the relation

$$\partial_{p_1}C_{32}(x;q,t)=(1+t+q+tq)C_{31}(x;q,t)+C_{22}(x;q,t)+(q^2-1)\;\Phi \qquad 1.24$$

where for convenience we have denoted by Φ the Frobenius characteristic of $\boxplus\!\square\wedge\boxplus$. Here again the left-hand side should have been $\mathbf{flip}_{32}\partial_{p_1}C_{32}$, but 1.9 makes it right the way it is.

Note that if we follow the other alternative and choose $\mathcal{B}_{02}=\boxplus$, then we must take $\mathcal{B}_{11}$ to be a basis of

$$\mathbf{flip}_{31}^{-1}\left[\left(\mathbf{M}_{31}\wedge\mathbf{M}_{22}\right)^{\perp}\wedge\mathbf{M}_{31}\right]\;.$$

Since we can also show that

$$\mathbf{flip}_{31}\left(\mathbf{M}_{31}\wedge\mathbf{M}_{22}\right) \;=\; \left(\mathbf{M}_{31}\wedge\mathbf{M}_{22}\right)^{\perp}\wedge\mathbf{M}_{31}\;, \qquad 1.25$$

we can again set

$$\mathcal{B}_{11} \;=\; \boxplus\!\square\wedge\boxplus\;.$$

These choices yield

$$\partial_{p_1}C_{32}(x;q,t)=(1+t+q)C_{31}(x;q,t)+(1+q^2)C_{22}(x;q,t)+(tq-1)\;\Phi\;. \qquad 1.26$$

Subtracting 1.24 from 1.26 we deduce that these two expansions of $\partial_{p_1} C_{32}(x;q,t)$ are one and the same if and only if

$$\Phi \;=\; \frac{t\, C_{31} - q\, C_{22}}{t-q} \;. \tag{1.27}$$

It develops that this is a consequence of 1.22, and 1.25. Indeed these two relations yield that $\mathbf{M}_{31}$ and $\mathbf{M}_{22}$ decompose as follows (as bigraded S_4-modules):

$$\begin{aligned}\mathbf{M}_{31} &= \mathbf{M}_{31} \wedge \mathbf{M}_{22} \oplus \mathbf{flip}_{31} \mathbf{M}_{31} \wedge \mathbf{M}_{22} \;, \\ \mathbf{M}_{22} &= \mathbf{M}_{31} \wedge \mathbf{M}_{22} \oplus \mathbf{flip}_{22} \mathbf{M}_{31} \wedge \mathbf{M}_{22} \;.\end{aligned}$$

In particular we must also have that

$$\begin{aligned} C_{31}(x;q,t) &= \Phi + \mathbf{flip}_{31}\, \Phi &= \Phi + tq^3 \downarrow \Phi \;, \\ C_{22}(x;q,t) &= \Phi + \mathbf{flip}_{22}\, \Phi &= \Phi + t^2q^2 \downarrow \Phi \;. \end{aligned} \tag{1.28}$$

Now it is easily seen that these relations are equivalent to 1.27. The reader should recognize at this point that 1.22 and 1.25 are but two special instances of I.29. In fact, in the notation of the introduction they may be written as

$$\mathbf{flip}_1\, \mathbf{M}_{32}^{11} = \mathbf{M}_{32}^{10} \qquad \text{and} \qquad \mathbf{flip}_2\, \mathbf{M}_{32}^{11} = \mathbf{M}_{32}^{01} \;.$$

It goes without saying that we can verify by computer the validity of 1.22, 1.25 and 1.27. We can also easily check that the symmetric function given by

$$\frac{t\, \widetilde{H}_{31} - q\, \widetilde{H}_{22}}{t-q}$$

is indeed Schur positive and that the $C = \widetilde{H}$ conjecture is in fact true for 32, 31 and 22. Nevertheless, it is more illuminating to show how all of these identities may be proved using representation theory.

Recalling that

$$\Delta_{22} = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \end{pmatrix}, \quad \Delta_{31} \;=\; \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix} \;,$$

we get

$$-\partial_{x_4}\partial_{y_4}\Delta_{22} \;=\; \frac{1}{2}\,\partial_{y_4}^2\Delta_{31} \;=\; \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

Now from a well known result of A. Young (see Theorem 5.8 of [24]) we derive that the action of S_4 on this element generates a 3-dimensional irreducible representation with character χ^{211} and weight tq which is shared by $\mathbf{M}_{31}$ and $\mathbf{M}_{22}$. Similarly, we see that its partial derivatives

$$\partial_{x_3} \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} = -\det \begin{pmatrix} 1 & 1 \\ y_1 & y_2 \end{pmatrix} \;, \qquad \partial_{y_3} \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 \\ x_1 & x_2 \end{pmatrix} \;,$$

generate common 3-dimensional representations with character χ^{31} and weights q and t respectively. Now, we also have

$$\frac{1}{2}(\partial_{x_4}\partial_{y_3} - \partial_{x_3}\partial_{y_4})\Delta_{31} \;=\; 2\partial_{x_3}\partial_{x_4}\Delta_{22} \;=\; \det \begin{pmatrix} 1 & 1 \\ y_1 & y_2 \end{pmatrix} \det \begin{pmatrix} 1 & 1 \\ y_3 & y_4 \end{pmatrix} \;,$$

and this generates a common 2 dimensional representation with character χ^{22} and weight q^2. Including the trivial, we have thus identified a submodule of $\mathbf{M}_{31} \wedge \mathbf{M}_{22}$ with bigraded Frobenius characteristic

$$\Phi \;=\; S_4 + (t+q)\, S_{31} \;+\; tq\, S_{211} \;+\; q^2\, S_{22}\ .$$

Using I.19 and I.20 and the definition in 1.23 we then immediately derive that $\mathbf{M}_{31}$ and $\mathbf{M}_{22}$ contain submodules with respective Frobenius characteristics

$$\begin{aligned} \mathbf{flip}_{31}\, \Phi &= tq^3\, S_{1^4} + (q^3 + tq^2)\, S_{211} \;+\; q^2\, S_{31} \;+\; tq\, S_{22}\ , \\ \mathbf{flip}_{22}\, \Phi &= t^2q^2 S_{1^4} + (tq^2 + t^2q)\, S_{211} \;+\; tq\, S_{31} \;+\; t^2\, S_{22}\ . \end{aligned} \tag{1.29}$$

Since, Φ and $\mathbf{flip}_{31}\, \Phi$ have no common terms and each accounts for 12 dimensions we must conclude that together they must give the Frobenius characteristic of $\mathbf{M}_{31}$. A similar reasoning applies to Φ and $\mathbf{flip}_{22}\, \Phi$ and we can write

$$C_{31} \;=\; \Phi \;+\; \mathbf{flip}_{31}\, \Phi\ , \qquad C_{22} \;=\; \Phi \;+\; \mathbf{flip}_{22}\, \Phi\ , \tag{1.30}$$

Comparing the expressions in 1.29, we see that the possibility still remains that $\mathbf{M}_{31}$ and $\mathbf{M}_{22}$ may have in common an irreducible submodule with character χ^{211} and weight tq^2. However, this submodule is generated in $\mathbf{M}_{22}$ by the action of S_4 on the polynomial

$$P \;=\; \det \begin{pmatrix} 1 & 1 \\ \partial_{x_1} & \partial_{x_2} \end{pmatrix} \Delta_{22}$$

If this polynomial were to belong to $\mathbf{M}_{31}$ it would be killed by any element that kills Δ_{31}. In particular we should have $\partial_{x_4}\partial_{y_4} P = 0$. But we see that

$$\partial_{x_4}\partial_{y_4} P \;=\; \det \begin{pmatrix} 1 & 1 \\ \partial_{x_1} & \partial_{x_2} \end{pmatrix} \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \;=\; y_1 + y_2 - 2y_3 \neq 0\ .$$

This completes the proof of 1.22 and 1.25 and shows that Φ is none other than the Frobenius characteristic of the intersection of $\mathbf{M}_{31}$ and $\mathbf{M}_{22}$.

Remark 1. 1.1It is interesting to see what 1.24 reduces to when we express its right hand side entirely in terms of Φ. To this end we use the identities in 1.29 to replace C_{31} and C_{22} and obtain (using the notation introduced in I.21)

$$\begin{aligned} \partial_{p_1} C_{32} &= (1+t+q+tq)(\Phi + \mathbf{flip}_{31}\Phi) \;+\; (\Phi + \mathbf{flip}_{22}\Phi) \;+\; (q^2-1)\Phi \\ &= (1+t+q+tq+q^2)\Phi \;+\; (1+t+q+tq)tq^3 \downarrow \Phi \;+\; t^2q^2 \downarrow \Phi \\ &= (1+t+q+tq+q^2)\Phi \;+\; t^2q^4(1+\frac{1}{t}+\frac{1}{q}+\frac{1}{tq}+\frac{1}{q^2}) \downarrow \Phi\ . \end{aligned}$$

and this may be rewritten in the very suggestive form

$$\partial_{p_1} C_{32} \;=\; B_{32}(q,t)\, \Phi \;+\; \mathbf{flip}_{32}\big(B_{32}(q,t)\, \Phi\big)\ . \tag{1.31}$$

What we have discovered in this particular example holds true in full generality.

Proposition 1.1 *Let μ be a two-corner partition of $n+1$ and let $\alpha^{(1)}$ and $\alpha^{(2)}$ be the partitions obtained by removing one of the corners. Then*

$$\begin{aligned} \dim\Big((\mathbf{M}_{\alpha^{(1)}} \wedge \mathbf{M}_{\alpha^{(2)}})^{\perp} \wedge \mathbf{M}_{\alpha^{(1)}}\Big) &\leq \dim\big(\mathbf{M}_{\alpha^{(1)}} \wedge \mathbf{M}_{\alpha^{(2)}}\big)\ , \\ \dim\Big((\mathbf{M}_{\alpha^{(1)}} \wedge \mathbf{M}_{\alpha^{(2)}})^{\perp} \wedge \mathbf{M}_{\alpha^{(2)}}\Big) &\leq \dim\big(\mathbf{M}_{\alpha^{(1)}} \wedge \mathbf{M}_{\alpha^{(2)}}\big)\ . \end{aligned} \tag{1.32}$$

If the $n!$ conjecture holds for $\alpha^{(1)}$ and $\alpha^{(2)}$ and

$$\dim\left(\mathbf{M}_{\alpha^{(1)}} \wedge \mathbf{M}_{\alpha^{(2)}}\right) \;=\; \frac{n!}{2} \tag{1.33}$$

then equalities hold in 1.32 and the $(n+1)!$-conjecture holds for μ as well. Moreover, if

$$\begin{aligned}(\mathbf{M}_{\alpha^{(1)}} \wedge \mathbf{M}_{\alpha^{(2)}})^{\perp} \wedge \mathbf{M}_{\alpha^{(1)}} &= \mathbf{flip}_{\alpha^{(1)}}(\mathbf{M}_{\alpha^{(1)}} \wedge \mathbf{M}_{\alpha^{(2)}}) \;, \\ (\mathbf{M}_{\alpha^{(1)}} \wedge \mathbf{M}_{\alpha^{(2)}})^{\perp} \wedge \mathbf{M}_{\alpha^{(2)}} &= \mathbf{flip}_{\alpha^{(2)}}(\mathbf{M}_{\alpha^{(1)}} \wedge \mathbf{M}_{\alpha^{(2)}}) \;.\end{aligned} \tag{1.34}$$

Then, if we let Φ denote the bivariate Frobenius characteristic of $\mathbf{M}_{\alpha^{(1)}} \wedge \mathbf{M}_{\alpha^{(2)}}$, we also have the identity

$$\partial_{p_1} C_\mu(x;q,t) \;=\; B_\mu(q,t)\,\Phi \;+\; \mathbf{flip}_\mu\big(B_\mu(q,t)\,\Phi\big) \tag{1.35}$$

The proof of this will be given in section 4.

This given, we see that the validity of Conjecture I.2 (restated for the case of λ,μ predecessors of a two-corner partition) depends only on the verification of the equality in 1.33

2. The three corner case

We will better understand the formalism we have used to state properties (i)–(iv) of the heuristic, if we study in detail a sufficiently rich special case. To this end, we consider the case of a partition μ with 3 corners. Letting $\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}$ denote the predecessors of μ, we see that property (i) states that the space

$$\mathbf{V}_\mu = \mathbf{M}_{\alpha^{(1)}} \vee \mathbf{M}_{\alpha^{(2)}} \vee \mathbf{M}_{\alpha^{(3)}}$$

has a basis $\mathcal{B}$, with 3 subsets $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ such that

$$\mathbf{M}_{\alpha^{(1)}} = \mathcal{L}[\mathcal{B}_1], \qquad \mathbf{M}_{\alpha^{(2)}} = \mathcal{L}[\mathcal{B}_2], \qquad \mathbf{M}_{\alpha^{(3)}} = \mathcal{L}[\mathcal{B}_3] \;.$$

Moreover, each of these subsets breaks up as disjoint unions

$$\begin{aligned}\mathcal{B}_1 &= \mathcal{B}^{100} + \mathcal{B}^{110} + \mathcal{B}^{101} + \mathcal{B}^{111}, \\ \mathcal{B}_2 &= \mathcal{B}^{010} + \mathcal{B}^{110} + \mathcal{B}^{011} + \mathcal{B}^{111}, \\ \mathcal{B}_3 &= \mathcal{B}^{001} + \mathcal{B}^{011} + \mathcal{B}^{011} + \mathcal{B}^{111},\end{aligned}$$

where, for example,

$$\begin{aligned}\mathcal{B}^{100} &= \mathcal{B}_1 \cap \overline{\mathcal{B}}_2 \cap \overline{\mathcal{B}}_3, \\ \mathcal{B}^{110} &= \mathcal{B}_1 \cap \mathcal{B}_2 \cap \overline{\mathcal{B}}_3, \\ \mathcal{B}^{101} &= \mathcal{B}_1 \cap \overline{\mathcal{B}}_2 \cap \mathcal{B}_3, \\ \mathcal{B}^{111} &= \mathcal{B}_1 \cap \mathcal{B}_2 \cap \mathcal{B}_3,\end{aligned}$$

with $\overline{\mathcal{B}}_i = \mathcal{B} \setminus \mathcal{B}_i$.

All of this information is best visualized by the following Venn diagram.

Figure 1. Basis decomposition for 3 corners

By abuse of notation, we use the same imagery to represent the various subspaces $\mathbf{M}_\mu^\epsilon$. This given, we may visualize parts (ii) and (iii) of the heuristic by the diagram below.

Figure 2. Module decomposition and flips for 3 corners

Here, each of the regions corresponds to an S_n invariant subspace, and the arrows represent the effect of flip maps. We should add that arrows whithin the circle labelled $\mathbf{M}_{\alpha^{(i)}}$, represent the effect of $\mathbf{flip}_i$. We can thus visualize the geometric significance of (I.29), which in this case asserts that

$$\begin{array}{llll} \text{a)} & \mathbf{flip}_1\mathbf{M}^{100} \cong \mathbf{M}^{111}, & \text{a')} & \mathbf{flip}_1\mathbf{M}^{111} \cong \mathbf{M}^{100}, \\ \text{b)} & \mathbf{flip}_1\mathbf{M}^{110} \cong \mathbf{M}^{101}, & \text{b')} & \mathbf{flip}_1\mathbf{M}^{101} \cong \mathbf{M}^{110}, \\ \text{c)} & \mathbf{flip}_2\mathbf{M}^{110} \cong \mathbf{M}^{011}, & \text{c')} & \mathbf{flip}_2\mathbf{M}^{011} \cong \mathbf{M}^{110}, \\ \text{d)} & \mathbf{flip}_2\mathbf{M}^{010} \cong \mathbf{M}^{111}, & \text{d')} & \mathbf{flip}_2\mathbf{M}^{111} \cong \mathbf{M}^{010}, \\ \text{e)} & \mathbf{flip}_3\mathbf{M}^{101} \cong \mathbf{M}^{011}, & \text{e')} & \mathbf{flip}_3\mathbf{M}^{011} \cong \mathbf{M}^{101}, \\ \text{f)} & \mathbf{flip}_3\mathbf{M}^{001} \cong \mathbf{M}^{111}, & \text{f')} & \mathbf{flip}_3\mathbf{M}^{111} \cong \mathbf{M}^{001}. \end{array} \tag{2.1}$$

Of course these relations yield identities involving the corresponding bivariate characteristics as indicated in (I.28). For example, b'), c) and e') yield the following identities

$$\begin{array}{ll} 1) & T_1 \downarrow \Phi^{101} = \Phi^{110}, \\ 2) & T_2 \downarrow \Phi^{110} = \Phi^{011}, \\ 3) & T_3 \downarrow \Phi^{011} = \Phi^{101}. \end{array} \tag{2.2}$$

Note that combining 1) and 2) above, and using the involutory nature of $\downarrow$, we derive that

$$\Phi^{011} = T_2 \downarrow T_1 \downarrow \Phi^{101} = \frac{T_2}{T_1}\,\Phi^{101}.$$

Similarly, 2) and 3) of (2.2) yield

$$\Phi^{101} = T_3 \downarrow T_2 \downarrow \Phi^{110} = \frac{T_3}{T_2}\,\Phi^{110},$$

and these two relations give that

$$T_1\,\Phi^{011} = T_2\,\Phi^{101} = T_3\,\Phi^{110}. \tag{2.3}$$

Thus, if we denote $\Phi^{(2]}$ this common expression, we can write

$$\Phi^{011} = \frac{\Phi^{(2)}}{T_1}, \qquad \Phi^{101} = \frac{\Phi^{(2)}}{T_2}, \qquad \Phi^{110} = \frac{\Phi^{(2)}}{T_3}. \tag{2.4}$$

In the same manner, using a'), d') and f') of (2.1), we get

$$T_1 \downarrow \Phi^{111} = \Phi^{100}, \qquad T_2 \downarrow \Phi^{111} = \Phi^{010}, \qquad T_3 \downarrow \Phi^{111} = \Phi^{001}.$$

Thus, if we set

$$\Phi^{(1)} := T_1 T_2 T_3 \;\downarrow \Phi^{111},$$

we obtain that

$$\Phi^{100} = \frac{\Phi^{(1)}}{T_2 T_3}, \qquad \Phi^{010} = \frac{\Phi^{(1)}}{T_1 T_3}, \qquad \Phi^{001} = \frac{\Phi^{(1)}}{T_1 T_2}. \tag{2.5}$$

This leads us to discover the remarkable fact (mentioned in the introduction) that each Φ^{ϵ}_{μ}, up to a factor, depends only on the sum of the ϵ_i's. Postponing the proof of the general result to the next section, it will be good to see what can be further derived from these relations.

First of all, we should note that part (ii) of the SF heuristic implies that the $\mathbf{M}_{\alpha^{(i)}}$'s have the following decomposition into S_n-invariant submodules†.

$$\begin{aligned}
\mathbf{M}_{\alpha^{(1)}} &= \mathbf{M}^{100} \oplus \mathbf{M}^{101} \oplus \mathbf{M}^{110} \oplus \mathbf{M}^{111} \;, \\
\mathbf{M}_{\alpha^{(2)}} &= \mathbf{M}^{010} \oplus \mathbf{M}^{110} \oplus \mathbf{M}^{011} \oplus \mathbf{M}^{111} \;, \\
\mathbf{M}_{\alpha^{(3)}} &= \mathbf{M}^{001} \oplus \mathbf{M}^{101} \oplus \mathbf{M}^{011} \oplus \mathbf{M}^{111} \;.
\end{aligned} \tag{2.6}$$

It is clear that in general we have the following decomposition

$$\mathbf{M}_{\alpha^{(i)}} = \bigoplus_{\substack{\epsilon \\ \epsilon_i = 1}} \mathbf{M}^{\epsilon}_{\mu}.$$

Of course, all these relations yield corresponding relations for the associated bivariate Frobenius characteristics. For instance, those in (2.6) combined with (2.4) and (2.5) give

$$\begin{aligned}
\mathcal{F}(\mathbf{M}_{\alpha^{(1)}}) &= \frac{1}{T_2T_3}\Phi^{(1)} + \frac{1}{T_2}\Phi^{(2)} + \frac{1}{T_3}\Phi^{(2)} + \Phi^{(3)} \;, \\
\mathcal{F}(\mathbf{M}_{\alpha^{(2)}}) &= \frac{1}{T_1T_3}\Phi^{(1)} + \frac{1}{T_3}\Phi^{(2)} + \frac{1}{T_1}\Phi^{(2)} + \Phi^{(3)} \;, \\
\mathcal{F}(\mathbf{M}_{\alpha^{(3)}}) &= \frac{1}{T_1T_2}\Phi^{(1)} + \frac{1}{T_2}\Phi^{(2)} + \frac{1}{T_1}\Phi^{(2)} + \Phi^{(3)} \;,
\end{aligned}$$

where for consistency we have set

$$\Phi^{111} := \Phi^{(3)} \;.$$

† (see Figure 2)

Note that the $C = \widetilde{H}$ conjecture then yields the expansions

$$\begin{aligned} \widetilde{H}_{\alpha^{(1)}} &= \frac{1}{T_2T_3}\Phi^{(1)} + \frac{1}{T_2}\Phi^{(2)} + \frac{1}{T_3}\Phi^{(2)} + \Phi^{(3)} \ , \\ \widetilde{H}_{\alpha^{(2)}} &= \frac{1}{T_1T_3}\Phi^{(1)} + \frac{1}{T_3}\Phi^{(2)} + \frac{1}{T_1}\Phi^{(2)} + \Phi^{(3)} \ , \\ \widetilde{H}_{\alpha^{(3)}} &= \frac{1}{T_1T_2}\Phi^{(1)} + \frac{1}{T_2}\Phi^{(2)} + \frac{1}{T_1}\Phi^{(2)} + \Phi^{(3)} \ . \end{aligned} \tag{2.7}$$

which is I.38 for $m = 3$

Note further that the non singularity of the flip maps combined with the relations in (2.1) implies that

$$\dim \mathbf{M}^{100} = \dim \mathbf{M}^{010} = \dim \mathbf{M}^{001} = \dim \mathbf{M}^{111} \ , \qquad \dim \mathbf{M}^{110} = \dim \mathbf{M}^{101} = \dim \mathbf{M}^{011} \ .$$

Calling d_1 the first common dimension and d_2 the second, any of the equations in (2.6) gives that

$$n! = 2\, d_1 + 2\, d_2.$$

If we add to this, part (v) of the heuristic which gives that $\dim \mathbf{M}^{111} = n!/3$, we get

$$d_1 = \frac{n!}{3}, \qquad \text{and} \qquad d_2 = \frac{n!}{6}. \tag{2.8}$$

Since $\mathbf{M}_{\alpha^{(1)}} \cap \mathbf{M}_{\alpha^{(2)}} = \mathbf{M}^{110} + \mathbf{M}^{111}$, we deduce from (2.8) that

$$\dim \mathbf{M}_{\alpha^{(1)}} \cap \mathbf{M}_{\alpha^{(2)}} = \frac{n!}{2},$$

which shows that at least in this case part (v) of the heuristic is self-consistent.

Leaving all this aside for a moment, we shall next narrow our study to the case $\mu = 321$. We begin by showing here, that the heuristic enables us to push through the recursive approach to proving the $n!$ conjecture. We shall then complete the proof by the explicit construction of a basis $\mathcal{B}$ which satisfies properties (i)-(iv) required by the heuristic. In particular, since here

$$\alpha^{(1)} = \ydiagram{2,3} \ , \qquad \alpha^{(2)} = \ydiagram{1,1,3} \ , \qquad \alpha^{(3)} = \ydiagram{1,2,2} \ ,$$

we see that $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ must respectively give bases for

$$\mathbf{M}_{32} = \mathcal{L}[\partial_x^p \partial_y^q \Delta_{\ydiagram{2,3}}] \ , \qquad \mathbf{M}_{311} = \mathcal{L}[\partial_x^p \partial_y^q \Delta_{\ydiagram{1,1,3}}] \ , \qquad \mathbf{M}_{221} = \mathcal{L}[\partial_x^p \partial_y^q \Delta_{\ydiagram{1,2,2}}] \ .$$

Let us also keep in mind that according to I.33 we must have

$$\mathcal{B} \backslash \mathcal{B}_1 \subseteq \mathbf{M}_{32}^{\perp} \ , \qquad \mathcal{B} \backslash \mathcal{B}_2 \subseteq \mathbf{M}_{311}^{\perp} \ , \qquad \mathcal{B} \backslash \mathcal{B}_3 \subseteq \mathbf{M}_{221}^{\perp} \ . \tag{2.9}$$

Note now that in this case, Δ_μ is given by the determinant in (I.11) which may be written as

$$\Delta_{\ydiagram{1,2,3}} = D_{00} + y\, D_{01} + y^2 D_{02} + x\, D_{10} + xy\, D_{11} + x^2 D_{20}, \tag{2.10}$$

where for simplicity we have set $x_6 = x$ and $y_6 = y$, and D_{ij} represents the appropriately signed complementary minor of the element $x^i y^j$. Using 1.1 again, our strategy will be to construct a collection $\mathcal{B}_{321} \subset \mathbf{M}_{321}$ of the form

$$\begin{aligned} \mathcal{B}_{321} = \left\{ b(\partial)\, \Delta_{321} \right\}_{b \in \mathcal{B}_{00}} + \left\{ b(\partial)\, \partial_y\, \Delta_{321} \right\}_{b \in \mathcal{B}_{01}} + \left\{ b(\partial)\, \partial_y^2\, \Delta_{321} \right\}_{b \in \mathcal{B}_{02}} + \\ \left\{ b(\partial)\, \partial_x\, \Delta_{321} \right\}_{b \in \mathcal{B}_{10}} + \left\{ b(\partial)\, \partial_x \partial_y\, \Delta_{321} \right\}_{b \in \mathcal{B}_{11}} + \left\{ b(\partial)\, \partial_x^2\, \Delta_{321} \right\}_{b \in \mathcal{B}_{20}} \end{aligned} \tag{2.11}$$

where the $\mathcal{B}_{ij}$ are collections of polynomials in the variables $x_1, x_2, \ldots, x_5; y_1, y_2, \ldots, y_5$, to be determined from the equations that would have to assure that

1) $\mathcal{B}_{321}$ is an independent set,

2) the cardinality of $\mathcal{B}_{321}$ is $6!$,

To construct these equations, we assume that a basis $\mathcal{B}$ with properties (i)-(v) has been constructed, then determine how the subsets $\mathcal{B}_{ij} \subseteq \mathcal{B}$ must be chosen so that no matter how we pick $b_{ij} \in \mathcal{L}[\mathcal{B}_{ij}]$ the equality

$$\begin{aligned} b_{00}(\partial)\,\Delta_{321} + b_{01}(\partial)\,\partial_y\,\Delta_{321} + b_{02}(\partial)\,\partial_y^2\Delta_{321} + \\ + b_{10}(\partial)\,\partial_x\,\Delta_{321} + b_{11}(\partial)\,\partial_x\partial_y\,\Delta_{321} + b_{20}(\partial)\,\partial_x^2\Delta_{321} \;=\; 0 \end{aligned} \tag{2.12}$$

forces all the b_{ij} to vanish identically.

Note that from 2.10 we derive

$$\begin{array}{lcccccccccccc} \Delta_{321} & = & D_{00} & + & y\,D_{01} & - & y^2\Delta_{221} & + & x\,D_{10} & - & xy\,\Delta_{311} & + & x^2\Delta_{32} \\ \partial_y\,\Delta_{321} & = & & & D_{01} & - & 2y\,\Delta_{221} & & & - & x\,\Delta_{311} & & \\ \partial_y^2\Delta_{321} & = & & & & - & 2\,\Delta_{221} & & & & & & \\ \partial_x\,\Delta_{321} & = & & & & & & & D_{10} & - & y\,\Delta_{311} & + & 2x\,\Delta_{32} \\ \partial_x\partial_y\,\Delta_{321} & = & & & & & & & & - & \Delta_{311} & & \\ \partial_x^2\Delta_{321} & = & & & & & & & & & & & 2\,\Delta_{32} \end{array} \tag{2.13}$$

where we have used the fact that $D_{02} = -\Delta_{32}$, $D_{11} = -\Delta_{311}$ and $D_{20} = \Delta_{221}$. Thus equating to 0 the coefficients of $x^i y^j$ in (2.12), we can replace it by the following system of 6 equations

$$\begin{array}{lllllll} \text{a)}\ b_{00}(\partial)D_{00} & +b_{01}(\partial)D_{01} & -2b_{02}(\partial)\Delta_{221} & +b_{10}(\partial)D_{10} & -b_{11}(\partial)\Delta_{311} & +2b_{20}(\partial)\Delta_{32} & =0 \\ \text{b)} & b_{00}(\partial)D_{01} & -2b_{01}(\partial)\Delta_{221} & & -b_{10}(\partial)\Delta_{311} & & =0 \\ \text{c)} & & b_{00}(\partial)\Delta_{221} & & & & =0 \\ \text{d)} & & & b_{00}(\partial)D_{10} & -b_{01}(\partial)\Delta_{311} & +2b_{10}(\partial)\Delta_{32} & =0 \\ \text{e)} & & & & b_{00}(\partial)\Delta_{311} & & =0 \\ \text{f)} & & & & & b_{00}(\partial)\Delta_{32} & =0 \end{array} \tag{2.14}$$

Our guiding principle in looking for such sets $\mathcal{B}_{i,j}$, will be to choose them as "large" as possible in a manner that still forces equations in 2.14 to have as unique solution $b_{i,j} = 0$ for all i, j. To this end, note that whatever our choice of $\mathcal{B}_{00}$ is in $\mathbf{M}_{\alpha^{(1)}} \vee \mathbf{M}_{\alpha^{(2)}} \vee \mathbf{M}_{\alpha^{(3)}}$ equations c), e) and f) force b_{00} to be orthogonal to itself and therefore identically zero. Hence our maximal choice is

$$\mathcal{B}_{00} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \ .$$

To proceed, we need to adopt a convention. For a polynomial $P \in \mathcal{L}[\mathcal{B}]$ we shall let $P^{\epsilon_1\epsilon_2\epsilon_3}$ denote the projection of P into $\mathbf{M}^{\epsilon_1\epsilon_2\epsilon_3}$. Putting it in another way, if

$$P \;=\; \sum_{b\in\mathcal{B}} c_b\, b$$

then

$$P^{\epsilon_1\epsilon_2\epsilon_3} \;=\; \sum_{b\in\mathcal{B}^{\epsilon_1\epsilon_2\epsilon_3}} c_b\, b$$

Thus, whatever $\mathcal{B}_{10}$ and $\mathcal{B}_{01}$ are chosen to be in $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$, our elements b_{10} and b_{10} will have the decompositions

$$\begin{aligned} b_{10} &= b_{10}^{100} + b_{10}^{110} + b_{10}^{010} + b_{10}^{111} + b_{10}^{101} + b_{10}^{011} + b_{10}^{001}\ , \\ b_{01} &= b_{01}^{100} + b_{01}^{110} + b_{01}^{010} + b_{01}^{111} + b_{01}^{101} + b_{01}^{011} + b_{01}^{001}\ , \end{aligned} \tag{2.15}$$

Note that since the spaces $M^{\epsilon_1\epsilon_2\epsilon_3}$ are orthogonal to each other b_{01} or b_{10} will vanish if and only if each of their components $b_{01}^{\epsilon_1\epsilon_2\epsilon_3}$ and $b_{10}^{\epsilon_1\epsilon_2\epsilon_3}$ separately vanishes. Now, without any assumptions regarding the nature of the determinants D_{ij} occurring in 2.14, our only method for showing the vanishing of one of these components is to force some of its flips to vanish. Under these conditions, the only way we can show b_{10}^{001} to vanish is to force

$$b_{10}^{001}(\partial)\Delta_{\ydiagram{1,2,2}} = 0 \ .$$

However, we see no way to extract such a result out of 2.14 since b_{10} only acts on $\Delta_{\ydiagram{1,1,3}}$ and $\Delta_{\ydiagram{2,3}}$. In conclusion, we would be unable to force the vanishing of b_{10} if we allow $b_{10}^{001} \neq 0$. This given, the most we can include in $\mathcal{B}_{10}$ is all of $\mathcal{B}_1 \cup \mathcal{B}_2$. An analogous reasoning yields that the most we can include in $\mathcal{B}_{01}$ is all of $\mathcal{B}_2 \cup \mathcal{B}_3$. This leaves us with

$$\begin{aligned} b_{10} &= b_{10}^{100} + b_{10}^{110} + b_{10}^{010} + b_{10}^{111} + b_{10}^{101} + b_{10}^{011} \ , \\ b_{01} &= b_{01}^{110} + b_{01}^{010} + b_{01}^{111} + b_{01}^{101} + b_{01}^{011} + b_{01}^{001} \ . \end{aligned} \tag{2.16}$$

Note next that setting $b_{00} = 0$ in 2.14 reduces the 2^{nd} and 4^{th} equations to

$$\begin{aligned} &\text{b)} \quad 2b_{01}(\partial)\,\Delta_{\ydiagram{1,2,2}} + b_{10}(\partial)\,\Delta_{\ydiagram{1,1,3}} = 0 \ , \\ &\text{d)} \quad -b_{01}(\partial)\,\Delta_{\ydiagram{1,1,3}} + 2b_{10}(\partial)\,\Delta_{\ydiagram{2,3}} = 0 \ . \end{aligned} \tag{2.17}$$

In other words we must have

$$\begin{aligned} &\text{b)} \quad 2\,\mathbf{flip}_{\ydiagram{1,2,2}}\, b_{01} = -\mathbf{flip}_{\ydiagram{1,1,3}}\, b_{10} \ , \\ &\text{d)} \quad \mathbf{flip}_{\ydiagram{1,1,3}}\, b_{01} = 2\,\mathbf{flip}_{\ydiagram{2,3}}\, b_{10} \ . \end{aligned}$$

Projecting these equations in each of our spaces $\mathbf{M}^{\epsilon_1\epsilon_2\epsilon_3}$ and using the relations in 2.1 as equalities we obtain seven pairs of equations whose consequences are easily visualised from the two tables given below.

by $\mathbf{flip}_{\ydiagram{1,2,2}}$		Spaces		by $\mathbf{flip}_{\ydiagram{1,1,3}}$
$\{0\}$		$\mathbf{M}^{100}$		$\{0\}$
$\{0\}$		$\mathbf{M}^{110}$	$\longleftarrow$	b_{10}^{011}
$\{0\}$		$\mathbf{M}^{010}$	$\longleftarrow$	b_{10}^{111}
b_{01}^{011}	$\longrightarrow$	$\mathbf{M}^{101}$		$\{0\}$
b_{01}^{001}	$\longrightarrow$	$\mathbf{M}^{111}$	$\longleftarrow$	b_{10}^{010}
b_{01}^{101}	$\longrightarrow$	$\mathbf{M}^{011}$	$\longleftarrow$	b_{10}^{110}
b_{01}^{111}	$\longrightarrow$	$\mathbf{M}^{001}$		$\{0\}$

,

by $\mathbf{flip}_{\ydiagram{1,1,3}}$		Spaces		by $\mathbf{flip}_{\ydiagram{2,3}}$
$\{0\}$		$\mathbf{M}^{100}$	$\longleftarrow$	b_{10}^{111}
b_{01}^{011}	$\longrightarrow$	$\mathbf{M}^{110}$	$\longleftarrow$	b_{10}^{101}
b_{01}^{111}	$\longrightarrow$	$\mathbf{M}^{010}$		$\{0\}$
$\{0\}$		$\mathbf{M}^{101}$	$\longleftarrow$	b_{10}^{110}
b_{01}^{010}	$\longrightarrow$	$\mathbf{M}^{111}$	$\longleftarrow$	b_{10}^{100}
b_{01}^{110}	$\longrightarrow$	$\mathbf{M}^{011}$		$\{0\}$
$\{0\}$		$\mathbf{M}^{001}$		$\{0\}$

For instance, the second row of the table in the left states that $\mathbf{flip}_{\ydiagram{1,1,3}}$ sends b_{10}^{011} into $\mathbf{M}^{110}$ and at the same time there is no component of b_{01} that is sent into $\mathbf{M}^{110}$ by $\mathbf{flip}_{\ydiagram{1,2,2}}$. Thus, as long as b_{10} and b_{01} are are given by 2.16, equation b) forces $b_{10}^{011} = 0$. On the other hand, the second row of the table on the right, simply says that $\mathbf{flip}_{\ydiagram{1,1,3}}$ sends b_{01}^{011} into $\mathbf{M}^{110}$ and at the same time $\mathbf{flip}_{\ydiagram{2,3}}$ sends b_{10}^{101} into the same space. So we can't deduce any implications at this stage from this particullar row. Proceeding in this manner we immediately derive from equation b) and the table on the left that $b_{10}^{011} = b_{10}^{111} = b_{01}^{011} = b_{01}^{111} = 0$. Similarly, equation d) and the table on the right imply that $b_{10}^{111} = b_{10}^{110} = b_{01}^{111} = b_{01}^{110} = 0$. Less trivially, if we feed the information we get from one table into the other, we see that the vanishing of b_{10}^{110} forced by the fourth row of the table on the right, makes no component of $\mathbf{flip}_{\ydiagram{1,1,3}}\, b_{10}$ available to match the image of b_{01}^{101} under $\mathbf{flip}_{\ydiagram{1,2,2}}$. So from the sixth row of the table on the left we deduce that $b_{01}^{101} = 0$. A similar reasoning based on the second row of the table on the right yields that $b_{10}^{101}=0$. This given, without committing ourselves to any particular choices of $\mathcal{B}_{01}$ and $\mathcal{B}_{10}$, (other than 2.15), our equations reduce the decompositions in 2.15 to

$$\begin{aligned} b_{10} &= b_{10}^{100} + b_{10}^{010} \ , \\ b_{01} &= b_{01}^{010} + b_{01}^{001} \ , \end{aligned}$$

Setting all the remaining components equal to zero reduces both our tables to a single row:

$$\begin{array}{ccccc} \text{by } \mathbf{flip}_{\tiny\yng(1,2,2)} & & \text{Spaces} & & \text{by } \mathbf{flip}_{\tiny\yng(1,1,3)} \\ b_{01}^{001} & \rightarrow & \mathbf{M}^{111} & \leftarrow & b_{10}^{010} \end{array} \ , \qquad \begin{array}{ccccc} \text{by } \mathbf{flip}_{\tiny\yng(1,1,3)} & & \text{Spaces} & & \text{by } \mathbf{flip}_{\tiny\yng(2,3)} \\ b_{01}^{010} & \rightarrow & \mathbf{M}^{111} & \leftarrow & b_{10}^{100} \end{array}$$

Now the non vanishing of b_{10}^{100} , b_{10}^{010} and b_{01}^{010} , b_{01}^{001} is perfectly consistent with the equations in 2.17. So it is pretty clear that we can't choose at the same time $\mathcal{B}_{10}=\mathcal{B}_1\cup\mathcal{B}_2$ and $\mathcal{B}_{01}=\mathcal{B}_2\cup\mathcal{B}_3$. On the other hand if we take the subset $\mathcal{B}^{001}+\mathcal{B}^{010}$ out of $\mathcal{B}_{01}$ or the subset $\mathcal{B}^{100}+\mathcal{B}^{010}$ out of $\mathcal{B}_{10}$ then equation b) and d) will do the rest and force $b_{01}=b_{10}=0$. It turns out that either choice leads to the construction of a basis for $\mathbf{M}_{\tiny\yng(1,2,3)}$. To be definite we shall take

$$\mathcal{B}_{10}=\mathcal{B}_1\cup\mathcal{B}_2 \qquad \text{and} \qquad \mathcal{B}_{01}=\mathcal{B}^{110}+\mathcal{B}^{111}+\mathcal{B}^{011}+\mathcal{B}^{101} \tag{2.18}$$

Our choices so far force $b_{00}=b_{10}=b_{01}=0$ and the system 2.14 now reduces to the single equation

$$-\,2b_{02}(\partial)\Delta_{\tiny\yng(1,2,2)} \ - \ b_{11}(\partial)\Delta_{\tiny\yng(1,1,3)} \ + \ 2b_{20}(\partial)\Delta_{\tiny\yng(2,3)} \ = \ 0 \ . \tag{2.19}$$

Remarkably, we can chose $\mathcal{B}_{02}$, $\mathcal{B}_{11}$ and $\mathcal{B}_{20}$ in an optimal way and still guarantee that this single equation will force $b_{02}=b_{11}=b_{20}=0$. Note that at this stage, the maximal choices for $\mathcal{B}_{02}$, $\mathcal{B}_{11}$ and $\mathcal{B}_{20}$ are $\mathcal{B}_3$, $\mathcal{B}_2$ and $\mathcal{B}_1$ respectively. Adopting the *greedy algorithm* strategy we chose

$$\mathcal{B}_{20} \ = \ \mathcal{B}_1 \ . \tag{2.20}$$

In order that the second term in 2.19 does not produce a component that could be canceled by the third term we need $b_{11}(\partial)\Delta_{\tiny\yng(1,1,3)}$ to fall out of $\mathbf{M}_{\tiny\yng(2,3)}$. A look at Figure 2. suggests that $\mathcal{B}_{11}$ should then lie in $\mathbf{M}^{110}\oplus\mathbf{M}^{111}$. The maximal choice is then

$$\mathcal{B}_{11} \ = \ \mathcal{B}^{110} \ + \ \mathcal{B}^{111} \ .$$

This done we must assure that $b_{02}(\partial)\Delta_{\tiny\yng(1,2,2)}$ falls out of $\mathbf{M}_{\tiny\yng(1,1,3)}$ and $\mathbf{M}_{\tiny\yng(2,3)}$. Again from Figure 2. we get that we must take

$$\mathcal{B}_{02} \ = \ \mathcal{B}^{111} \ .$$

With this final choice we have assured that the system in 2.17 forces all b_{ij} to vanish and consequently establish the independence of the system given in 2.11. We are left with checking that its cardinality is indeed 6!. Now our choices have been

Basis	*Cardinality*
$\mathcal{B}_{00}=\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3=\mathcal{B}_1+\mathcal{B}_2\cap(\mathcal{B}\backslash\mathcal{B}_1)+\mathcal{B}^{001}$	$5! \ + \ \frac{5!}{2} \ + \ \frac{5!}{3}$
$\mathcal{B}_{10}=\mathcal{B}_1+\mathcal{B}^{010}+\mathcal{B}^{011}$	$5! \ + \ \frac{5!}{3} \ + \ \frac{5!}{6}$
$\mathcal{B}_{01}=\mathcal{B}^{110}+\mathcal{B}^{111}+\mathcal{B}^{101}+\mathcal{B}^{011}$	$3\times\frac{5!}{6} \ + \ \frac{5!}{3}$
$\mathcal{B}_{20}=\mathcal{B}_1$	$5!$
$\mathcal{B}_{11}=\mathcal{B}^{110}+\mathcal{B}^{111}$	$\frac{5!}{2}$
$\mathcal{B}_{02}=\mathcal{B}^{111}$	$\frac{5!}{3}$

Here all these cardinalities result from the assumption that the $n!$ conjecture holds true for $\ydiagram{1,2,2}$, $\ydiagram{1,1,3}$, $\ydiagram{2,3}$ and that the dimensions of the spaces $\mathbf{M}^{\epsilon_1\epsilon_2\epsilon_3}$ are as given by 2.8. As we can see the count is indeed $6 \times 5!$ as desired. Thus, to prove the $n!$ conjecture for the partition $(3, 2, 1)$ and obtain a basis for the module $\mathbf{M}_{321}$ we need only exhibit the collection $\mathcal{B}$ with properties (i)-(iv) of the SF heuristic.

Using the dimensions given in 2.8 we can easily see that $\mathcal{B}$ will consist of $5! \times 13/6 = 260$ elements. This given, we shall limit ourselves here to exhibiting the basic ingredients from which $\mathcal{B}$ may be easily constructed. Note first that we need only exhibit $\mathcal{B}^{111}$ and $\mathcal{B}^{110}$ since (via 2.1) all the other $\mathcal{B}^{\epsilon_1\epsilon_2\epsilon_3}$ may be recovered by successive flips. Secondly, note that if we obtain a complete decomposition of the modules $\mathbf{M}^{111}$ and $\mathbf{M}^{110}$ into irreducible constituents, then we need only exhibit a collection of cyclic elements generating these constituents. In fact, if the cyclic elements are given as Young idempotents acting on polynomials, then $\mathbf{M}^{111}$ and $\mathbf{M}^{110}$ may be constructed by selecting from the S_5-orbits of these cyclic elements those corresponding to standard tableaux idempotents. In the following tables we list such a set of cyclic elements as sums of products of determinants. This description is a special case of a construction of basis for these intersections given by F. Bergeron and S. Hamel in [4]. We recall that if $a_1 < a_2 < \cdots < a_m$ is a sequence of integers then Young uses the symbol $[a_1, a_2, \ldots, a_m]$ to denote the formal sum of all elements of the symmetric group $S_{\{a_1,a_2,\ldots,a_m\}}$. Likewise Young uses the symbol $[a_1, a_2, \ldots, a_m]'$ to denote the sum of the elements of $S_{\{a_1,a_2,\ldots,a_m\}}$ multiplied their sign. In this notation, the Young idempotent correspoding to the standard tableau

$$T = \begin{array}{|c|c|c|} \cline{1-1} 6 \\ \cline{1-2} 4 & 5 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$$

is simply given by the group algebra expression

$$[1, 4, 6]'[2, 5]'\ [1, 2, 3][4, 5]$$

Young calls $[1, 2, 3][4, 5]$ the <u>row group</u> of T and $[1, 4, 6]'[2, 5]'$ the <u>signed column group</u>. Young in [24] makes the following important observation. Namely, if γ is any element of the group algebra of S_n and V is a vector space spanned by some symbols on which S_n acts then, for any $v \in V$, the action of S_n on the submodule generated by the element $\gamma\, v$ is identical to the action of S_n on the left ideal generated by the group algebra element $\gamma\, \Sigma_v$ where Σ_v denotes the formal sum of the elements of S_n that stabilize v. This observation should be kept in mind when reading our table below. For instance, suppose we are to study the representation resulting from the action of S_6 on the polynomial

$$P(x; y) = \det \begin{bmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{bmatrix} \det \begin{bmatrix} 1 & 1 \\ y_4 & y_5 \end{bmatrix} .$$

We first notice that it may be written in the form

$$P(x; y) = [1, 2, 3]'[4, 5]'\, y_2\, x_3\, y_5 \ .$$

Then observe that, since the stabilizer of $y_2\, x_3\, y_5$ in S_6 is $[1, 4, 6]\ [2, 5]$, we may also write

$$P(x; y) = \frac{1}{12}\ [1, 2, 3]'[4, 5]'[1, 4, 6][2, 5]\, y_2 x_3 y_5 \ . \qquad 2.21$$

This makes it evident that $P(x, y)$ generates an irreducible representation of S_6 with character $\chi^{3,2,1}$. Note further that, since P is a bihomogeneous polynomial of x-degree 1 and y-degree 2, the submodule generated by P would contribute the term $tq^2 S_{321}$ to the bivariate Frobenius characteristic of any bigraded S_6-module that contains P. It will be convenient to refer to tq^2 as the *weight* of this representation and to the partition 321 as its *shape*. Finally, we also immediately read from 2.21 that a basis for the submodule generated by P is given by P and the 15 polynomials obtained by replacing in 2.21 the idempotent $[1, 2, 3]'[4, 5]'[1, 4, 6][2, 5]$ by the idempotents corresponding to the other 14 standard tableaux of shape 321. Another example may be helpful in checking some of the entries in the tables below. Suppose that we are to study the S_5-module generated by the action of S_5 on the polynomial

$$Q(x,y) \;=\; \det\begin{pmatrix} 1 & 1 & 1 \\ y_1^2 & y_2^2 & y_3^2 \\ x_1 & x_2 & x_3 \end{pmatrix} \;-\; 2\,y_5\,\det\begin{pmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

Then it is easily seen that we have

$$[4,5]'Q(x,y) \;=\; -2\,\det\begin{pmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{pmatrix}\det\begin{pmatrix} 1 & 1 \\ y_4 & y_5 \end{pmatrix} \;=\; -\frac{1}{2}\,[1,2,3]'[4,5]'[1,4][2,5]\,y_2x_3y_5$$

as well as

$$\begin{aligned} \frac{1}{2}\,[4,5]Q(x,y) \;&=\; \det\begin{pmatrix} 1 & 1 & 1 \\ y_1^2 & y_2^2 & y_3^2 \\ x_1 & x_2 & x_3 \end{pmatrix} \;-\; \det\begin{pmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{pmatrix}\;(y_4+y_5) \\ &=\; [1,2,3]'\big(\,y_2^2x_3 \;-\; y_2x_3\,(y_1+y_4+y_5)\,\big) \\ &=\; \frac{1}{6}\,[1,2,3]'[1,4,5]\big(\,y_2^2x_3 \;-\; y_2x_3\,(y_1+y_4+y_5)\,\big)\;. \end{aligned} \qquad 2.22$$

We thus see that Q generates a bigraded S_5-module with bivariate Frobenius characteristic

$$t\,q^2S_{2,2,1} \;+\; t\,q^2S_{3,1,1}\;. \qquad 2.23$$

This given, it should not be difficult to check that the following two tables give all the cyclic elements needed to generate our bases $\mathcal{B}^{111}$ and $\mathcal{B}^{110}$.

TABLE FOR $\mathcal{B}^{111}$

Cyclic element	**Weight**	**Shape**
$\left\{\begin{matrix} \frac{1}{2}(\partial_{x_5}+\partial_{x_4})\partial_{y_4}\partial_{y_5}\Delta_{(3,2)} \\ \frac{1}{4}(\partial_{y_4}\partial_{x_5}^2+\partial_{y_5}\partial_{x_4}^2)\Delta_{(3,1,1)} \\ \frac{1}{2}\partial_{x_1}\partial_{x_2}\partial_{x_3}\Delta_{(2,2,1)} \end{matrix}\right\} = \det\begin{bmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{bmatrix}\det\begin{bmatrix} 1 & 1 \\ y_4 & y_5 \end{bmatrix}$	q^2t	$(2,2,1)$
$\left\{\begin{matrix} -\frac{1}{2}\partial_{y_1}\partial_{y_2}\partial_{y_3}\Delta_{(3,2)} \\ -\frac{1}{4}(\partial_{x_4}\partial_{y_5}^2+\partial_{x_5}\partial_{y_4}^2)\Delta_{(3,1,1)} \\ \frac{1}{2}(\partial_{y_5}+\partial_{y_4})\partial_{x_4}\partial_{x_5}\Delta_{(2,2,1)} \end{matrix}\right\} = \det\begin{bmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{bmatrix}\det\begin{bmatrix} 1 & 1 \\ x_4 & x_5 \end{bmatrix}$	qt^2	$(2,2,1)$
$\left\{\begin{matrix} \frac{1}{2}\partial_{y_3}\partial_{y_4}\partial_{y_5}^2\Delta_{(3,2)} \\ \frac{1}{4}(\partial_{y_3}\partial_{x_4}+\partial_{x_3}\partial_{y_4})\partial_{y_5}^2\Delta_{(3,1,1)} \\ \frac{1}{2}\partial_{y_3}\partial_{y_4}\partial_{x_5}^2\Delta_{(2,2,1)} \end{matrix}\right\} = \det\begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}\det\begin{bmatrix} 1 & 1 \\ x_3 & x_4 \end{bmatrix}$	t^2	$(3,2)$

TABLE FOR $\mathcal{B}^{111}$ *continued*

Cyclic element	**Weight**	**Shape**
$\left\{\begin{matrix} \frac{1}{2}(\partial_{y_1}\partial_{y_2}+\partial_{y_3}\partial_{y_4})\partial_{x_5}\partial_{y_5}\Delta \\ \frac{1}{4}(\partial_{x_1}\partial_{x_2}+\partial_{x_3}\partial_{x_4})\partial^2_{y_5}\Delta \\ -\frac{1}{2}(\partial_{x_1}\partial_{x_2}+\partial_{x_3}\partial_{x_4})\partial_{x_5}\partial_{y_5}\Delta \end{matrix}\right\} = [1,2]'[3,4]'(y_2x_4+x_2y_4)$	qt	
$\left\{\begin{matrix} \frac{1}{2}\partial_{x_3}\partial_{x_4}\partial^2_{y_5}\Delta \\ \frac{1}{4}(\partial_{y_3}\partial_{x_4}+\partial_{x_3}\partial_{y_4})\partial^2_{x_5}\Delta \\ \frac{1}{2}\partial_{x_3}\partial_{x_4}\partial^2_{x_5}\Delta \end{matrix}\right\} = \det\begin{bmatrix} 1 & 1 \\ y_1 & y_2 \end{bmatrix}\det\begin{bmatrix} 1 & 1 \\ y_3 & y_4 \end{bmatrix}$	q^2	
$\left\{\begin{matrix} \frac{1}{2}\partial_{x_4}\partial_{y_4}\partial^2_{y_5}\Delta \\ \frac{1}{4}\partial^2_{x_5}\partial^2_{y_4}\Delta \\ \frac{1}{2}\partial_{x_4}\partial_{y_4}\partial^2_{x_5}\Delta \end{matrix}\right\} = \det\begin{bmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{bmatrix}$	qt	
$\det\begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}$	t	
$\det\begin{bmatrix} 1 & 1 \\ y_1 & y_2 \end{bmatrix}$	q	
1	1	

TABLE FOR $\mathcal{B}^{011}$

Cyclic element	**Weight**	**Shape**
$\left\{\begin{matrix} [3,4,5]'(\partial_{y_3}\partial_{x_4}\partial_{x_5}\partial_{y_5}) \\ \frac{1}{2}[3,4,5]'(\partial_{y_3}\partial_{x_4}\partial^2_{x_5}) \end{matrix}\right\} = [1,2]'(-6y_2^2+4(y_1+y_3+y_4+y_5)y_2)$	q^2	
$\left\{\begin{matrix} \frac{1}{2}(\partial_{y_4}-\partial_{x_5})\partial_{x_4}\partial_{y_5} \\ \frac{1}{4}(\partial^2_{x_4}\partial_{y_5}-\partial^2_{x_5}\partial_{y_4}) \end{matrix}\right\} = [1,2,3]'\left(-y_2^2x_3+2(y_1+y_4+y_5)y_2x_3\right)$	q^2t	
$\left\{\begin{matrix} \partial_{x_5}\partial_{y_5}\Delta \\ \frac{1}{2}\partial^2_{x_5}\Delta \end{matrix}\right\} = \det\begin{bmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix}$	q^3t	
$\left\{\begin{matrix} \partial_{x_4}\partial_{x_5}\partial_{y_5} \\ \frac{1}{2}\partial_{x_4}\partial_{x_5}\partial_{y_5} \end{matrix}\right\} = \begin{bmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ y_1^2 & y_2^2 & y_3^2 \end{bmatrix}$	q^3	

At this point it is good to see to what extent the results of this section are consistent with the explicit calculations made by Macdonald in the original paper [20]. We are referring here to the tables of the coefficients $K_{\lambda\mu}(q,t)$ which are also given in [21]. After making the changes of scale

$$K_{\lambda\mu}(q,t) \;\rightarrow\; \widetilde{K}_{\lambda\mu}(q,t) = T_\mu \, K_{\lambda\mu}(q,1/t) \;,$$

and using the expansion in I.8 we can easily compute all of the polynomials $\widetilde{H}_\mu(x;q,t)$ for all $|\mu| \leq 6$. This given, let us subtract the the second equation in 2.7 multiplied by $1/T_2$ from the first multiplied by $1/T_1$, obtaining

$$\frac{1}{T_1}\widetilde{H}_{\alpha^{(1)}} \;-\; \frac{1}{T_2}\,\widetilde{H}_{\alpha^{(2)}} \;=\; \Big(\frac{1}{T_1} \;-\; \frac{1}{T_2}\Big)\Big(\frac{1}{T_3}\Phi^{(2)} \;+\; \Phi^{(3)}\Big) \;.$$

Which gives

$$\begin{aligned}\Big(\frac{1}{T_1}\widetilde{H}_{\alpha^{(1)}} \;-\; \frac{1}{T_2}\,\widetilde{H}_{\alpha^{(2)}}\Big)\Big/\Big(\frac{1}{T_1} \;-\; \frac{1}{T_2}\Big) \;&=\; \frac{1}{T_3}\Phi^{(2)} \;+\; \Phi^{(3)} \\ &=\; \Phi^{110}_\mu + \Phi^{111}_\mu \;=\; \mathcal{F}\big(\mathbf{M}_{\alpha^{(1)}} \cap \mathbf{M}_{\alpha^{(2)}}\big) \;.\end{aligned} \qquad 2.24$$

Similarly, the second and third equations in 2.7 give

$$\begin{aligned}\Big(\frac{1}{T_2}\widetilde{H}_{\alpha^{(2)}} \;-\; \frac{1}{T_3}\,\widetilde{H}_{\alpha^{(3)}}\Big)\Big/\Big(\frac{1}{T_2} \;-\; \frac{1}{T_3}\Big) \;&=\; \frac{1}{T_1}\Phi^{(2)}_\mu \;+\; \Phi^{(3)}_\mu \\ &=\; \Phi^{011}_\mu + \Phi^{111}_\mu \;=\; \mathcal{F}\big(\mathbf{M}_{\alpha^{(2)}} \cap \mathbf{M}_{\alpha^{(3)}}\big) \;.\end{aligned} \qquad 2.25$$

Subtracting 2.25 multiplied by $1/T_3$ from 2.24 multiplied by $1/T_1$ and dividing the result by $\big(\frac{1}{T_1} - \frac{1}{T_3}\big)$ gives

$$\Phi^{(3)} = \;=\; \frac{\frac{1}{T_1}\Big(\frac{1}{T_1}\widetilde{H}_{\alpha^{(1)}} - \frac{1}{T_2}\,\widetilde{H}_{\alpha^{(2)}}\Big)\Big/\Big(\frac{1}{T_1} - \frac{1}{T_2}\Big) \;-\; \frac{1}{T_3}\Big(\frac{1}{T_2}\widetilde{H}_{\alpha^{(2)}} - \frac{1}{T_3}\,\widetilde{H}_{\alpha^{(3)}}\Big)\Big/\Big(\frac{1}{T_2} - \frac{1}{T_3}\Big)}{\Big(\frac{1}{T_1} \;-\; \frac{1}{T_3}\Big)} \;. \qquad 2.26$$

Using Macdonald's tables to compute the left-hand side of 2.26 for $\alpha^{(1)} = {32}$, $\alpha^{(2)} = {311}$ and $\alpha^{(3)} = {221}$ gives

$$\begin{aligned}&\frac{\frac{1}{q^2}\Big(\frac{1}{q}\widetilde{H}_{32} \;-\; \frac{1}{t}\,\widetilde{H}_{311}\Big)\Big/\Big(\frac{1}{q} \;-\; \frac{1}{t}\Big) \;-\; \frac{1}{t^2}\Big(\frac{1}{q}\widetilde{H}_{311} \;-\; \frac{1}{t}\,\widetilde{H}_{221}\Big)\Big/\Big(\frac{1}{q} \;-\; \frac{1}{t}\Big)}{\frac{1}{q^2} \;-\; \frac{1}{t^2}} \;= \\ &\qquad =\; \Phi^{111}_{321} \;=\; S_4 + (t+q)S_{41} + (t^2+tq+q^2)S_{32} + tqS_{311} + (qt^2+tq^2)S_{221}\end{aligned}$$

which is easily seen to be in perfect agreement with the Frobenius characteristic of

$$(\mathbf{M}_{32} \cap \mathbf{M}_{311} \cap \mathbf{M}_{221})$$

as may be put together from our tables giving $\mathcal{B}^{111}$. Similarly, from 2.24 we derive that

$$\Phi^{110}_{321} = \Big(\frac{1}{q}\widetilde{H}_{32} \;-\; \frac{1}{t}\,\widetilde{H}_{311}\Big)\Big/\Big(\frac{1}{q} \;-\; \frac{1}{t}\Big) \;-\; \Phi^{111}_{321} \;=\; q^2 S_{41} + q^2(t+q)\,S_{311} + q^3 t\,S_{2111} \;.$$

Which is again easily seen to agree with our table for $\mathcal{B}^{110}$.

We shall see in the next section that divided difference formulas as in 2.24, 2.25 and 2.26 are but special cases of a general identity giving the Frobenius characteristic of the intersection of any subset of the modules $\mathbf{M}_{\alpha^{(1)}}, \mathbf{M}_{\alpha^{(2)}}, \ldots, \mathbf{M}_{\alpha^{(m)}}$.

3. General identities

We begin by extending the arguments that gave 2.4 and 2.5 to the general case. Here and in the following we adopt the same notational conventions we made in the introduction. We work with a fixed partition μ with m corners, with predecessors $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(m)}$ ordered as we indicated in the introduction. We shall assume that the SF heuristic is valid for μ and that a basis $\mathcal{B}$ has been constructed with the required properties.

Proposition 3.1 *For every $1 \leq k \leq m$ there exists a Schur positive function $\Phi_\mu^{(k)}$ such that for all words $\epsilon = \epsilon_1\epsilon_2\cdots\epsilon_m$ with $\sum_{i=1}^m \epsilon_i = k$ we have*

$$\Phi_\mu^\epsilon \;=\; \frac{\Phi^{(k)}}{\prod_{\epsilon_i=0} T_i} \tag{3.1}$$

Proof. For a moment let us set

$$\Psi_\mu^\epsilon \;=\; T^{\tilde\epsilon}\,\Phi_\mu^\epsilon \tag{3.2}$$

with

$$T^{\tilde\epsilon} \;=\; \prod_{i=1}^m T_i^{\tilde\epsilon_i} \;=\; \prod_{\epsilon_i=0} T_i \ . \tag{3.3}$$

This given, to show 3.1 we need only establish that Ψ_μ^ϵ depends only on $\sum_{i=1}^m \epsilon_i$. Note that if $\epsilon = 1111\cdots 1 = 1^m$ then we may take $\Phi^{(m)} := \Phi_\mu^{1^m}$. In all other cases ϵ will contain at least a 0 and a 1. Choose a pair i, j such that $\epsilon_i = 1$ and $\epsilon_j = 0$. Note that two applications of I.30 yield that

$$\downarrow_j \downarrow_i \ \Phi_\mu^\epsilon \;=\; \Phi_\mu^{\tau_j\tau_i\epsilon} \ . \tag{3.4}$$

However, from the definition I.28 we immediately derive that (if $i < j$)

$$\tau_j\tau_i\epsilon \;=\; \epsilon_1\cdots\epsilon_{i-1}\mathbf{0}\,\epsilon_{i+1}\cdots\epsilon_{j-1}\mathbf{1}\,\epsilon_{j+1}\cdots\epsilon_m \;=\; (i,j)\,\epsilon \ , \tag{3.5}$$

where (i,j) denotes the transposition that interchanges the i^{th} and j^{th} letters of a word. On the other hand I.27 gives

$$\downarrow_j \downarrow_i \ \Phi_\mu^\epsilon \;=\; \downarrow_j (T_i \ \downarrow \Phi_\mu^\epsilon) \;=\; \frac{T_j}{T_i}\ \Phi_\mu^\epsilon \ . \tag{3.6}$$

Combining 3.4, 3.5 and 3.6 we get

$$\Phi_\mu^{(i,j)\epsilon} \;=\; \frac{T_j}{T_i}\ \Phi_\mu^\epsilon \ . \tag{3.7}$$

Since $\tilde\epsilon_i = 0$ and $\tilde\epsilon_j = 1$ we can write

$$T^{(i,j)\tilde\epsilon} \;=\; \frac{T_i}{T_j}\ T^{\tilde\epsilon} \ .$$

Thus multiplying both sides of 3.7 by $T^{(i,j)\tilde\epsilon}$ we finally obtain

$$\Psi_\mu^{(i,j)\epsilon} \;=\; T^{(i,j)\tilde\epsilon}\ \Phi_\mu^{(i,j)\epsilon} \;=\; T^{\tilde\epsilon}\ \Phi_\mu^\epsilon \;=\; \Psi_\mu^\epsilon \ .$$

Thus Ψ_μ^ϵ remains unchanged when we arbitrarily permute the letters of ϵ, proving the proposition.

To proceed we need some further notational convenions. To begin with, note that we can associate to the word $\epsilon = \epsilon_1\epsilon_2\cdots\epsilon_m$ the subset $S = \{1 \leq i \leq m : \epsilon_i = 1\}$. Conversely, for $S \subseteq \{1, 2, \ldots, m\}$ we let $\epsilon(S)$ be the word ϵ such that $\epsilon_i = 1$ iff $i \in S$. This given, it will be convenient to set

$$\Phi_\mu^{=S} := \Phi_\mu^{\epsilon(S)} \ .$$

We should note that this notation is consistent with the fact that

$$\Phi_\mu^{\epsilon(S)} \;=\; \mathcal{F}\big(\bigcap_{i\in S}\mathbf{M}_{\alpha^{(i)}} \cap \bigcap_{j\notin S}\mathbf{M}^{\perp}_{\alpha^{(j)}}\big) \;.$$

In the same vein we let

$$\Phi_\mu^{\supseteq S} \;=\; \mathcal{F}\big(\bigcap_{i\in S}\mathbf{M}_{\alpha^{(i)}}\big) \;.$$

Note that since

$$\bigcap_{i\in S}\mathbf{M}_{\alpha^{(i)}} \;=\; \bigoplus_{T\supseteq S}\mathbf{M}^{\epsilon(T)} \;,$$

we deduce that

$$\Phi_\mu^{\supseteq S} \;=\; \sum_{T\supseteq S}\Phi_\mu^{=T} \;. \tag{3.8}$$

Proposition 3.2 *For every subset $S\subseteq\{1,2,\ldots,m\}$ we have*

$$\Phi_\mu^{\supseteq S} \;=\; \sum_{k=0}^{m}\Phi^{(k)}e_{m-k}\Big[\frac{1}{T_1}+\frac{1}{T_2}+\cdots+\frac{1}{T_m} \;-\; \sum_{i\in S}\frac{1}{T_i}\Big] \;. \tag{3.9}$$

Proof. Using formula 3.1, the identity in 3.8 reduces to

$$\Phi_\mu^{\supseteq S} \;=\; \sum_{k=0}^{m}\Phi^{(k)}\sum_{T\supseteq S\;\&|T|=k}\;\prod_{i\notin T}\frac{1}{T_i}$$

and this may also be rewritten as in 3.9.

It will be convenient to denote by $\mathbf{\Sigma}_\mu$ the vector space spanned by the symmetric functions $\widetilde{H}_{\alpha^{(i)}}$ for $i=1,\ldots,m$. In symbols

$$\mathbf{\Sigma}_\mu \;=\; \mathcal{L}[\widetilde{H}_{\alpha^{(1)}},\widetilde{H}_{\alpha^{(2)}}\;\ldots,\widetilde{H}_{\alpha^{(m)}}] \tag{3.10}$$

We will make extensive use here of the operator ∇, acting on symmetric polynomials, which gives

$$\nabla\,\widetilde{H}_\lambda \;=\; T_\lambda\,\widetilde{H}_\lambda \qquad (\,\forall\;\lambda\,)\;. \tag{3.11}$$

Since it can be shown (see [7]) that the symmetric polynomials $\widetilde{H}_\lambda$ ($\lambda\vdash n$) form a basis (of the space of hogeneous symmetric functions of degree n), formula 3.11 may be used as the definition of ∇.

We should note that, using the notation adopted in the introduction, we can write

$$\nabla\widetilde{H}_{\alpha^{(i)}} \;=\; T_i\;\widetilde{H}_{\alpha^{(i)}} \;. \tag{3.12}$$

Note further that if a_i and l_i respectively denote the coarm and coleg of the cell we must remove from μ to get $\alpha^{(i)}$ then setting

$$x_i = t^{l_i}q^{a_i} \qquad\qquad (\,i=1,\ldots,m\,) \tag{3.13}$$

we have

$$T_i \;=\; T_\mu/x_i \;. \tag{3.14}$$

In particular we see that the monomials $T_1,T_2,\ldots,T_m$ are all distinct. This enables us to write every element of $\mathbf{\Sigma}_\mu$ as a polynomial in ∇ applied to a single element of $\mathbf{\Sigma}$. In fact, any element

$$\Phi \;=\; \sum_{j=1}^{m}c_j\;\widetilde{H}_{\alpha^{(j)}}$$

with all $c_i \neq 0$ may be used for this purpose. However, the most convenient one is the element

$$\Phi_\mu \;=\; \sum_{j=1}^{m} \Big(\prod_{s=1,s\neq j}^{m} \frac{1}{(1-T_j/T_s)} \Big) \; \widetilde{H}_{\alpha^{(j)}} \;, \tag{3.15}$$

for, we have the following

Proposition 3.3 *If*

$$\Phi \;=\; \sum_{j=1}^{m} c_j \; \widetilde{H}_{\alpha^{(j)}}$$

then

$$\Phi \;=\; P(\nabla)\, \Phi_\mu$$

with

$$P(\nabla) \;=\; \sum_{i=1}^{m} c_i \prod_{s=1,s\neq i}^{m} (1-\nabla/T_s)$$

Proof. From 3.12 it follow that

$$\prod_{s=1,s\neq i}^{m} \Big(1-\frac{\nabla}{T_s}\Big) \; \Phi_\mu = \Big(\prod_{s=1,s\neq i}^{m} \frac{1}{(1-T_i/T_s)} \Big) \prod_{s=1,s\neq i}^{m} \Big(1-\frac{T_i}{T_s}\Big) \; \widetilde{H}_{\alpha^{(i)}} = \widetilde{H}_{\alpha^{(i)}} \;, \tag{3.16}$$

and the result follows by linearity.

Surprisingly, it develops that Φ_μ is none other than $\Phi^{(m)}$ itself, namely the polynomial giving the bigraded Frobenius characteristic of the intersection of the modules $\mathbf{M}_{\alpha^{(i)}}$. More generally, we have the following remarkable identities:

Theorem 3.1 *For $k = 1, .., m$*

$$\Phi^{(k)} \;=\; (-\nabla)^{m-k} \, \Phi_\mu \;, \tag{3.17}$$

and for any $S \subseteq \{1, 2, \dots, m\}$

$$\Phi^{\supseteq S} \;=\; \prod_{i\notin S} \Big(1 - \frac{\nabla}{T_i} \Big) \; \Phi_\mu \;. \tag{3.18}$$

We also have for $i = 1, .., m$

$$\begin{aligned} a)\quad \widetilde{H}_{\alpha^{(i)}}(x;q,t) \;&=\; \prod_{s=1,s\neq i}^{m} \Big(1-\frac{\nabla}{T_s}\Big) \; \Phi_\mu \;, \\ b)\quad \widetilde{H}_{\alpha^{(i)}}(x;q,t) \;&=\; \sum_{k=1}^{m} \Phi^{(k)} \; e_{m-k}\Big[\frac{1}{T_1} + \frac{1}{T_2} + \cdots + \frac{1}{T_m} \;-\; \frac{1}{T_i} \Big] \;. \end{aligned} \tag{3.19}$$

Thus the $\Phi^{(k)}$ form a basis for $\mathbf{\Sigma}_\mu$. They may be also be explicitely computed by means of the following formula:

$$\Phi^{(k)} \;=\; \sum_{j=1}^{m} \Big(\prod_{s=1,s\neq j}^{m} \frac{1}{(1-T_j/T_s)} \Big) \; (-T_j)^{m-k} \; \widetilde{H}_{\alpha^{(j)}} \;. \tag{3.20}$$

Proof. Formula 3.19 a) is 3.16 and 3.19 b) is the special case $S = \{i\}$ of 3.9. This not only implies that the $\Phi^{(k)}$ are a basis for $\mathbf{\Sigma}_\mu$ but also that the matrix

$$\Big\| e_{m-k}\Big[\frac{1}{T_1} + \cdots + \frac{1}{T_m} \;-\; \frac{1}{T_i} \Big] \Big\|_{k,i=1}^{m} \tag{3.21}$$

is non-singular. Expanding the right-hand side of 3.19 b) as a polynomial in ∇ we derive that

$$\widetilde{H}_{\alpha^{(i)}}(x;q,t) \;=\; \sum_{k=1}^{m} (-\nabla)^{m-k}\, \Phi_\mu \; e_{m-k}\Big[\frac{1}{T_1}+\frac{1}{T_2}+\cdots+\frac{1}{T_m} \;-\; \frac{1}{T_i}\Big]\ .$$

Comparing this with 3.19 b) and using the non-singularity of the matrix in 3.21, we derive that 3.17 must hold true for all $k = 1,..,m$. This given, formula 3.20 follows immediately by applying $(-\nabla)^{m-k}$ to both sides of 3.15 and using 3.12. In the same vein we see that, using 3.17, 3.9 can be rewritten in the form

$$\Phi_{\mu}^{\supseteq S} \;=\; \sum_{k=0}^{m-|S|} (-\nabla)^k\, e_k\Big[\sum_{i\notin S}\frac{1}{T_i}\Big]\ ,$$

This gives 3.18 and completes our proof.

As a corollary of 3.18 we obtain a recursive way of computing Φ_μ as well as any of the $\Phi_{\mu}^{\supseteq S}$.
For given quantities $y_1, y_2, y_3, \ldots\ ; A_1, A_2, A_3, \ldots$ we recursively set (for $k > 2$)

$$\Delta[y_1,\ldots,y_k;A_1,\ldots,A_k] \;=\; \frac{y_1\,\Delta[y_1,\ldots,y_{k-1};A_1,\ldots,A_{k-1}] \;-\; \Delta[y_2,\ldots,y_k;A_2,\ldots,A_k]\,y_k}{y_1 \;-\; y_k}$$

with

$$\Delta[y_1,y_2;A_1,A_2] \;=\; \frac{y_1\,A_1 \;-\; A_2\,y_2}{y_1 \;-\; y_2} \qquad (\text{ for } \quad k=2\ )$$

Proposition 3.4 *For* $S = \{1 \le i_1 < i_2 < \cdots < i_k \le m\}$ *we have*

$$\Phi_{\mu}^{\supseteq S} \;=\; \Delta\Big[\frac{1}{T_{i_1}},\ldots,\frac{1}{T_{i_k}};\widetilde{H}_{\alpha^{i_1}},\ldots,\widetilde{H}_{\alpha^{i_k}}\Big] \qquad 3.22$$

Proof. Let T be a subset of $\{1,2,\ldots,m\}$ and let i be smaller and j bigger than any element of T, then note that

$$\begin{aligned}\frac{1}{T_i}\prod_{s\notin T\cup\{i\}}\Big(1-\frac{\nabla}{T_s}\Big)-\frac{1}{T_j}\prod_{s\notin T\cup\{j\}}\Big(1-\frac{\nabla}{T_s}\Big) \;&=\; \Big(\frac{1}{T_i}\Big(1-\frac{\nabla}{T_j}\Big)-\frac{1}{T_j}\Big(1-\frac{\nabla}{T_i}\Big)\Big)\prod_{s\notin T\cup\{i,j\}}\Big(1-\frac{\nabla}{T_s}\Big)\\ &=\; \Big(\frac{1}{T_i} \;-\; \frac{1}{T_j}\Big)\prod_{s\notin T\cup\{i,j\}}\Big(1-\frac{\nabla}{T_s}\Big)\ .\end{aligned} \qquad 3.23$$

Applying both sides to Φ_μ and using 3.18 gives

$$\Big(\frac{1}{T_i}\ \Phi_{\mu}^{\supseteq T\cup\{i\}} \;-\; \frac{1}{T_j}\ \Phi_{\mu}^{\supseteq T\cup\{j\}}\Big)\Big/\Big(\frac{1}{T_i} \;-\; \frac{1}{T_j}\Big) \;=\; \Phi_{\mu}^{\supseteq T\cup\{i,j\}} \qquad 3.24$$

and 3.22 follows by induction on k and our definition of the divided difference operator Δ.

Of course all of these expressions and identities can be viewed solely as part of the theory of Macdonald polynomials. However, the SF heuristic associates a meaning to them which suggests manipulations, results and conjectures that are difficult to predict without it. For instance, the case $T = \emptyset$ of 3.23 is

$$\Big(\frac{1}{T_i}\ \widetilde{H}_{\alpha^{(i)}} \;-\; \frac{1}{T_j}\ \widetilde{H}_{\alpha^{(j)}}\Big)\Big/\Big(\frac{1}{T_i} \;-\; \frac{1}{T_j}\Big) \;=\; \Phi_{\mu}^{\supseteq\{i,j\}} \qquad 3.25$$

and the SF heuristic states that the divided difference on the left-hand side must be Schur positive since it should give the Frobenius characteristic of the intersection of the modules $\mathbf{M}_{\alpha^{(i)}}$ and $\mathbf{M}_{\alpha^{(j)}}$. Of course the further divided differences given by 3.23 should turn out to be Schur positive as well. Now these particular facts can be verified by computer much more extensively and for much larger partitions than the partitions for which we can carry out the verification of the SF heuristic. Although such experimental verifications give support to the SF heuristic, more substantial support comes from results (predicted by SF) which we can actually prove in full generality from within the theory of Macdonald polynomials. For instance, it can be shown (see [4]) that for all $\lambda = j1^{n-j}$ and many other infinite families of partitions, the coefficients of S_λ in $\Phi_\mu^{(k)}$ is (as expected) in $\mathbf{N}[q,t]$. In fact, Bergeron-Hamel derive many explicit plethystic expressions for these coefficients from those given by Garsia-Tesler in [12]. To emphasize this point we will carry out a calculation which supports part (v) of SF. This is the assertion that the dimension of of any k of the modules $\mathbf{M}_{\alpha^{(i)}}$ is $n!/k$. The verification of this property is based on the following.

Proposition 3.5 *For any $S = \{1 \le i_1 < i_2 < \cdots < i_k \le m\}$ we have*

$$\Phi_\mu^{\supseteq S} \;=\; \sum_{i_s \in S} \Bigg(\prod_{\substack{i_r \in S \\ i_r \neq i_s}} \frac{1}{1 - T_{i_s}/T_{i_r}} \Bigg) \widetilde{H}_{\alpha^{(i_s)}} \tag{3.26}$$

In particular, under SF, the Hilbert series of the module $\bigcap_{s=1}^{k} \mathbf{M}_{\alpha^{(i_s)}}$ is given by the rational expression

$$F_{\supseteq S}(q,t) \;=\; \sum_{i_s \in S} \Bigg(\prod_{\substack{i_r \in S \\ i_r \neq i_s}} \frac{1}{1 - T_{i_s}/T_{i_r}} \Bigg) \partial_{p_1}^{n} \widetilde{H}_{\alpha^{(i_s)}} \tag{3.27}$$

Proof. Substituting in 3.18 Φ_μ as given by 3.15 and using 3.12 we obtain

$$\Phi_\mu^{\supseteq S} \;=\; \sum_{i \in S} \Bigg(\prod_{\substack{j=1 \\ j \neq i}}^{m} \frac{1}{1 - T_i/T_j} \Bigg) \prod_{r \notin S} \Big(1 - T_i/T_r\Big)\; \widetilde{H}_{\alpha^{(i)}}$$

making the appropriate cancellations gives 3.26 and 3.27 follows by taking the n^{th} derivative of both sides of 3.26 with respect to p_1.

Of course, on the validity of SF, the expression on the right-hand side of 3.27 should evaluate to a polynomial with positive integer coefficients and the dimension of the module $\bigcap_{s=1}^{k} \mathbf{M}_{\alpha^{(i_s)}}$ could then be simply obtained by evaluating $F_{\supseteq S}(q,t)$ at $t = q = 1$. What we shall show here is that for any $\mu \vdash n+1$ and S of cardinality k we have

$$\lim_{q \to 1} F_{\supseteq S}(q,1) \;=\; \frac{n!}{k} \tag{3.28}$$

Using one of the specializations computed by Macdonald in [21] (see [10]) it can be shown that for any partition $\beta = (\beta_1, \beta_2, \ldots, \beta_l)$ we have

$$\widetilde{H}_\beta(x;q,1) \;=\; \prod_{i=1}^{l} (q)_{\beta_i}\, h_{\beta_i}\big[\tfrac{X}{1-q}\big] \ . \tag{3.29}$$

where for any index s we set $(q)_s = (1-q)(1-q^2)\cdots(1-q^s)$. Assuming the row that contains the corner cell we remove from μ to obtain $\alpha^{(i)}$ has length a_i we deduce from 3.29 that

$$\widetilde{H}_{\alpha^{(i)}}(x;q,1) \;=\; \widetilde{H}_\gamma(s;q,1)\,(q)_{a_i-1} h_{a_i-1}\big[\tfrac{X}{1-q}\big] \prod_{\substack{j=1 \\ j \neq i}}^{m} (q)_{a_j} h_{a_j}\big[\tfrac{X}{1-q}\big] \tag{3.30}$$

where γ is the partition obtained by removing from μ all the rows that contain corner cells.

Now it is easily seen that

$$T_{\alpha^{(i)}}/T_{\alpha^{(j)}}\big|_{t=1} \;=\; q^{a_j}/q^{a_i} \;.$$

Thus from 3.27 we derive

$$F_{\supseteq S}(q,1) \;=\; \sum_{i_s\in S}\Bigg(\prod_{\substack{i_r\in S\\ i_r\neq i_s}} \frac{q^{a_{i_s}}}{q^{a_{i_s}}-q^{a_{i_r}}}\Bigg)\;\partial_{p_1}^n \widetilde{H}_{\alpha^{(i_s)}}(x;q,1)\;. \qquad 3.31$$

Note that if $a_1+a_2+\cdots+a_m-1=r$ then $|\gamma|=n-r$. Let us set for a moment

$$F_\gamma(q,t) \;=\; \partial_{p_1}^{n-r}\widetilde{H}_\gamma(x;q,t)\;.$$

This given, applying $\partial_{p_1}^n$ to both sides of 3.30 and using Leibnitz formula we obtain

$$\begin{aligned}\partial_{p_1}^n\,\widetilde{H}_{\alpha^{(i)}}(x;q,1) \;&=\; \frac{n!}{(n-r)!}\,F_\gamma(q,1)\,\frac{[a_i-1]_q!}{(a_i-1)!}\prod_{\substack{j=1\\ j\neq i}}^{m}\frac{[a_j]_q!}{a_j!}\\ &=\; \frac{n!}{(n-r)!}\,F_\gamma(q,1)\Big(\prod_{j=1}^{m}\frac{[a_j]_q!}{a_j!}\Big)\,\frac{(a_i)}{[a_i]_q}\;.\end{aligned} \qquad 3.32$$

Where as customary, for any integer l we set $[l]_q=(1-q^l)/(1-q)$ and $[l]_q!=(q)_l/(1-q)^l$.

Setting $i=i_s$ in 3.32 and substituting in 3.31 we may write

$$\frac{F_{\supseteq S}(q,t)}{\frac{n!}{(n-r)!}\,F_\gamma(q,1)\Big(\prod_{j=1}^m\frac{[a_j]_q!}{a_j!}\Big)} \;=\; \sum_{i_s\in S}\Bigg(\prod_{\substack{i_r\in S\\ i_r\neq i_s}}\frac{q^{a_{i_s}}}{q^{a_{i_s}}-q^{a_{i_r}}}\Bigg)\,\frac{(a_{i_s})}{[a_{i_s}]_q}\;.$$

Now we know (see [10]) that $\lim_{q\to1}\,F_\gamma(q,1)=(n-r)!$. Moreover it is easy to see that all the ratios $[a_j]_q!/a_j!$ tend to 1 as $q\to1$. Thus we see that the identity in 3.28 is a simple consequence of the following elementary fact:

Lemma 3.1 *For any distinct integers $y_1,y_2,\ldots,y_k$ we have*

$$\lim_{q\to1}\,(1-q)\sum_{s=1}^{k}\Bigg(\prod_{\substack{1\le r\le k\\ r\neq s}}\frac{q^{y_s}}{q^{y_s}-q^{y_r}}\Bigg)\,\frac{y_s}{1-q^{y_s}} \;=\; \frac{1}{k} \qquad 3.33$$

Proof. We give a sketch of the argument since the details are easily filled. Our point of departure is the partial fraction decomposition

$$\prod_{s=1}^{k}\frac{1}{1-t\,q^{y_s}} \;=\; \sum_{s=1}^{k}\Bigg(\prod_{\substack{1\le r\le k\\ r\neq s}}\frac{q^{y_s}}{q^{y_s}-q^{y_r}}\Bigg)\,\frac{1}{1-t\,q^{y_s}}\;. \qquad 3.34$$

Now the trick in proving 3.33 is to use the expansion

$$\frac{y_s}{1-q^{y_s}} \;=\; \frac{1}{\log_e 1/q}\sum_{p\geq1}\frac{1}{p}\,(1-q^{y_s})^{p-1} \qquad 3.35$$

and noting that, since the term multiplying $y_s/(1-q^{y_s})$ in 3.33 has only a pole of order $k-1$ at $q=1$, in calculating our limit we need only the first k terms of the series in 3.35.

Since 3.34 gives that for any integer $a \geq 0$

$$\sum_{s=1}^{k} \Big(\prod_{\substack{1\leq r\leq k \\ r\neq s}} \frac{q^{y_s}}{q^{y_s}-q^{y_r}} \Big) \, q^{a y_s} \;=\; h_a[q^{y_1}, q^{y_2}, \ldots, q^{y_k}] \;\rightarrow\; \binom{a+k-1}{a} \qquad (\text{ as } \quad q \rightarrow 1 \text{) },$$

we see that 3.33 is reduced to showing that

$$\sum_{p=1}^{k} \frac{1}{p} \sum_{a=0}^{p-1} (-1)^a \binom{p-1}{a} \binom{a+k-1}{a} \;=\; \frac{1}{k} \; .$$

However, this turns out to be an immediate consequence of the classical Gauss summation formula.

Our proof of 3.28 is thus complete and we have verified that property (v) of the SF heuristic is consistent with Macdonald theory.

We terminate this section with three curious identities which, on the validity of SF, imply further Schur positivity results.

Theorem 3.2

$$\mathcal{F} \bigvee_{i=1}^{m} \mathbf{M}_{\alpha^{(i)}} \;=\; \sum_{k=1}^{m} \Phi_\mu^{(k)} \, e_{m-k}\Big[\frac{1}{T_1}+\frac{1}{T_2}+\cdots+\frac{1}{T_m}\Big] \;=\; \frac{\nabla \, \Phi_\mu^{(1)}}{T_1 T_2 \cdots T_m} \;, \qquad 3.36$$

$$\sum_{k=1}^{m} \Phi^{(k)} \, e_{m+1-k}\Big[\frac{1}{T_1}+\frac{1}{T_2}+\cdots+\frac{1}{T_m}\Big] \;=\; \nabla^{-1}\Phi_\mu \;. \qquad 3.37$$

Proof. We clearly have the decomposition

$$\bigvee_{i=1}^{m} \mathbf{M}_{\alpha^{(i)}} \;=\; \bigoplus_{\substack{S \subseteq \{1,2,..,m\} \\ S \neq \emptyset}} \mathbf{M}^{\epsilon(S)} \;.$$

Thus

$$\begin{aligned} \mathcal{F} \bigvee_{i=1}^{m} \mathbf{M}_{\alpha^{(i)}} \;&=\; \sum_{\substack{S \subseteq \{1,2,..,m\} \\ S \neq \emptyset}} \Phi_\mu^{=S} \\ &=\; \sum_{k=1}^{m} \Phi^{(k)} \sum_{\substack{S \subseteq \{1,2,..,m\} \\ |S| = k}} \prod_{i \notin S} \frac{1}{T_i} \;. \end{aligned}$$

This shows the first equality in 3.36. For the second equality we note that we can write

$$\begin{aligned} \sum_{k=1}^{m} \Phi^{(k)} \, e_{m-k}\Big[\frac{1}{T_1}+\frac{1}{T_2}+\cdots+\frac{1}{T_m}\Big] \;&=\; \sum_{k=1}^{m} (-\nabla)^{m-k} \, \Phi_\mu \; e_{m-k}\Big[\frac{1}{T_1}+\frac{1}{T_2}+\cdots+\frac{1}{T_m}\Big] \\ &=\; \Big\{ \prod_{i=1}^{m} \Big(1-\frac{\nabla}{T_i}\Big) \;-\; (-1)^m \, \frac{\nabla^m}{T_1 T_2 \cdots T_m} \Big\} \; \Phi_\mu \\ &=\; (-1)^{m-1} \, \frac{\nabla^m}{T_1 T_2 \cdots T_m} \, \Phi_\mu \end{aligned}$$

and the result follows from 3.17 for $k = 1$.

This given, applying ∇^{-1} to the second equality in 3.36, and using again the equalities in 3.17 we get:

$$- \sum_{k=1}^{m-1} \Phi_\mu^{(k+1)} e_{m-k} \left[\frac{1}{T_1} + \frac{1}{T_2} + \cdots + \frac{1}{T_m} \right] \; + \; \nabla^{-1} \Phi_\mu \; = \; \frac{\Phi_\mu^{(1)}}{T_1 T_2 \cdots T_m} \; ,$$

and 3.37 follows by rearranging terms and changing the index of summation.

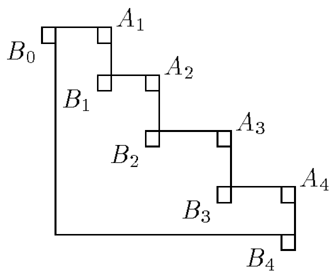
To state our next result we need further ingredients. Let us recall that in 3.13 we have set $x_i = t^{l_i} q^{a_i}$ with l_i and a_i denoting the coleg and coarm of the cell we must remove from μ to get $\alpha^{(i)}$. We may refer to x_i as the "weight" of the i^{th} "outer" corner cell of μ. We shall need here also the monomials

$$u_i \; = \; t^{l_{i+1}} q^{a_i} \qquad (\, i = 1, 2, \ldots, m-1 \,) \tag{3.38}$$

giving the weights of the "inner" corner cells. We shall also set

$$u_0 \; = \; t^{l_1}/q \quad , \quad u_m \; = \; q^{a_m}/t \quad , \quad x_0 \; = \; 1/tq \tag{3.39}$$

To appreciate the geometric significance of these weights, in the figure below we illustrate a 4-corner case with outer corner cells labelled A_1, A_2, A_3, A_4 and inner corner cells labelled B_1, B_2, B_3, B_4.



We recall (see [7]) that one of the Pieri rules given by Macdonald may be written in the form

$$\partial_{p_1} \widetilde{H}_\mu \; = \; \sum_{\nu \to \mu} c_{\mu\nu}(q,t) \, \widetilde{H}_\nu$$

where ∂_{p_1} denotes the Hall inner product adjoint of multiplication by p_1 and $\nu \to \mu$ is to indicate that the sum is to be carried out over the partitions ν which immediately precede μ in the Young order. In our present notation we can write this identity in the form

$$\partial_{p_1} \widetilde{H}_\mu \; = \; \sum_{i=1}^{m} c_{\mu\alpha^{(i)}}(q,t) \, \widetilde{H}_{\alpha^{(i)}} \tag{3.40}$$

It is well known and it is easy to show that if Ψ is the Frobenius characteristic of an S_n module $\mathbf{M}$ then $\partial_{p_1} \Psi$ gives the Frobenius characteristic of the module $\mathbf{M}$ *restricted to* S_{n-1}. Thus, under the $C = \widetilde{H}$ conjecture the polynomial $\partial_{p_1} \widetilde{H}_\mu$ should give the Frobenius characteristic of $\mathbf{M}_\mu$ as an S_{n-1}-module.

Our next and final task in this section is to compute the expansion of this characteristic in terms of our polynomials $\Phi_\mu^{(k)}$. This computation became possible only recently due to the discovery in [12] that the Macdonald Pieri coefficients $c_{\mu\nu}$ reduce to a remarkably simple form when expressed in terms of the weights of the "outer" and "inner" corner cells of μ. This result may be stated as follows.

Proposition 3.6 *For any partition μ with predecessors $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(m)}$ and corner weights $x_0, x_1, \ldots, x_m$, $u_0, u_1, \ldots, u_m$ given by 3.13 , 3.38 and 3.39 we have*

$$\partial_{p_1} \widetilde{H}_\mu \;=\; \frac{1}{M} \sum_{i=1}^{m} \frac{1}{x_i} \frac{\prod_{s=0}^{m}(u_s - x_i)}{\prod_{s=1,s\neq i}^{m}(x_s - x_i)} \; \widetilde{H}_{\alpha^{(i)}} \;, \tag{3.41}$$

where for convenience we have set

$$M = (1 - 1/t)(1 - 1/q) \;. \tag{3.42}$$

The proof is given in [12] (see Proposition 2.4 there).

Combining this result with our expansions of $\widetilde{H}_{\alpha^{(i)}}$ we obtain the following remarkable identities.

Theorem 3.3 *Under the same hypotheses*

$$\partial_{p_1} \widetilde{H}_\mu \;=\; \frac{1}{M}\, \frac{T_\mu}{\nabla} \left\{ \prod_{s=0}^{m} \Big(1 - \frac{\nabla}{T_\mu}\, u_s\Big) - \prod_{s=0}^{m} \Big(1 - \frac{\nabla}{T_\mu}\, x_s\Big) \right\} \Phi_\mu \;, \tag{3.43}$$

or equivalently:

$$\partial_{p_1} \widetilde{H}_\mu \;=\; \sum_{k=1}^{m} \frac{\Phi_\mu^{(k)}}{T_\mu^{m-k}}\, \frac{e_{m+1-k}[x_0 + \cdots + x_m] - e_{m+1-k}[u_0 + \cdots + u_m]}{(1-1/t)(1-1/q)} \tag{3.44}$$

Proof. It is easy to verify from the definitions 3.13 , 3.38 and 3.39 that we have

$$x_0 x_1 \cdots x_m \;=\; u_0 u_1 \cdots u_m \;. \tag{3.45}$$

Thus the expression

$$\frac{1}{z} \left\{ \prod_{s=0}^{m} \Big(1 - z\, u_s\Big) - \prod_{s=0}^{m} \Big(1 - z\, x_s\Big) \right\}$$

defines a polynomial in z of degree $m - 1$. Using the Lagrange interpolation formula at $z = x_1, x_2, \ldots, x_m$ we then get that

$$\frac{1}{z} \left\{ \prod_{s=0}^{m} \Big(1 - z\, u_s\Big) - \prod_{s=0}^{m} \Big(1 - z\, x_s\Big) \right\} \;=\; \sum_{i=1}^{m} x_i \prod_{s=0}^{m} \Big(1 - u_s/x_i\Big) \frac{\prod_{s=1,\neq i}^{m} \Big(1 - z\, x_s\Big)}{\prod_{s=1,\neq i}^{m} \Big(1 - x_s/x_i\Big)}$$

and this may also be rewritten as

$$\frac{1}{z} \left\{ \prod_{s=0}^{m} \Big(1 - z\, u_s\Big) - \prod_{s=0}^{m} \Big(1 - z\, x_s\Big) \right\} \;=\; \sum_{i=1}^{m} \frac{1}{x_i} \frac{\prod_{s=0}^{m}(x_i - u_s)}{\prod_{s=1,\neq i}^{m}(x_i - x_s)} \prod_{s=1,\neq i}^{m} \Big(1 - z\, x_s\Big) \;. \tag{3.46}$$

Setting $z = \frac{\nabla}{T_\mu}$ and applying both sides to Φ_μ gives

$$\frac{T_\mu}{\nabla} \left\{ \prod_{s=0}^{m} \Big(1 - \frac{\nabla}{T_\mu}\, u_s\Big) - \prod_{s=0}^{m} \Big(1 - \frac{\nabla}{T_\mu}\, x_s\Big) \right\} \Phi_\mu \;=\; \sum_{i=1}^{m} \frac{1}{x_i} \frac{\prod_{s=0}^{m}(x_i - u_s)}{\prod_{s=1,\neq i}^{m}(x_i - x_s)} \prod_{s=1,\neq i}^{m} \Big(1 - \frac{\nabla}{T_\mu}\, x_s\Big) \Phi_\mu \;. \tag{3.47}$$

Since $x_s/T_\mu = 1/T_s$ for $s = 1, 2, \ldots, m$ we see from 3.19 a) that

$$\prod_{s=1,\neq i}^{m} \Big(1 - \frac{\nabla}{T_\mu}\, x_s\Big) \Phi_\mu \;=\; \widetilde{H}_{\alpha^{(i)}} \;.$$

Thus 3.42 follows by combining 3.41 and 3.47.

This given, note that expanding the products on the right-hand side of 3.43 we get

$$\partial_{p_1}\widetilde{H}_{\alpha^{(i)}} \;=\; \frac{T_\mu}{\nabla}\sum_{k=1}^{m}\Big(-\frac{\nabla}{T_\mu}\Big)^{m+1-k}\Phi_\mu \;\; \frac{e_{m+1-k}[u_0+\cdots+u_m]-e_{m+1-k}[x_0+\cdots+x_m]}{(1-1/t)(1-1/q)}\;,$$

and 3.43 follows from the identities in 3.17.

Remark 2. 3.1 Routine manipulations starting from the definition in I.9 yield that †

$$(1-1/t)(1-1/q)\,B_\mu(q,t) \;=\; x_0+x_1+\cdots+x_m \;-\; u_0-u_1-\cdots-u_m\;. \qquad 3.48$$

This given, we see from 3.44 that the coefficient of $\Phi_\mu^{(m)}=\Phi_\mu$ in $\partial_{p_1}\widetilde{H}_\mu$ is precisely by $B_\mu(q,t)$.

A simple calculation based on Property (iii) of SF (see eq. I.30) yield that for any $k=1,2,..,m$ we have

$$\downarrow\,\Phi_\mu^{(k)} \;=\; \frac{\Phi_\mu^{(m+1-k)}}{T_1T_2\cdots T_m} \qquad 3.49$$

Note also that replacing t,q by $1/t,1/q$ in 3.48, and using 3.45, we derive that

$$\begin{aligned}(1-t)(1-q)\,B_\mu(1/q,1/t) \;&=\; \frac{1}{x_0}+\frac{1}{x_1}+\cdots+\frac{1}{x_m} \;-\; \frac{1}{u_0}-\frac{1}{u_1}-\cdots-\frac{1}{u_m}\\ &=\; \frac{e_m[x_0+\cdots+x_m] \;-\; e_m[u_0+\cdots+u_m]}{x_0x_1\cdots x_m}\end{aligned}$$

Since $x_0=1/tq$ and $x_i=T_\mu/T_i$ we deduce that

$$\frac{1}{T_\mu^{m-1}}\,\frac{e_m[x_0+\cdots+x_m] \;-\; e_m[u_0+\cdots+u_m]}{(1-1/t)(1-1/q)} \;=\; T_\mu\,B_\mu(1/q,1/t)\,\frac{1}{T_1T_2\cdots T_m}\;.$$

Thus using 3.49 we see that we may write 3.43 in the form

$$\partial_{p_1}\widetilde{H}_\mu \;=\; B_\mu(q,t)\,\Phi_\mu \;+\;\cdots\;+\; T_\mu\downarrow B_\mu(q,t)\,\Phi_\mu\;. \qquad 3.50$$

Note that in the two-corner case there are no further terms and under the $C=\widetilde{H}$-conjecture this identity yields

$$\partial_{p_1}C_\mu(x;q,t) \;=\; B_\mu(q,t)\,\Phi_\mu \;+\; T_\mu\downarrow B_\mu(q,t)\,\Phi_\mu\;. \qquad 3.51$$

which is formula 1.35. Thus we obtain here the general case of an identity we first encountered with $\mu=32$ (see 1.31).

We also see that the coefficients of $\Phi^{(1)}$ and $\Phi^{(m)}$ in 3.43 are polynomials in $q,t,1/q,1/t$ with positive integer coefficients. This is also true for the coefficients of the remaining $\Phi^{(k)}$. The basic result here may be stated as follows.

Theorem 3.4 *If $x_0,x_1,\ldots,x_m$ and $u_0,u_1,\ldots,u_m$ are the corner weights of a partition then for any $k=1,2,\ldots,m$ the expression*

$$B_\mu^{(k)}(q,t) \;=\; \frac{e_k[x_0+\cdots+x_m]-e_k[u_0+\cdots+u_m]}{(1-1/t)(1-1/q)}$$

evaluates to a polynomial in q,t with positive integer coefficients.

† See Proposition 2.3 in [12]

Proof. We owe the following very simple induction argument to Glen Tesler. Note first that for $m=1$ we have

$$x_0 = 1/qt\ ,\quad x_1 = t^{l_1} q^{a_1}\ ;\quad u_0 = t^{l_1}/q\ ,\quad u_1 = q^{a_1}/t\ .$$

Thus

$$B^{(1)}_\mu(q,t) \;=\; \frac{1/qt + t^{l_1}q^{a_1} - t^{l_1}/q - q^{a_1}/t}{(1-1/t)(1-1/q)} \;=\; [l_1+1]_t\,[a_1+1]_q\ .$$

Let us then assume the result true for all partitions with $m-1$ corners. For a partition μ with m corners and weights given by 3.13, 3.38 and 3.39 we write (see 3.42)

$$B^{(k)}_\mu = \frac{e_k[x_0 + X_{m-1} + x_m] - e_k[\tilde{x}_0 + X_{m-1} + u_m]}{M} + \frac{e_k[\tilde{x}_0 + X_{m-1} + u_m] - e_k[u_0 + U_{m-1} + u_m]}{M}$$

where for convenience we have set

$$\tilde{x}_0 = t^{l_m}/q\quad ;\quad X_{m-1} = x_1 + x_2 + \cdots + x_{m-1}\ ,\quad U_{m-1} = u_1 + u_2 + \cdots + u_{m-1}\ . \qquad 3.52$$

Now we have

$$\begin{aligned}
\frac{e_k[x_0 + X_{m-1} + x_m] - e_k[\tilde{x}_0 + X_{m-1} + u_m]}{M} &= \sum_{s=1}^{2} \frac{e_s[x_0 + x_m] - e_s[\tilde{x}_0 + u_m]}{M}\, e_{k-s}[X_{m-1}] \\
&= \frac{x_0 + x_m - \tilde{x}_0 - u_m}{M}\ e_{k-1}[X_{m-1}] \\
&= [l_m+1]_t[a_m+1]_q\ e_{k-1}[X_{m-1}]\ ,
\end{aligned}$$

where the second equality is due to the relation $x_0 x_m = \tilde{x}_0 u_m$.

Similarly we deduce that

$$\begin{aligned}
&\frac{e_k\left[\tilde{x}_0 + X_{m-1} + u_m\right] - e_k\left[u_0 + U_{m-1} + u_m\right]}{M} \\
&\qquad = \frac{e_k\left[\tilde{x}_0 + X_{m-1}\right] - e_k[u_0 + U_{m-1}]}{M} + \frac{e_{k-1}\left[\tilde{x}_0 + X_{m-1}\right] - e_{k-1}[u_0 + U_{m-1}]}{M}\ u_m
\end{aligned} \qquad 3.53$$

Note that the auxiliary monomial $\tilde{x}_0$ we introduced in 3.52 may be viewed as the weight of the cell $\tilde{A}_0$ that is immediately to the left of the diagram of μ and is in the same row as the cell A_m with weight x_m. Let $\tilde{\mu}$ denote the $m-1$-corner partition obtained by removing from μ all the rows on or below the cells $\tilde{A}_0$ and A_m. After drawing the diagrams of μ and $\tilde{\mu}$, a minute of reflection should reveal that the monomials

$$\frac{\tilde{x}_0}{t^{l_m}}\ ,\ \frac{x_1}{t^{l_m}}\ ,\ \ldots,\ \frac{x_{m-1}}{t^{l_m}}\quad ;\quad \frac{u_0}{t^{l_m}}\ ,\ \frac{u_1}{t^{l_m}}\ ,\ \ldots,\ \frac{u_{m-1}}{t^{l_m}}$$

are none other than the corner weights of the partition $\tilde{\mu}$.

This given, we can rewrite 3.53 in the form

$$\frac{e_k\left[\tilde{x}_0 + X_{m-1} + u_m\right] - e_k\left[u_0 + U_{m-1} + u_m\right]}{M} \;=\; t^{kl_m}\ B^{(k)}_{\tilde{\mu}}(q,t) \;+\; u_m\, t^{(k-1)l_m} B^{(k-1)}_{\tilde{\mu}}(q,t)\ .$$

In summary we have shown that

$$B^{(k)}_\mu(q,t) \;=\; [l_m+1]_t[a_m+1]_q\, e_{k-1}[X_{m-1}] \;+\; t^{kl_m}\ B^{(k)}_{\tilde{\mu}}(q,t) \;+\; u_m\, t^{(k-1)l_m} B^{(k-1)}_{\tilde{\mu}}(q,t)\ .$$

which proves the positive integrality of $B^{(k)}_\mu(q,t)$ and completes our induction argument.

We shall see in the next section that the positivity of these coefficients expresses a remarkable combinatorial and representation theoretical process intimately connected with the SF heuristic..

4. Dissecting $\mathbf{M}_\mu$ as an S_{n-1}-module

We have seen in section 3 that, at least in the case of the partition $\mu = 321$, it is possible to use the bases $\mathcal{B}^{\epsilon_1\epsilon_2\epsilon_3}$ to construct a basis which decomposes $\mathbf{M}_{321}$ into a direct sum of six left regular representations of S_5. In summary for each cell $(i,j) \in 321$ we constructed a basis $\mathcal{B}_{ij}$ as a union of bases $\mathcal{B}^{\epsilon_1\epsilon_2\epsilon_3}$ then the basis for $\mathbf{M}_{321}$ was taken to be of the form

$$\mathcal{B}_{321} \;=\; \bigcup_{(i,j)\in 321} \mathcal{B}_{ij}(\partial)\, \partial_{x_6}^i \partial_{y_6}^j \Delta_{321} \;. \tag{4.1}$$

Note that our construction yields that the space $\mathbf{M}_{ij}$ spanned by the basis $\mathcal{B}_{ij}$ is a direct sum of the S_5-modules $\mathbf{M}^{\epsilon_1\epsilon_2\epsilon_3}$ and thus is itself an S_5-module. Using this fact, it is not difficult to derive that our construction also yields the direct sum decomposition

$$\mathbf{M}_{321}\Big|_{s_5} \;=\; \bigoplus_{(i,j)\in 321} \mathbf{flip}_{\Delta_{321}(ij)}\; \mathbf{M}_{ij} \;, \tag{4.2}$$

where for convenience we have set

$$\Delta_{321}(ij) \;=\; \partial_{x_6}^i \partial_{y_6}^j \Delta_{321} \;. \tag{4.3}$$

To be sure, the polynomial $\Delta_{321}(ij)$ is not one of our Δ_μ. Nevertheless it is an S_5 alternant and this is all that is needed for the corresponding **flip** map to have all of the properties we need. In particular, letting Φ_{ij} denote the bigraded Frobenius characteristic of $\mathbf{M}_{ij}$, we can deduce from 4.2 that the bigraded Frobenius characteristic $C_{321}(x;q,t)$ satisfies the equation

$$\partial_{p_1} C_{321}(x;q,t) \;=\; \sum_{(i,j)\in 321} \frac{T_{321}}{t^i q^j} \downarrow \Phi_{ij} \;. \tag{4.4}$$

Since for any partition μ we have $T_\mu \downarrow C_\mu(x,q,t) = C_\mu(x,q,t)$, applying $\downarrow$ to both sides of 4.4 and multiplying by T_{321} we derive that

$$\partial_{p_1} C_{321}(x;q,t) \;=\; \sum_{(i,j)\in 321} \Phi_{ij}\, t^i q^j \;. \tag{4.5}$$

Looking back at the display of bases $\mathcal{B}_{ij}$ given in section 2, and using the notation we introduced in section 3, we can easily deduce that in this case

$$\begin{aligned}
\Phi_{00} &= \Phi^{(3)} + \tfrac{1}{T_{321}}\big(x_1+x_2+x_3\big)\Phi^{(2)} + \tfrac{1}{T_{321}^2}\big(x_1x_2+x_1x_3+x_2x_3\big)\Phi^{(1)}\\
\Phi_{01} &= \Phi^{(3)} + \tfrac{1}{T_{321}}\big(x_1+x_2+x_3\big)\Phi^{(2)}\\
\Phi_{10} &= \Phi^{(3)} + \tfrac{1}{T_{321}}\big(x_1+x_2+x_3\big)\Phi^{(2)} + \tfrac{1}{T_{321}^2}\big(x_1x_3+x_2x_3\big)\Phi^{(1)}\\
\Phi_{20} &= \Phi^{(3)} + \tfrac{1}{T_{321}}\big(x_2+x_3\big)\Phi^{(2)} + \tfrac{1}{T_{321}^2}\, x_2x_3\, \Phi^{(1)}\\
\Phi_{11} &= \Phi^{(3)} + \tfrac{1}{T_{321}}\, x_3\, \Phi^{(2)}\\
\Phi_{02} &= \Phi^{(3)}
\end{aligned} \tag{4.6}$$

To test the consistency of all our conjectures we should want to verify that substituting these expressions for the Φ_{ij} in 4.5 we do obtain the same final expression for $\partial_{p_1} C_{321}(x;q,t)$ that can be computed by means of 3.44. Taking account that in this case we have

$$T_{321} = t^3q^3\;;\quad \begin{matrix} x_o = 1/tq\,, & x_1 = t^2\,, & x_2 = tq\,, & x_3 = q^2\,,\\ u_o = t^2/q\,, & u_1 = t\,, & u_2 = q\,, & u_3 = q^2/t\,, \end{matrix}$$

it can be somewhat tediously verified that, in fact, 4.5 and 3.44 do turn out to be in complete agreement. We shall not do so here since, very shortly, we will prove a general result that includes this verification as a particular case. The assignment of modules $\mathbf{M}_{ij}$ that yields the Frobenius characteristics Φ_{ij} can best be visualized by representing each $\mathbf{M}_{ij}$ by the appropriate union of pieces of the Venn diagram for $\mu = 321$ and placing the corresponding figure in the cell (i, j). This results in the diagram given below.

Upon close examination of the process that led to these choices of the modules $\mathbf{M}_{ij}$ for the partition $\mu = 321$, Mark Haiman and Nantel Bergeron put together a scheme for constructing an assignment of modules $\mathbf{M}_{ij}$ to the cells any Ferrer's diagram. They conjectured that if μ is an m-corner partition of n, and the corresponding bases $\mathcal{B}_{ij}$ are made up of unions of the bases $\mathcal{B}^{\epsilon_1\epsilon_2\cdots\epsilon_m}$, then the collection

$$\mathcal{B}_\mu \;=\; \bigcup_{(i,j)\in\mu} \mathcal{B}_{ij}(\partial)\, \partial_{x_6}^{i} \partial_{y_6}^{j} \Delta_\mu \;. \tag{4.7}$$

should turn out to be a basis for $\mathbf{M}_\mu$. Moreover, if Φ_{ij} denotes the Frobenius characteristic of $\mathbf{M}_{ij}$, then we should also have that

$$\partial_{p_1} C_\mu(x; q, t) \;=\; \sum_{(i,j)\in\mu} \Phi_{ij}\; t^i q^j \;. \tag{4.8}$$

This is of course in complete analogy with 4.1 and 4.4. Although we cannot prove any of this at the present time, we are nevertheless in a position to show that their construction leads to a formula for $\partial_{p_1} C_\mu(x; q, t)$ which is in full generality consistent with Macdonald theory and the $C = \widetilde{H}$ conjecture. To describe the Bergeron-Haiman Algorithm, which here and after we shall simply refer as refer as the BH algorithm, we shall use the same notation we introduced in section 3. The algorithm proceeds one row at the time starting from the top row. The first step is to assign to each cell of the top row the module $\mathbf{M}_1$. Inductively, the assignements for each row are obtained from the assignments for the previous row by exactly the same proceedure. There are however two cases. Calling A and B the two successive rows, if they have the same length then the assignments for B are the same as those for A. If A has length a and B has length b and $c = b - a \geq 1$ then let

$$\mathbf{A}_1 \;,\; \mathbf{A}_2, \;,\; \ldots \;,\; \mathbf{A}_a$$

be the assignments for the successive cells of row A (from left to right) and set

$$\mathbf{A}'_s \;=\; \begin{cases} \mathbf{A}_s & \text{for } 1 \leq s \leq a\;, \\ \vee_{i=1}^{m} \mathbf{M}_i & \text{for } s \leq 0\;, \\ \{0\} & \text{for } a+1 \leq s \leq b\;. \end{cases} \tag{4.9}$$

Then, the s^{th} cell (from left to right) of row B gets assigned the module

$$\mathbf{B}_s \;=\; \mathbf{A}'_s \;\vee \left(\mathbf{A}'_{s-c} \cap \mathbf{C}\,\right) \tag{4.10}$$

where $\mathbf{C} = \mathbf{M}_{(i)}$ if the cell at end of row B is the i^{th} corner cell of μ. It is important to note that if

$$\mathbf{A}_1 \supseteq \mathbf{A}_2 \supseteq \cdots \supseteq \mathbf{A}_a$$

then 4.9 gives that

$$\mathbf{A}'_{1-c} \supseteq \mathbf{A}'_{2-c} \supseteq \cdots \supseteq \mathbf{A}'_{b-c} \qquad 4.11$$

and 4.10 yields that

$$\mathbf{B}_1 \supseteq \mathbf{B}_2 \supseteq \cdots \supseteq \mathbf{B}_b$$

Since we start by assigning the same module to all the cells of the first row, we see that this process will assign to each successive row a decreasing sequence of S_{n-1}-modules each of which may be decomposed into a direct sum of the modules $\mathbf{M}^{\epsilon_1\epsilon_2\cdots\epsilon_m}$.

Now, according to formula 4.5, if A is the j^{th} row of μ , the contribution to $\partial_{p_1}\widetilde{H}_\mu(x;q,t)$ coming from the Frobenius characteristics assigned to the cells of row A will be t^{j-1} times the polynomial

$$L_A \;=\; \mathcal{F}\mathbf{A}_1 \;+\; q\,\mathcal{F}\mathbf{A}_2 \;+\; \cdots \;+\; q^{a-1}\mathcal{F}\mathbf{A}_a \qquad 4.12$$

while the contribution of row B will be t^j times the polynomial

$$L_B \;=\; \mathcal{F}\mathbf{B}_1 \;+\; q\,\mathcal{F}\mathbf{B}_2 \;+\; \cdots \;+\; q^{b-1}\mathcal{F}\mathbf{B}_b \;. \qquad 4.13$$

We need to know how to compute L_B starting from L_A. Note that if $\mathbf{M}_1$ and $\mathbf{M}_2$ are two S_n-submodules of a given S_n-module $\mathbf{M}$ then we necessarily have

$$\mathcal{F}\left(\mathbf{M}_1 \vee \mathbf{M}_2\right) \;=\; \mathcal{F}\,\mathbf{M}_1 \;+\; \mathcal{F}\,\mathbf{M}_2 \;-\; \mathcal{F}\left(\mathbf{M}_1 \wedge \mathbf{M}_2\right)$$

This given, taking account of the inclusion $A'_{s-c} \supseteq A'_s$, we deduce from 4.10 that

$$\mathcal{F}\mathbf{B}_s \;=\; \mathcal{F}\mathbf{A}'_s \;+\; \mathcal{F}\left(\mathbf{A}'_{s-c} \cap \mathbf{C}\,\right) \;-\; \mathcal{F}\left(\mathbf{A}'_s \cap \mathbf{C}\,\right)$$

Summing from 1 to b and taking account of 4.9 we get

$$\begin{aligned} L_B \;&=\; \sum_{s=1}^{a} q^{s-1}\mathcal{F}\mathbf{A}_s \;+\; \sum_{s=1}^{c} q^{s-1}\mathcal{F}\,\mathbf{C} \;+\; \sum_{s=c+1}^{b} q^{s-1}\big(\mathbf{A}_{s-c} \cap \mathbf{C}\,\big) \;-\; \sum_{s=1}^{a} q^{s-1}\mathcal{F}\big(\mathbf{A}_s \cap \mathbf{C}\,\big) \\ &=\; L_A \;+\; \frac{q^c-1}{q-1}\,\mathcal{F}\,\mathbf{C} \;+\; (q^c-1)\sum_{s=1}^{a} q^{s-1}\mathcal{F}\big(\mathbf{A}_s \cap \mathbf{C}\,\big)\;. \end{aligned} \qquad 4.14$$

We have seen in section 3 that the Frobenius characteristic of any direct sum of the modules $\mathbf{M}^{\epsilon_1\epsilon_2\cdots\epsilon_m}$ may be expressed as a polynomial in ∇ applied to Φ_μ. It develops that when we express L_A and L_B in this manneer the terms corresponding to $\mathcal{F}\left(\mathbf{A}_s \cap \mathbf{C}\,\right)$ take a particularly simple form. To this end let Π_A and Π_B denote the unique polynomials of degree $< m$ such that

$$L_A = \Pi_A(\nabla)\,\Phi_\mu \qquad \text{and} \qquad L_B = \Pi_B(\nabla)\,\Phi_\mu \qquad 4.15$$

To compute Π_B from Π_A according to 4.14, we need to make a precise assumption as to which corner cell is at the end of row B. So let us suppose that it is the i^{th} corner cell so that $\mathbf{B} = \mathbf{M}_{\alpha^{(i)}}$. In that case, we see that the inductive process, expressed by the recursion in 4.10, forces all the modules $\mathbf{B}_s$ to be direct sums of the submodules

$$\mathbf{M}^{\epsilon_1\epsilon_2\cdots\epsilon_i} \;=\; \mathbf{M}^{\epsilon_1}_{\alpha^{(1)}} \wedge \mathbf{M}^{\epsilon_2}_{\alpha^{(2)}} \wedge \cdots \wedge \mathbf{M}^{\epsilon_i}_{\alpha^{(i)}}$$

with the exponents ϵ_s taking the values $1,0$ according to the conventions we made at the introduction. This is because we assign $\mathbf{M}_{\alpha^{(1)}}$ to the top row, and in constructing the assignments for the row that contains the i^{th} corner we only intersect with $\mathbf{M}_{\alpha^{(i)}}$. This given the crucial step in solving the recursion in 4.14 is given by the following result.

Proposition 4.1 *If* $\mathbf{M}$ *is a direct sum of the modules* $\mathbf{M}^{\epsilon_1\epsilon_2\cdots\epsilon_{i-1}}$ *then its Frobenius characteristic is of the form*

$$\mathcal{F}\,\mathbf{M} \;=\; \Pi(\nabla)\,\Phi_\mu$$

with $\Pi(x)$ *a polynomial divisible by* $\prod_{j\geq i}(1-x/T_j)$ *and the Frobenius characteristic of* $\mathbf{M}\wedge\mathbf{M}_{\alpha^{(i)}}$ *is given by the formula*

$$\mathcal{F}\,\big(\,\mathbf{M}\wedge\mathbf{M}_{\alpha^{(i)}}\big) \;=\; \Pi'(\nabla)\,\Phi_\mu \tag{4.16}$$

with

$$\Pi'(\nabla) \;=\; \frac{\Pi(\nabla)}{1-\nabla/T_i}\;. \tag{4.17}$$

Proof. Clearly it is sufficient to prove the assertion for $\mathbf{M}=\mathbf{M}^{\epsilon_1\epsilon_2\cdots\epsilon_{i-1}}$. But in this case we have

$$\mathbf{M}^{\epsilon_1\epsilon_2\cdots\epsilon_{i-1}} \;=\; \bigoplus_{\eta_i=0}^{1}\bigoplus_{\eta_{i+1}=0}^{1}\cdots\bigoplus_{\eta_{i+1}=0}^{1}\mathbf{M}^{\epsilon_1\epsilon_2\cdots\epsilon_{i-1}\eta_i\eta_{i+1}\cdots\eta_m}\;.$$

Thus, from 3.1 and 3.17 we derive that

$$\begin{aligned}\mathcal{F}\,\mathbf{M}^{\epsilon_1\epsilon_2\cdots\epsilon_{i-1}} \;&=\; \sum_{\eta_i=0}^{1}\sum_{\eta_{i+1}=0}^{1}\cdots\sum_{\eta_{i+1}=0}^{1}\prod_{j=1}^{i-1}\Big(\frac{-\nabla}{T_j}\Big)^{1-\epsilon_j}\prod_{j=i}^{m}\Big(\frac{-\nabla}{T_j}\Big)^{1-\eta_j}\quad\Phi_\mu\\ &=\; \prod_{j=1}^{i-1}\Big(\frac{-\nabla}{T_j}\Big)^{1-\epsilon_j}\prod_{j=i}^{m}\Big(1-\frac{\nabla}{T_j}\Big)\quad\Phi_\mu\end{aligned} \tag{4.18}$$

This proves the divisibility by $\prod_{j\geq i}(1-x/T_j)$. Now, since

$$\mathbf{M}^{\epsilon_1\epsilon_2\cdots\epsilon_{i-1}}\wedge\mathbf{M}^{(\alpha_i)} \;=\; \mathbf{M}^{\epsilon_1\epsilon_2\cdots\epsilon_{i-1}1}$$

in the same manner we obtain that

$$\begin{aligned}\mathcal{F}\,\Big(\mathbf{M}^{\epsilon_1\epsilon_2\cdots\epsilon_{i-1}}\wedge\mathbf{M}^{(\alpha_i)}\Big) \;&=\; \Big(\prod_{j=1}^{i-1}\Big(\frac{-\nabla}{T_j}\Big)^{1-\epsilon_j}\Big)\Big(\frac{-\nabla}{T_i}\Big)^{1-1}\prod_{j=i+1}^{m}\Big(1-\frac{\nabla}{T_j}\Big)\quad\Phi_\mu\\ &=\; \left(\prod_{j=1}^{i-1}\Big(\frac{-\nabla}{T_j}\Big)^{1-\epsilon_j}\prod_{j=i}^{m}\Big(1-\frac{\nabla}{T_j}\Big)\Big/\Big(1-\frac{\nabla}{T_i}\Big)\right)\quad\Phi_\mu\;.\end{aligned}$$

Comparing with 4.18 we see that 4.16 with 4.17 does hold in this case. This completes our proof.

If $\mathbf{M}$ is a direct sum of the modules $\mathbf{M}^{\epsilon_1\epsilon_2\cdots\epsilon_m}$, by a slight abuse of notation, we let $\Pi_{\mathbf{M}}(z)$ denote the unique polynomial of degree $< m$ such that $\mathcal{F}\,\mathbf{M} \;=\; \Pi_{\mathbf{M}}(\nabla)\Phi_\mu$. This given, Proposition 1.1 has the following beautiful corollary.

Proposition 4.2 *If* B *is the row that contains the* i^{th} *corner of* μ *then*

$$\Pi_B(z) \;=\; \Pi_A(z)\,\frac{q^c-z/T_i}{1-z/T_i} \;+\; \frac{q^c-1}{q-1}\prod_{j=1,j\neq i}^{m}\big(1-\frac{z}{T_j}\big) \tag{4.19}$$

Proposition F rom 4.14 and 4.15 we derive that

$$\Pi_B(\nabla)\Phi_\mu \;=\; \Pi_A(\nabla)\Phi_\mu \;+\; \frac{q^c-1}{q-1}\,\Pi_{\mathbf{C}}(\nabla)\Phi_\mu \;+\; (q^c-1)\sum_{i=1}^{a} q^{s-1}\Pi_{\mathbf{A}_s\cap\mathbf{C}}(\nabla)\Phi_\mu \;. \tag{4.20}$$

Since by assumption $\mathbf{C}=\mathbf{M}_{\alpha^{(i)}}$, *formula 3.19 a) gives*

$$\Pi_{\mathbf{C}}(z) \;=\; \prod_{j=1,j\neq i}^{m}\big(1-\frac{z}{T_j}\big)\;.$$

Recalling that the BH algorithm constructs A_s *as a direct sum of the modules* $\mathbf{M}^{\epsilon_1\epsilon_2\cdot\epsilon_{i-1}}$, *we can apply Proposition 4.1 and obtain that*

$$\Pi_{\mathbf{A}_s\cap\mathbf{C}}(z) \;=\; \Pi_{\mathbf{A}_s}(z)\;\frac{1}{1-z/T_i}\;.$$

Thus 4.20 yields

$$\Pi_B(z) \;=\; \Pi_A(z) \;+\; \frac{q^c-1}{q-1}\prod_{j=1,j\neq i}^{m}\big(1-\frac{z}{T_j}\big) \;+\; \frac{q^c-1}{1-z/T_i}\left(\sum_{i=1}^{a} q^{s-1}\Pi_{\mathbf{A}_s}(z)\right)\;,$$

and 4.19 follows because from 4.12 we get that

$$\sum_{i=1}^{a} q^{s-1}\Pi_{\mathbf{A}_s}(z) \;=\; \Pi_A(z)\;.$$

Note that, since the the BH algorithm keeps the same cell assignments when the row length doesn't change, the polynomial $\Pi_A(z)$ will be the same as that constructed from the row that contains the $(i-1)^{st}$ corner cell. Recalling our definitions of corner weights given in 3.13 and 3.38, we see that the monomial q^c appearing in 4.19 is none other than the ratio x_i/u_{i-1}. Moreover, since Thus if we change our notation and denote Π_A by Π_{i-1} and Π_B by Π_i, the recursion in 4.19 becomes

$$\Pi_i(z) \;=\; \Pi_{i-1}(z)\,\frac{x_i/u_{i-1}-z/T_i}{1-z/T_i} \;+\; \frac{x_i/u_{i-1}-1}{q-1}\prod_{j=1,j\neq i}^{m}\big(1-\frac{z}{T_j}\big)\;.$$

Recalling that we have $T_\mu/T_i = x_i$, we may rewrite this as

$$\Pi_i(z) \;=\; \Pi_{i-1}(z)\,\frac{x_i}{u_{i-1}}\frac{1-u_{i-1}\,z/T_\mu}{1-x_i\,z/T_\mu} \;+\; \frac{1}{u_{i-1}}\frac{x_i-u_{i-1}}{q-1}\prod_{j=1,j\neq i}^{m}\big(1-x_i\,z/T_\mu\big)\;. \tag{4.21}$$

To solve this recursion, it is convenient to work with the expressions

$$\Gamma_i(z) \;=\; \frac{\Pi_i(\,z\,T_\mu)}{\prod_{j=1}^{m}(1-x_j\,z)} \qquad (\,for \quad i=1\ldots m\;)\;, \tag{4.22}$$

so that 4.21 reduces to

$$\Gamma_i(z) \;=\; \Gamma_{i-1}(z)\,\frac{x_i}{u_{i-1}}\,\frac{1-u_{i-1}\,z}{1-x_i\,z} \;+\; \frac{1}{u_{i-1}}\frac{x_i-u_{i-1}}{q-1}\,\frac{1}{1-x_i\,z}\;. \tag{4.23}$$

Standard methods of the theory of difference equations yield that the solution of this recurrence may be expressed in the form

$$\Gamma_i(z) \;=\; a\,\Gamma_i^{(1)}(z) \;+\; \Gamma_i^{(2)}(z)\;. \tag{4.24}$$

where $\Gamma_i^{(1)}(z)$ and $\Gamma_i^{(2)}(z)$ are respectively solutions of the homogeneous and non-homogeneous equations, and a is a constant that is the determined by the initial conditions. Now, it is easily verified that we can take

$$\Gamma_i^{(1)}(z) \;=\; \frac{x_i}{u_{i-1}}\cdots\frac{x_1}{u_o}\;\frac{1-u_{i-1}z}{1-x_iz}\cdots\frac{1-u_oz}{1-x_1z} \qquad \text{and} \qquad \Gamma_i^{(2)}(z) \;=\; \frac{1}{1-q}$$

Substituting this in 4.24 gives that

$$\Gamma_1(z) \;=\; a\,\frac{x_1}{u_o}\;\frac{1-u_oz}{1-x_1z} \;+\; \frac{1}{1-q}\ . \tag{4.25}$$

Now since the BH algorithm assigns $\mathbf{M}_{\alpha^{(1)}}$ to every cell of the top row, and

$$\mathcal{F}\mathbf{M}_{\alpha^{(1)}} = \prod_{j=2}^{m}(1-\nabla/T_j)\Phi_\mu\ ,$$

we must have

$$\Pi_1(z) \;=\; (1+q+q^2+\cdots+q^{a_1})\,\prod_{j=2}^{m}(1-x_j\,z/T_\mu)\ ,$$

where a_1 is the coarm of the 1^{st} corner cell and therefore a_1+1 is the length of the top row. Thus, since $q^{a_1+1}=x_1/u_o$, 4.22 gives

$$\Gamma_1(z) \;=\; \frac{1}{u_o}\;\frac{u_o-x_1}{1-q}\,\frac{1}{1-x_1z}\ . \tag{4.26}$$

Equating the righthand sides of 4.25 and 4.26 and solving for a gives

$$a \;=\; -\frac{1}{1-q}$$

Using this in 4.24 we finally obtain that

$$\Gamma_i(z) \;=\; -\frac{1}{1-q}\;\frac{x_i}{u_{i-1}}\cdots\frac{x_1}{u_o}\;\frac{1-u_{i-1}z}{1-x_iz}\cdots\frac{1-u_oz}{1-x_1z} \;+\; \frac{1}{1-q}\ . \tag{4.27}$$

Note that from our notation † it follows that there are exactly l_i-l_{i+1} rows that have the same length as that which contains the i^{th} corner cell. Thus, under the BH algorithm, each of these rows contributes to the right hand side of 4.4 a term of the form $t^s\Pi_i(\nabla)\Phi_\mu$ with $l_{i+1}<s\le l_i$.

In summary the BH algorithm yields that

$$\begin{aligned}\delta_{p_1}\,\widetilde{H}_\mu(x;q,t) \;&=\; \sum_{i=1}^{m} t^{1+l_{i+1}}\Big(1+t+\cdots+t^{l_i-l_{i+1}-1}\Big)\ \Pi_i(\nabla)\Phi_\mu \\ &=\; \frac{1}{1-t}\sum_{i=1}^{m} t^{1+l_{i+1}}\big(\,1-t^{l_i-l_{i+1}}\big)\ \Pi_i(\nabla)\Phi_\mu\end{aligned} \tag{4.28}$$

Note that since $t^{1+l_m}=x_m/u_m$ and $t^{l_j-l_{i+j}}=x_j/u_j$ we may rewrite 4.28 as

$$\partial_{p_1}\widetilde{H}_\mu(x;q,t) \;=\; \frac{1}{1-t}\sum_{i=1}^{m}\frac{x_m}{u_m}\cdots\frac{x_{i+1}}{u_{i+1}}\Big(1-\frac{x_i}{u_i}\Big)\ \Pi_i(\nabla)\Phi_\mu\ . \tag{4.29}$$

† Recall that the exponent l_i in $x_i=t^{l_i}q^{a_i}$ gives the coleg of the i^{th} corner cell

Now, 4.27 gives that

$$(1-q)\sum_{i=1}^{m}\frac{x_m}{u_m}\cdots\frac{x_{i+1}}{u_{i+1}}\left(1-\frac{x_i}{u_i}\right)\Gamma_i(z) \;=\;$$

$$= \; -\sum_{i=1}^{m}\frac{x_m}{u_m}\cdots\frac{x_{i+1}}{u_{i+1}}\left(1-\frac{x_i}{u_i}\right)\frac{x_i}{u_{i-1}}\cdots\frac{x_1}{u_o}\;\frac{1-u_{i-1}z}{1-x_iz}\cdots\frac{1-u_oz}{1-x_1z} \qquad 4.30$$

$$+\;\sum_{i=1}^{m}\frac{x_m}{u_m}\cdots\frac{x_{i+1}}{u_{i+1}}\left(1-\frac{x_i}{u_i}\right)\;.$$

Calling, for a moment, $\mathbf{S_1}$ and $\mathbf{S_2}$ the first and second sums on the righthand side, and recalling from 3.45 that $x_ox_1\cdots x_m = u_ou_1\cdots u_m$, we may write

$$\begin{aligned}
\mathbf{S_1} \;&=\; -\sum_{i=1}^{m}\frac{u_i}{x_o}\left(1-\frac{x_i}{u_i}\right)\frac{1-u_{i-1}z}{1-x_iz}\cdots\frac{1-u_oz}{1-x_1z}\\
&=\; -\sum_{i=1}^{m}\frac{u_i}{x_o}\;\frac{1-u_{i-1}z}{1-x_iz}\cdots\frac{1-u_oz}{1-x_1z}\;+\;\sum_{i=1}^{m}\frac{x_i}{x_o}\;\frac{1-u_{i-1}z}{1-x_iz}\cdots\frac{1-u_oz}{1-x_1z}\\
&=\; \frac{1}{x_oz}\left\{\sum_{i=1}^{m}\;\frac{(1-u_iz)\cdots(1-u_oz)}{(1-x_iz)\cdots(1-x_1z)}\;-\;\sum_{i=1}^{m}\frac{(1-u_{i-1}z)\cdots(1-u_oz)}{(1-x_{i-1}z)\cdots(1-x_1z)}\right\}\\
&=\; \frac{1}{x_oz}\left\{\frac{(1-u_mz)\cdots(1-u_oz)}{(1-x_mz)\cdots(1-x_1z)}\;-\;(1-u_oz)\right\}\;.
\end{aligned}$$

At the same time we have (again because of 3.45)

$$\begin{aligned}
\mathbf{S_2} \;&=\; \sum_{i=1}^{m}\frac{x_m}{u_m}\cdots\frac{x_{i+1}}{u_{i+1}}\;-\;\sum_{i=1}^{m}\frac{x_m}{u_m}\cdots\frac{x_i}{u_i}\\
&=\; 1\;-\;\frac{x_m}{u_m}\cdots\frac{x_1}{u_1}\;=\;1-\frac{u_o}{x_o}\;=\;\frac{(1-u_oz)\;-\;(1-x_oz)}{x_oz}
\end{aligned}$$

Using these expressions for $\mathbf{S_1}$ and $\mathbf{S_2}$ in 4.30 gives

$$\sum_{i=1}^{m}\frac{x_m}{u_m}\cdots\frac{x_{i+1}}{u_{i+1}}\left(1-\frac{x_i}{u_i}\right)\Pi_i(z)\;=\;\frac{1}{1-q}\;\frac{1}{x_oz}\left\{\frac{(1-u_mz)\cdots(1-u_oz)}{(1-x_mz)\cdots(1-x_1z)}\;-\;(1-x_oz)\right\}\;.$$

Thus from 4.22 we get (recalling that $x_o = 1/qt$)

$$\sum_{i=1}^{m}\frac{x_m}{u_m}\cdots\frac{x_{i+1}}{u_{i+1}}\left(1-\frac{x_i}{u_i}\right)\Pi_i(z)\;=\;\frac{1}{(1-1/t)(1-1/q)}\;\frac{T_\mu}{z}\left\{\prod_{i=0}^{m}(1-u_i\,\frac{z}{T_\mu})\;-\;\prod_{i=0}^{m}(1-x_i\,\frac{z}{T_\mu})\right\}$$

So that 4.29 becomes

$$\partial_{p_1}\widetilde{H}_\mu(x;q,t)\;=\;\frac{1}{(1-1/t)(1-1/q)}\;\frac{T_\mu}{\nabla}\left\{\prod_{i=0}^{m}\Big(1-u_i\,\frac{\nabla}{T_\mu}\Big)\;-\;\prod_{i=0}^{m}\Big(1-x_i\,\frac{\nabla}{T_\mu}\Big)\right\}\Phi_\mu\;,$$

which is in complete agreement with 3.43. It is indeed remarkable that these two quite distinct paths lead exactly to the same formula. In fact, in this section we have basically deduced it only from the combinatorics of the BH algorithm. On the other hand, in section 3. we made heavy use of Science Fiction identities as well as various identities of the theory of Macdonald polynomials. This yields simultaneous support to the BH algorithm, the SF heuristic and the $C=\widetilde{H}$ conjecture. Since it all fits together so well, it is difficult to believe that it results from purely accidental circumsances.

Needeless to say, the BH algorithm applied to any of the special cases we have studied in this paper does produce a basis for the corresponding module $\mathbf{M}_\mu$. It may be also successfully applied to some infinite family of partitions. As an instance in point we shall treat the case of 2-corner partitions. So let μ have l_a rows of length a and l_b rows of length b with $c = b - a > 0$. Let μ be a partition of $n+1$ and α and β denote the two partitions of n obtained by removing one of the two corners of μ. More precisely, we set $\alpha = a^{l_a-1}b^{l_b}$ and $\beta = a^{l_a}b^{l_b-1}$. Moreover, let us decompose the diagram of μ into 4 parts as follows:

For c > a

$$\begin{cases} UPR & = \left\{(r,s) \;:\; \begin{matrix} l_b \le r < l_b + l_a \\ 0 \le s < a \end{matrix}\right\} \\ INS & = \left\{(r,s) \;:\; \begin{matrix} 0 \le r < l_b \\ 0 \le s < a \end{matrix}\right\} \\ MID & = \left\{(r,s) \;:\; \begin{matrix} 0 \le r < l_b \\ a \le s < c \end{matrix}\right\} \\ OUT & = \left\{(r,s) \;:\; \begin{matrix} 0 \le r < l_b \\ c \le s < b \end{matrix}\right\} \end{cases}$$

and

For c ≤ a

$$\begin{cases} UPR & = \left\{(r,s) \;:\; \begin{matrix} l_b \le r < l_b + l_a \\ 0 \le s < a \end{matrix}\right\} \\ INS & = \left\{(r,s) \;:\; \begin{matrix} 0 \le r < l_b \\ 0 \le s < c \end{matrix}\right\} \\ MID & = \left\{(r,s) \;:\; \begin{matrix} 0 \le r < l_b \\ c \le s < a \end{matrix}\right\} \\ OUT & = \left\{(r,s) \;:\; \begin{matrix} 0 \le r < l_b \\ a \le s < b \end{matrix}\right\} \end{cases}$$

Now let $\mathcal{B}_{A\cup B}$ be a basis for $\mathbf{M}_\alpha \vee \mathbf{M}_\beta$, let $\mathcal{B}_{A\cap B}$ be a basis for $\mathbf{M}_\alpha \wedge \mathbf{M}_\beta$ and let $\mathcal{B}^*_{A^\perp\cap B}$ denote a collection whose flip with respect to Δ_β forms a basis for the submodule $\mathbf{M}_\alpha^\perp \wedge \mathbf{M}_\beta$. This given, we have the following result.

Theorem 4.1 *If the $n!$ conjecture holds true for α and β and the intersection of the two modules $\mathbf{M}_\alpha$ and $\mathbf{M}_\beta$ has dimension $n!/2$ then $\dim \mathbf{M}_\mu$ has $(n+1)!$ dimensions and a basis for $\mathbf{M}_\mu$ is given by the collection*

$$\mathcal{B}_\mu \;=\; \sum_{(r,s)\in\mu} \left\{ b(\partial)\partial^r_{x_{n+1}}\partial^s_{y_{n+1}}\Delta_\mu \right\}_{b\in\mathcal{B}_{r,s}} \qquad 4.31$$

with the following choices:

$$\mathcal{B}_{r,s} = \mathcal{B}_A \;\; for \;\; (r,s)\in UPR\;,$$

$$\mathcal{B}_{r,s} = \mathcal{B}_{A\cup B} \;\; for \;\; (r,s)\in INS \quad and \quad \mathcal{B}_{r,s} = \mathcal{B}^*_{A^\perp\cap B} \;\; for \;\; (r,s)\in OUT$$

while for $(r,s)\in MID$

$$\mathcal{B}_{r,s} \;=\; \begin{cases} \mathcal{B}_B & when \;\; c > a \\[2ex] \mathcal{B}_A & when \;\; c \le a \end{cases}$$

Proof. It is not difficult to see that the BH algorithm yields precisely the choices indicated above except that for $(r,s)\in OUT$ it gives $\mathcal{B}_{r,s} = \mathcal{B}_{A\cap B}$. Of course if we assume the validity of the SF heuristic then a choice is guaranteed to exist for a $\mathcal{B}_{A\cap B}$ whose flip with respect to Δ_β is in fact a basis for $\mathbf{M}_\alpha^\perp \cap \mathbf{M}_\beta$. However, in the two corner case, we need not assume anything more than $\dim \mathbf{M}_\alpha \cap \mathbf{M}_\beta = n!/2$. Since a simple count shows that, with these choices, $\mathcal{B}_\mu$ has $(n+1)!$ elements, we need only show that for any choices of $b_{r,s} \in \mathcal{L}[\mathcal{B}_{r,s}]$ we cannot have

$$\sum_{(r,s)\in\mu} b_{r,s}(\partial)\partial^r_{x_{n+1}}\partial^s_{y_{n+1}}\Delta_\mu \;=\; 0 \tag{4.32}$$

without $b_{r,s} = 0$ identically for all $(r,s)\in\mu$. Proceeding as we did in the several examples we have previously considered, we can derive $n+1$ equations from 4.32 by equating to zero the coefficients of $x^{r_o}_{n+1}y^{s_o}_{n+1}$ for each $(r_o,s_o)\in\mu$. These equations are all of the form

$$\sum_{\substack{(r',s')\in\mu\\(r'+r_o,s'+s_o)\in\mu}} b_{(r',s')}(\partial)D_{r'+r_o,s'+s_o} \;=\; 0 \tag{4.33}$$

where $D_{r,s}$ denotes a suitable multiple of the cofactor of the monomial $x^r_{n+1}y^s_{n+1}$ in the matrix whose determinant gives Δ_μ. For convenience let us write $(r',s')\prec(r,s)$ if and only if $(r',s')\neq(r,s)$ and $r'\leq r$ and $s'\leq s$.

Now, a close examination of few special cases reveals that the equations in 4.33 may be successively solved to yield one of the following four types of conditions

(1) For a $b_{r,s}$ with $(r,s)\in INS$: (after showing that $b_{r',s'}=0$ for all $(r',s')\prec(r,s)$)

$$b_{r,s}(\partial)\Delta_\alpha = 0 \qquad \text{and} \qquad b_{r,s}(\partial)\Delta_\beta = 0 \tag{4.34}$$

(2) For a $b_{r,s}$ with $(r,s)\in MID$: (after showing that $b_{r',s'}=0$ for all $(r',s')\prec(r,s)$)

$$\begin{cases} b_{r,s}(\partial)\Delta_\beta = 0 & \text{if } c>a\\[2ex] b_{r,s}(\partial)\Delta_\alpha = 0 & \text{if } c\leq a\end{cases} \tag{4.35}$$

(3) For some $b_{r,s}$ with $(r,s)\in UPR$: (after showing that $b_{r',s'}=0$ for all $(r',s')\prec(r,s)$)

$$b_{r,s}(\partial)\Delta_\alpha = 0 \tag{4.36}$$

(4) Alternatively, for a pair b_{r_1,s_1}, b_{r_2,s_2} with $(r_1,s_1)\in UPR$ and $(r_2,s_2)\in OUT$

(after showing that $b_{r',s'}=0$ for all $(r',s')\prec(r_1,s_1)$ and for all $(r',s')\prec(r_2,s_2)$)

$$b_{r_1,s_1}(\partial)\Delta_\alpha \;+\; b_{r_2,s_2}(\partial)\Delta_\beta = 0\ . \tag{4.37}$$

Now in any of the cases (1), (2), and (3) the condition forces $b_{r,s}=0$. For instance in case (1), the choice $\mathcal{B}_{r,s}=\mathcal{B}_{A\cup B}$ yields that $b_{r,s}\in\mathbf{M}_\alpha\vee\mathbf{M}_\beta$ and at the same time the condition in 4.34 yields that $b_{r,s}\in\big(\mathbf{M}_\alpha\vee\mathbf{M}_\beta\big)^\perp$. Thus $b_{r,s}$ must be orthogonal to itself and therefore it must identically vanish. The conclusion in the other two cases are similarly obtained since the corresponding choice of $\mathcal{B}_{r,s}$ combined with 4.35 or 4.36, as the case may be, forces $b_{r,s}$ to be orthogonal to itself and therefore equal to zero. In case (4), since $\mathcal{B}_{r_1,s_1}=\mathcal{B}_A$, the polynomial $b_{r_1,s_1}(\partial)\Delta_\alpha$ may turn out be an arbitrary element of $\mathbf{M}_\alpha$. So the only way to force b_{r_1,s_1} to vanish is to assure that 4.37 forces $b_{r_1,s_1}(\partial)\Delta_\alpha=0$, and that will certainly be forced by 4.37 if $b_{r_2,s_2}(\partial)\Delta_\beta\in\mathbf{M}^\perp_\alpha$. Now the latter is guaranteed by our choice of $\mathcal{B}_{r_2,s_2}=\mathcal{B}^*_{A^\perp\cap B}$ for all $(r_2,s_2)\in OUT$. This completes our proof.

We should note that the role played by the condition $\dim(\mathbf{M}_\alpha\wedge\mathbf{M}_\beta)=n!/2$ in the above proof is simply to assure that the orthogonal complement of $\mathbf{M}_\alpha\wedge\mathbf{M}_\beta$ in $\mathbf{M}_\beta$ has also dimension $n!/2$ and this is the additional condition that needs to be satisfied so that in the final count $\mathcal{B}_\mu$ has exactly $(n+1)!$ elements.

There are a number of other identities and conjectures that may be derived from the SF heuristics, these will be included in a forthcoming joint work with N. Bergeron, M. Haiman. and G. Tessler [3] where many of the identities derived here are shown to have a remarkably suggestive combinatorial interpretation. We should also refer the reader to the work of C. Chang [2] where Theorem 4.1 is used to give a unified proof of the $n!$ conjecture for hook, two-row and two-column shapes.

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