A NEW FORMULA FOR MACDONALD POLYNOMIALS (DRAFT)

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1. Introduction

Let $J_\mu(X;q,t)$ be the *integral form* Macdonald polynomials as defined in Macdonald's book [13], Ch. VI, (8.3), and let

$$\tilde{H}_\mu(X;q,t) = t^{n(\mu)} J_\mu\left[\frac{X}{1-t^{-1}}; q, t^{-1}\right] \tag{1.1}$$

be the transformed variants which were found to be useful in some of our earlier work [1, 2, 3, 4, 5, 7]. Here $n(\mu)$ is the familiar statistic $\sum_i (i-1)\mu_i$ [13] (I,1.5), and the square brackets denote *plethystic substitution*, a valuable notational device which we briefly review in Section 2. Let ∇ be the linear operator defined in terms of the basis $\tilde{H}_\mu$ by

$$\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu')} \tilde{H}_\mu, \tag{1.2}$$

where μ' denotes the conjugate partition. This operator plays a crucial role in the developments relating Macdonald polynomials to geometry and symmetric group representation theory; for example the conjectured formula for the Frobenius series of the diagonal harmonics as in [3, 8] can be expressed most simply as

$$\nabla e_n,$$

and the q,t-Catalan numbers of [3, 9] as

$$\langle \nabla e_n, e_n \rangle,$$

where $\langle -,- \rangle$ is the standard Hall inner product [13] (I,4.5).

Let ω denote the usual involution on symmetric functions [13] (I,2.7) which exchanges the elementary symmetric functions e_r and the complete symmetric functions h_r, and set

$$\Omega(X) = \sum_{n\geq 0} h_n(X) = \prod_i \frac{1}{1-x_i}. \tag{1.3}$$

As usual, the *diagram* of a partition μ is the set $d(\mu) = \{(i,j) \in \mathbf{N}\times\mathbf{N} : j < \mu_{i+1}\}$, and we introduce for each μ the corresponding diagram generating function

$$B_\mu(q,t) = \sum_{(i,j)\in d(\mu)} t^i q^j. \tag{1.4}$$

Date: September 26, 1998.
1991 *Mathematics Subject Classification.* 05E05.
Key words and phrases. Macdonald polynomials.
Supported in part by N.S.F. Mathematical Sciences grant DMS-9400934.

For convenience, because these quantities will appear frequently, we also set

$$M = (1-q)(1-t), \quad A_\mu = 1 - MB_\mu. \tag{1.5}$$

Our chief object in this note is to prove *Glenn's formula:*

$$\tilde{H}_\mu[X+1] = \Omega[\frac{X}{M}]\nabla^{-1}\omega\Omega[\frac{-XA_\mu}{M}]. \tag{1.6}$$

This apparently simple formula has astonishing consequences. Several important results in the theory of Macdonald polynomials can be deduced from it with hardly any effort, as we demonstrate in Section 4. Namely, (1) we recover the plethystic formulas of [7] for Macdonald's q,t-Kostka coefficients $K_{\lambda\mu}(q,t)$, in a simpler and more effective form than in [7]; (2) we obtain a new and simple proof of the theorem [6, 7, 10, 11, 14] that the coefficients $K_{\lambda\mu}$ are polynomials; (3) we recover the vanishing theorem of Knop and Sahi [12, 14], in a strong "extended vanishing" form, with an exact formula for the vanishing polynomials and a natural interpretation of the value at the points where they do not vanish; and (4) we see that the curious and remarkable Koornwinder-Macdonald reciprocity formula [13] (VI,6.6) is a simple specialization of (1.6).

On top of all that, our proof of Glenn's formula itself is not difficult and uses no machinery other than well-known symmetric function theory and some plethystic operator manipulations. The starting point consists entirely of standard facts about Macdonald polynomials derived in [13], translated into corresponding facts about $\tilde{H}_\mu$.

2. Notation and basic facts

We work with the algebra Λ of symmetric functions in a formal infinite alphabet $X = x_1, x_2, \ldots$, with coefficients in the field of rational functions $\mathbf{Q}(q,t)$ of two parameters q,t. When it is relevant, we denote by $\Lambda_{\mathbf{Z}}$ the algebra of symmetric functions with coefficients in $\mathbf{Z}$. We write Λ^d for the space of symmetric functions homogeneous of degree d in the variables X, and likewise $\Lambda^{>d}$, $\Lambda^{\leq d}$, with the obvious meaning.

If A is a formal Laurent series in variables $a_1, a_2, \ldots$, which may include the parameters q,t, we define the *plethystic substitution* $f[A]$ of A into a symmetric function f by setting

$$p_k[A] = A_{(a_i \mapsto a_i^k, \forall i)}, \tag{2.1}$$

for the k-th power-sum p_k, and extending to all f by requiring $f \mapsto f[A]$ to be a homomorphism of $\mathbf{Q}(q,t)$-algebras. This is well-defined and unique since Λ is freely generated by the power-sums p_k. We make the convention that inside the plethystic bracket, X stands for $x_1 + x_2 + \cdots$, so that, as one sees immediately from the definition, $f[X] = f(X)$. A caution that must be observed with this notation is that it is not correct to substitute arbitrary values for the variables a_i inside the plethystic bracket. For example, if $f \in \Lambda^d$, then it is true that

$$f[zX] = z^d f[X],$$

but *false* that $f[-X] = (-1)^d f[X]$, that is, we cannot set $z = -1$. In fact,

$$f[-X] = (-1)^d \omega f[X],$$

as follows immediately from $\omega p_k = (-1)^{k+1} p_k$.

The Macdonald polynomials J_μ and transformed variants $\tilde{H}_\mu$, the operator ∇, the quantities M, B_μ, and A_μ, and the complete symmetric function generator $\Omega(X)$ are defined as in the introduction. We denote by a prime the replacement of q, t by q^{-1}, t^{-1}, so

$$M' = (1-q^{-1})(1-t^{-1}) = M/qt, \quad B'_\mu = B_\mu(q^{-1}, t^{-1}), \tag{2.2}$$

and so on.

In addition to the Hall inner product $\langle -, - \rangle$ we need the transformed inner product

$$\langle f, g \rangle_* = \langle f[-MX], g(X) \rangle = \langle f(X), g[-MX] \rangle. \tag{2.3}$$

Note that for any A (here, $-M$) the plethystic subsitution $X \mapsto AX$ is self-adjoint, by the orthogonality of the power sums, so (2.3) really is symmetric, as indicated. We denote by τ_A the plethystic translation operator

$$\tau_A f(X) = f[X + A]. \tag{2.4}$$

Recall the *Cauchy identity*, which in plethystic notation reads

$$\Omega[XY] = \sum_\mu u_\mu[X] v_\mu[Y] \tag{2.5}$$

where $\{u_\mu\}$ and $\{v_\mu\}$ are Hall-dual bases of Λ. It has the basis-free formulation

$$\langle \Omega[AX] f, g \rangle = \langle f, g[X+A] \rangle, \tag{2.6}$$

which says that τ_A is Hall adjoint to $\Omega[AX]$. Rewriting (2.6) as

$$\langle \Omega[-AX/M] f[-X/M], g \rangle_* = \langle f[-X/M], g[X+A] \rangle_*, \tag{2.7}$$

we see that τ_A is $*$-adjoint to $\Omega[-AX/M]$. We define ∂_1 to be the coefficient of z in the operator τ_z:

$$\partial_1 f = f[X+z]\big|_z, \tag{2.8}$$

so ∂_1 is a homogeneous operator of degree -1, Hall adjoint to $e_1 = \Omega[zX]\big|_z$ (the operator of multiplication by e_1). Similarly, ∂_1 is $*$-adjoint to $-e_1/M$.

For completeness, we remark that (2.6) can be derived as follows: first, (2.5) immediately yields

$$\langle \Omega[AX], g \rangle = g[A],$$

and then (2.6) follows by taking $f = \Omega[BX]$, $g = \Omega[CX]$, and using the identity $\Omega[X+Y] = \Omega[X]\Omega[Y]$. Since B and C are arbitrary this proves (2.6) for all f, g.

Finally we denote by D_k, for all integers k, the operators defined by the generating function

$$D(z) = \sum_{k \in \mathbf{Z}} D_k z^k, \quad \text{where} \quad D(z) f = f[X + M/z]\Omega[-zX]. \tag{2.9}$$

We also set $\hat{D}_k = (-1)^k \omega D'_k \omega$, or equivalently $(\hat{D}_k f)[-X] = D'_k(f[-X])$. Their generating function is then

$$\hat{D}(z) f = f[X - M'/z]\Omega[zX]. \tag{2.10}$$

We now list the basic facts we require about the polynomials $\tilde{H}_\mu$. All are straightforward translations of properties of J_μ stated in [13]. For most of them we will only state the fact we need and the result from [13] which implies it. We explain the derivation only where it involves steps which are not completely obvious.

Property 1. $D_0\tilde{H}_\mu = A_\mu \tilde{H}_\mu$.

Proof. Set $\tilde{E} = 1-(1-t)E$, where E is Macdonald's operator [13] (VI,4.3). Then by [13] (VI,4.7c) we have

$$\tilde{E}J_\mu = A_\mu(q,t^{-1})J_\mu, \tag{2.11}$$

since J_μ is just a scalar multiple of Macdonald's P_μ. On the other hand by [13] (VI,4.1) and (VI,3.4), we can compute $\tilde{E}$ on symmetric functions in n variables $x_1,\dots,x_n$ by

$$\begin{aligned}\tilde{E}f &= t^{-n}f - t^{-n}(1-t)\sum_{i=1}^{n}\prod_{j\neq i}\frac{tx_i-x_j}{x_i-x_j}f[X-x_i+qx_i]\\ &= f[X+(q-1)/z]\left(t^{-n}-t^{-n}(1-t)\sum_{i=1}^{n}\frac{1}{1-zx_i}\prod_{j\neq i}\frac{tx_i-x_j}{x_i-x_j}\right)\Bigg|_{z^0}.\end{aligned}$$

Since the quantity in parentheses is none other than the partial fraction expansion of

$$\Omega[z(1-t^{-1})X] = \prod_{i=1}^{n}\frac{1-zx_i/t}{1-zx_i},$$

we obtain the plethystic expression for $\tilde{E}$ as

$$\tilde{E}f = f[X+(q-1)/z]\Omega[z(1-t^{-1})X]\Big|_{z^0}.$$

Thus we can write (2.11) as

$$J_\mu[X+(q-1)/z]\Omega[z(1-t^{-1})X]\Big|_{z^0} = A_\mu(q,t^{-1})J_\mu[X].$$

Plethystically substituting $X/(1-t)$ for X, then replacing t by t^{-1}, and finally replacing z by z/t (as we may since we are taking the z^0 coefficient), this reduces to $D_0\tilde{H}_\mu = A_\mu\tilde{H}_\mu$ as desired. □

Property 2. $\tilde{H}_\mu[1] = 1$.

Proof. More generally, [13] (VI,8.8) is equivalent to $\tilde{H}_\mu[1-z] = \Omega[-zB_\mu]$. □

Property 3. $\omega\tilde{H}_\mu = t^{n(\mu)}q^{n(\mu')}\tilde{H}'_\mu$*; and hence* $\hat{D}_0\tilde{H}_\mu = A'_\mu\tilde{H}_\mu$.

Proof. Equivalent to [13] (VI,8.5). □

Property 4. $\langle\tilde{H}_\mu,\tilde{H}_\nu\rangle_* = \delta_{\mu\nu}(-1)^{|\mu|}h_\mu$*, where*

$$h_\mu = \prod_{s\in d(\mu)}(1-q^{-a(s)}t^{l(s)+1})(1-q^{a(s)+1}t^{-l(s)}), \tag{2.12}$$

and $a(s)$, $l(s)$ denote the arm and leg of the cell s in the diagram of μ, that is the number of cells strictly to the right of, and above s, respectively, the diagram being oriented in the French manner with the origin at lower left.

Proof. Equivalent to [13] (VI,4.6b) and (VI,6.19), using the definition (VI,8.3) of J_μ. Note that Macdonald's inner product $\langle f, g\rangle_{q,t}$ is $\langle f[X\frac{1-q}{1-t}], g\rangle$ in plethystic notation. □

Property 5. *$\tilde{H}_\mu(X; 0, t) = t^{n(\mu)} Q_\mu[X/(1-t^{-1}); t^{-1}]$, where $Q_\mu(X; t)$ is the Hall-Littlewood polynomial* [13] *(III,2.11). In particular $\tilde{H}_\mu(X; q, t)$ has no pole at $q = 0$, or more precisely, its coefficients with respect to any basis of $\Lambda_{\mathbf{Z}}$ have no pole at $q = 0$.*

Proof. This is [13] (VI,8.4ii). □

Property 6. *$\tilde{H}_\mu(X; 1, t) = h_{\mu'}[X/(1-t)] \prod_i (1-t)\cdots(1-t^{\mu'_i})$; in particular $\tilde{H}_\mu(X; q, t)$ has no pole at $q = 1$.*

Proof. This is [13] (VI,8.4iii). □

Our last property involves certain *Pieri coefficients* which we must first describe. Let μ be a partition whose diagram has k corners $(r_1, s_1), \ldots, (r_k, s_k)$, and set

$$z_i = t^{r_i} q^{s_i}, \quad Z = z_1 + \cdots + z_k. \tag{2.13}$$

Then μ has $k+1$ *co-corners*, just inside each concavity in the boundary of its diagram, counting the junctions with the x and y axes as concavities too. To be precise, if the corners are ordered northwest to southeast, so $r_1 > \cdots > r_k$, then the co-corners are $(r_1, s_0), \ldots, (r_{k+1}, s_k)$, with the convention that $s_0 = r_{k+1} = -1$. We set

$$y_i = t^{r_{i+1}} q^{s_i}, \quad Y = y_0 + \cdots + y_k. \tag{2.14}$$

Note that we have

$$A_\mu/qt = Y - Z \quad \text{and} \quad \frac{\prod_{i=0}^k y_i}{\prod_{i=1}^k z_k} = \frac{1}{qt}. \tag{2.15}$$

Then the Pieri coefficient $c_{\mu\nu}$, when ν is obtained by removing the r-th corner from the diagram of μ, is defined by

$$c_{\mu\nu} = \frac{z_r \prod_i (1 - y/z_r)}{(1-q^{-1})(1-t^{-1}) \prod_{i\neq r}(1 - z_i/z_r)} = \frac{z_r}{M'} \Omega[(Z - z_r - Y)/z_r]. \tag{2.16}$$

Property 7. *$\partial_1 \tilde{H}_\mu = \sum_{\nu\to\mu} c_{\mu\nu} \tilde{H}_\nu$, where the sum ranges over partitions ν gotten by removing a corner from the diagram of μ.*

Proof. Follows from the case $r = 1$ of [13] (VI,6.24iv) using the adjointness of ∂_1 and e_1. In detail, one has to multiply together various product expressions coming from the Pieri rule (VI,6.24iv) itself, from the definition of J_μ in terms of P_μ (VI,8.3), and from $\langle \tilde{H}_\nu, \tilde{H}_\nu\rangle_*$ and $\langle \tilde{H}_\mu, \tilde{H}_\mu\rangle_*$ as given by Property 4, and then cancel common factors. We leave this as an exercise for the reader. □

The alert reader may recognize the formula (2.16) defining $c_{\mu\nu}$ as a term a partial fraction expansion. We record here the result obtained by making this observation precise.

Lemma 1. *We have the identity*

$$\Omega[-uA_\mu] = 1 - u + M \sum_{\nu\to\mu} c_{\mu\nu} \frac{u}{1-uqtT_{\mu/\nu}}, \tag{2.17}$$

where $T_{\mu/\nu} = t^i q^j$ for the cell (i,j) removed from μ to leave ν, i.e., $T_{\mu/\nu} = z_r$ in the notation of (2.16).

Proof. Using the notation Z and Y as above, we expand

$$\Omega[u(Z-Y)] = \frac{\prod_{i=0}^{k}(1-uy_i)}{\prod_{i=1}^{k}(1-uz_i)} \tag{2.18}$$

in partial fractions, obtaining

$$\Omega[u(Z-Y)] = 1 - \frac{u}{qt} + \sum_r \frac{uz_r}{1-uz_r} \frac{\prod_i (1-y_i/z_r)}{\prod_{i\neq r}(1-z_i/z_r)}. \tag{2.19}$$

Note that the form of this is correct, since the numerator in (2.18) has degree one more than the denominator, and $uz_r/(1-uz_r)$ differs from $1/(1-uz_r)$ by a constant. The coefficients are found as usual by letting u approach 0, $1/z_r$, and ∞, respectively, using the second equation in (2.15) for the last case (linear term). But the r-th summand above is just $M' c_{\mu\nu} u/(1-uz_r)$, and replacing u with qtu, we obtain (2.17). □

3. Glenn's formula

For the proof of our main results, including Glenn's formula (1.6), we will use identities relating the various operators D_k, $\hat{D}_k$, ∇, τ_A, $\Omega[AX]$, ∂_1, and e_1. We begin this section by developing a series of such identities—more of them than we will actually use, since they fall out naturally from our calculations.

Lemma 2. *The operators D_k and D_{-k} are adjoint under the inner product $\langle -,- \rangle_*$.*

Proof. In terms of the generating function $D(z)$ we are to show

$$\langle D(z)f, g[-MX]\rangle = \langle f[-MX], D(1/z)g\rangle. \tag{3.1}$$

For this it suffices to take $f = \Omega[AX]$, $g = \Omega[BX]$, with A and B arbitrary. Then we have

$$\begin{aligned}\langle D(z)\Omega[AX], \Omega[-MBX]\rangle &= \langle \Omega[AX]\Omega[MA/z]\Omega[-zX], \Omega[-MBX]\rangle \\ &= \Omega[-MAB]\Omega[MA/z]\Omega[zMB]. \end{aligned}\tag{3.2}$$

Since the last expression is unaltered upon interchanging A with B and z with z^{-1}, it is also equal to

$$\langle \Omega[-MAX], D(1/z)\Omega[BX]\rangle,$$

as required. □

Corollary 1. *The operators $\hat{D}_k$ and $(qt)^k \hat{D}_{-k}$ are $*$-adjoint.*

Proof. Note that ω is self-adjoint for any of our inner products, so replacing q, t by q^{-1}, t^{-1} in the previous result shows that

$$\langle \hat{D}_k f, g[-M'X]\rangle = \langle f[-M'X], \hat{D}_{-k} g\rangle. \tag{3.3}$$

Since $\hat{D}_k$ is homogeneous of degree k we can assume f and g are homogeneous with $\deg g = d = k + \deg f$. Then multiplying both sides by $(qt)^d$ yields

$$\langle \hat{D}_k f, g[-MX]\rangle = \langle f[-MX], (qt)^k \hat{D}_{-k} g\rangle. \tag{3.4}$$

□

Lemma 3. *We have the commutator* $[D_k, \partial_r] = D_{k-1}\partial_{r-1}$*, where* $\partial_r = \tau_y|_{y^r}$*. In particular,*

$$\begin{array}{ll} [D_0, \partial_1] = D_{-1} & [e_1, D_0] = -MD_1 \\ [\hat{D}_0, \partial_1] = -\hat{D}_{-1} & [e_1, \hat{D}_0] = M'\hat{D}_1. \end{array} \tag{3.5}$$

Proof. We compute from the definitions

$$D(z)\tau_y f = f[X + y + M/z]\Omega[-zX] \tag{3.6}$$

$$\tau_y D(z) f = f[X + y + M/z]\Omega[-zX]\Omega[-zy]. \tag{3.7}$$

Now $\Omega[-zy] = 1 - zy$, so subtracting the second equation from the first yields the operator identity

$$[D(z), \tau_y] = zyD(z)\tau_y. \tag{3.8}$$

Taking the coefficient of $z^k y^r$ gives the main statement of the Lemma, and the first identity in (3.5). The second identity is its $*$-adjoint. The identities on the next line follow by replacing q, t with q^{-1}, t^{-1} and conjugating by ω. □

Corollary 2. *We have the commutators*

$$[D_k, \tau_1] = D_{k-1}\tau_1 \quad \textit{and} \quad [\Omega\left[\frac{-X}{M}\right], D_k] = \Omega\left[\frac{-X}{M}\right] D_{k+1}. \tag{3.9}$$

Proof. The first identity follows from (3.8) by setting $y = 1$ and taking the coefficient of z^k. The second identity is the $*$-adjoint of the first. □

Lemma 4. *We have the identities*

$$\begin{array}{ll} \partial_1 \nabla = \dfrac{1}{M}\nabla D_{-1} & \nabla e_1 = -D_1 \nabla \\ \nabla \partial_1 = -\dfrac{1}{M'}\hat{D}_{-1}\nabla & e_1 \nabla = \nabla \hat{D}_1. \end{array} \tag{3.10}$$

Proof. Recall we defined $T_{\mu/\nu}$ to be $t^i q^j$ when ν is obtained by deleting a corner (i, j) from the diagram of μ. This can also be expressed as

$$T_{\mu/\nu} = \frac{t^{n(\mu)} q^{n(\mu')}}{t^{n(\nu)} q^{n(\nu')}} = B_\mu - B_\nu = \frac{1}{M}(A_\nu - A_\mu). \tag{3.11}$$

Hence we have, using Property 7 (indeed, using only its form, not the exact value of $c_{\mu\nu}$)

$$\nabla^{-1}\partial_1\nabla\tilde{H}_\mu = \sum_\nu c_{\mu\nu}T_{\mu\nu}\tilde{H}_\nu = \frac{1}{M}[D_0,\partial_1]\tilde{H}_\mu = \frac{1}{M}D_{-1}\tilde{H}_\mu, \tag{3.12}$$

and similarly,

$$\nabla\partial_1\nabla^{-1}\tilde{H}_\mu = \sum_\nu c_{\mu\nu}T_{\mu\nu}^{-1}\tilde{H}_\nu = \frac{1}{M'}[\hat{D}_0,\partial_1]\tilde{H}_\mu = -\frac{1}{M'}\hat{D}_{-1}\tilde{H}_\mu. \tag{3.13}$$

Since the $\tilde{H}_\mu$ form a basis this establishes the operator identities

$$\nabla^{-1}\partial_1\nabla = \frac{1}{M}D_{-1}, \quad \nabla\partial_1\nabla^{-1} = -\frac{1}{M'}\hat{D}_{-1},$$

which are the identities on the left in (3.10). The identities on the right are their $*$-adjoints. □

With these identities established we are in a position to deduce our main results.

Theorem 1. $\Lambda^{>0} = e_1\Lambda + \hat{D}_1\Lambda$

Proof. In fact we will show that the operator products

$$E_\lambda = e_1^r\hat{D}_1 e_1^{\lambda_1-1}\hat{D}_1\cdots\hat{D}_1 e_1^{\lambda_k-1}1, \quad \lambda = (\lambda_1,\dots,\lambda_k,1^r),\ \lambda_i > 1 \tag{3.14}$$

form a basis of Λ. For this it is sufficient to show that those E_λ with $r=0$ span modulo $e_1\Lambda$. To this end we calcuate $\hat{D}_1 e_1^{m-1}f$, modulo $e_1\Lambda$, where f is arbitrary. By the definition of $\hat{D}_1$ this is

$$\hat{D}_1 e_1^{m-1}f = (e_1 - M'/z)^{m-1}f[X-M'/z]\Omega[zX]\Big|_z \tag{3.15}$$

$$\equiv (-M')^{m-1}f[X-M'/z]\Omega[zX]\Big|_{z^m} \qquad (\mathrm{mod}\ e_1\Lambda) \tag{3.16}$$

$$= (-M')^{m-1}h_m f(X) + (\text{terms in } (h_{m+1},h_{m+2},\dots)\Lambda). \tag{3.17}$$

Modulo $e_1\Lambda$ it follows that for λ with all parts >1,

$$\hat{D}_1 e_1^{\lambda_1-1}\hat{D}_1\cdots\hat{D}_1 e_1^{\lambda_k-1}1 = (-M')^{|\lambda|-k}h_\lambda + \cdots,$$

where the $\cdots$ indicates terms involving complete symmetric functions h_ν with $\nu > \lambda$ in lexicographic order. Hence the expressions on the left form a basis. □

Corollary 3. $\Lambda^{>0} = e_1\Lambda + D_1\Lambda$.

Proof. Change q,t to q^{-1},t^{-1} and conjugate by ω in Theorem 1. □

Theorem 2. *The operator ∇ is a polynomial operator, that is, $\nabla\Lambda_{\mathbf{Z}} \subseteq \mathbf{Z}[q,t]\Lambda_{\mathbf{Z}}$.*

Proof. The proof of Theorem 1 actually shows that the expressions E_λ belong to $\mathbf{Z}[q,t,(qtM)^{-1}]\Lambda_{\mathbf{Z}}$, and of course are a basis of it as a free $\mathbf{Z}[q,t,(qtM)^{-1}]$ module. Now using Lemma 4 we see that

$$\nabla E_\lambda = (-D_1)^r e_1(-D_1)^{\lambda_1-1}e_1\cdots e_e(-D_1)^{\lambda_k-1}1,$$

which shows that $\nabla \mathbf{Z}[q,t,(qtM)^{-1}]\Lambda_{\mathbf{Z}} \subseteq \mathbf{Z}[q,t,(qtM)^{-1}]\Lambda_{\mathbf{Z}}$, or equivalently, $\nabla \Lambda_{\mathbf{Z}} \subseteq \mathbf{Z}[q,t,(qtM)^{-1}]\Lambda_{\mathbf{Z}}$. In other words, at worst, ∇ introduces powers of q, t, $(1-q)$ and $(1-t)$ into the denominators.

To eliminate these denominators we must show that for $f \in \Lambda_{\mathbf{Z}}$, ∇f has no pole at $q=0$, $t=0$, $q=1$, or $t=1$. By symmetry, the cases $q=0,1$ are enough, and for these we can take f to be an element of any basis we choose of $\mathbf{Z}(t)\Lambda_{\mathbf{Z}}$.

For $q=0$ we take $f=\tilde{H}_\mu(X;0,t)$, as these form a basis by Property 5. If we write

$$\tilde{H}_\lambda(X;q,t)=\sum_\mu a_{\lambda\mu}(q,t)\tilde{H}_\mu(X;0,t),$$

then the coefficients $a_{\lambda\mu}$ have no pole at $q=0$, again by Property 5. For each degree d the matrix $[a_{\lambda\mu}]$ (λ, μ partitions of d) specializes to the identity at $q=0$, and in particular its determinant is non-zero at $q=0$. Hence the inverse matrix $[b_{\lambda\mu}]$ has no pole at $q=0$. We now have

$$\tilde{H}_\lambda(X;0,t)=\sum_\mu b_{\lambda\mu}\tilde{H}_\mu(X;q,t),$$

and

$$\nabla\tilde{H}_\lambda(X;0,t)=\sum_\mu t^{n(\mu)}q^{n(\mu')}b_{\lambda\mu}\tilde{H}_\mu(X;q,t).$$

Since nothing on the right hand side has a pole at $q=0$, this is all we need. The case $q=1$ is exactly the same except we take $f=\tilde{H}_\mu(X;1,t)$ and use Property 6.

□

Corollary 4. *The operator* ∇^{-1} *is a Laurent polynomial operator, i.e.,* $\nabla^{-1}\Lambda_{\mathbf{Z}} \subseteq \mathbf{Z}[q,t,q^{-1},t^{-1}]\Lambda_{\mathbf{Z}}$.

Proof. The eigenvalues of ∇ on Λ^d are $t^{n(\mu)}q^{n(\mu')}$ for $|\mu|=d$, so its determinant is a monomial in q and t, and the inverse of its matrix with respect to any basis of $\Lambda^d_{\mathbf{Z}}$ therefore has entries in $\mathbf{Z}[q,t,q^{-1},t^{-1}]$. □

Corollary 5. *The formula (15) in* [3], *conjectured to give the Frobenius series of the diagonal harmonics, is a polynomial in q and t.*

Proof. As shown in [3], the formula is the same as ∇e_n. □

Theorem 3. *We have the identity*

$$\tilde{H}_\mu[X+1]=\Omega[\frac{X}{M}]\nabla^{-1}\omega\Omega[\frac{-XA_\mu}{M}].$$

Proof. To begin with, we claim that the formula is equivalent to the inner product identity

$$\langle\Omega[-X/M]f,\tilde{H}_\mu\rangle_*=(\Gamma f)[A_\mu], \tag{3.18}$$

for all f, where $\Gamma=\omega\nabla^{-1}\tau_{-1}$. Indeed, by adjointess, the left hand side of (3.18) is equal to $\langle f,\tilde{H}_\mu[X+1]\rangle_*$, whose value for all f determines $\tilde{H}_\mu[X+1]$, while the right hand side is the value given for this by the formula in the theorem.

Now, assuming as we may that f is homogeneous of degree d, we prove (3.18), for all μ simultaneously, by induction on d. The induction basis $d=0$, $f=1$, reduces to $\tilde{H}_\mu[1]=1$, or Property 2. For $d>0$, Corollary 3 allows us to treat two cases: $f=D_1 g$, and $f=e_1 g$, where $\deg g = d-1$.

First consider the case $f=D_1 g$. Using the second identity in (3.9) and the induction hypothesis for g we have

$$\langle \Omega[-X/M] D_1 g, \tilde{H}_\mu\rangle_* = \langle \Omega[-X/M] D_0 g, \tilde{H}_\mu\rangle_* - \langle D_0 \Omega[-X/M] g, \tilde{H}_\mu\rangle_* \tag{3.19}$$

$$= \Gamma D_0 g[A_\mu] - \langle \Omega[-X/M] g, D_0 \tilde{H}_\mu\rangle_* \tag{3.20}$$

$$= \Gamma D_0 g[A_\mu] - A_\mu \Gamma g[A_\mu]. \tag{3.21}$$

We are to show this is equal to $\Gamma D_1 g[A_\mu]$, that is, we need the operator identity

$$\Gamma D_1 = \Gamma D_0 - e_1 \Gamma,$$

or

$$\omega \nabla^{-1} \tau_{-1} D_1 = \omega \nabla^{-1} \tau_{-1} D_0 - e_1 \omega \nabla^{-1} \tau_{-1}.$$

Operating on the left by $\tau_1 \nabla \omega$ and on the right by τ_1, and using $\nabla e_1 \nabla^{-1} = -D_1$, from Lemma 4, this reduces to

$$D_1 \tau_1 = D_0 \tau_1 + \tau_1 D_1,$$

which is the first identity in (3.9).

Next consider the case $f=e_1 g$. We have

$$\langle \Omega[-X/M] e_1 g, \tilde{H}_\mu\rangle_* = -M \langle \Omega[-X/M] g, \partial_1 \tilde{H}_\mu\rangle_* = -M \sum_{\nu \to \mu} c_{\mu\nu} \Gamma g[A_\nu],$$

by induction. Thus we have to prove

$$\Gamma e_1 g[A_\mu] = -M \sum_{\nu} c_{\mu\nu} \Gamma g[A_\nu]. \tag{3.22}$$

Now the left hand side is equal to

$$\omega \nabla^{-1} \tau_{-1} e_1 g = \omega \nabla^{-1} (e_1 - 1) \tau_{-1} g \tag{3.23}$$

$$= \omega (\hat{D}_1 - 1) \nabla^{-1} \tau_{-1} g \tag{3.24}$$

$$= -(D_1' + 1) \omega \nabla^{-1} \tau_{-1} g \tag{3.25}$$

$$= -(D_1' + 1) \Gamma g. \tag{3.26}$$

So the identity we need is

$$(D_1' + 1) h[A_\mu] = M \sum_{\nu} c_{\mu\nu} h[A_\nu], \tag{3.27}$$

for an arbitrary symmetric function h (which would be Γg in our case). Written out, (3.27) reads

$$h[A_\mu] + h[A_\mu + M'/z] \Omega[-z A_\mu]\big|_z = M \sum_{\nu} c_{\mu\nu} h[A_\nu]. \tag{3.28}$$

We are free to write h in the form $f[(X-A_\mu)/M]$ since the latter is perfectly arbitrary when f is. This converts (3.28) to the simpler

$$f[0]+f[1/qtz]\Omega[-zA_\mu]|_z = M\sum_\nu c_{\mu\nu} f[T_{\mu/\nu}]. \tag{3.29}$$

Now both sides of this vanish for $f\in(e_2,e_3,\dots)\Lambda$, since f is being evaluated on a single monomial. Hence it is sufficient to take $f=h_k$, or, doing this for all k at once with the help of a generating function, to take $f=\Omega[uX]$. Equivalent to this and slightly more convenient is to take $f=u\Omega[uqtX]$. Then the left hand side of (3.29) becomes

$$u+u\Omega[\frac{u}{z}]\Omega[-zA_\mu]\Big|_z = u+\frac{u}{z}\Omega[\frac{u}{z}]\Omega[-zA_\mu]\Big|_{z^0} \tag{3.30}$$

$$= u-1+\frac{1}{1-u/z}\Omega[-zA_\mu]\Big|_{z^0} \tag{3.31}$$

$$= u-1+\Omega[-uA_\mu]. \tag{3.32}$$

By Lemma 1 this is equal to

$$M\sum_\nu c_{\mu\nu}\frac{u}{1-uqtT_{\mu/\nu}},$$

which is the right hand side of (3.29). □

4. Applications

We now exhibit some of the remarkable consequences of formula (1.6), as promised in the introduction.

Recall that Macdonald's q,t-*Kostka coefficients* $K_{\lambda\mu}$ are defined through the expansion [13] (VI,8.11)

$$J_\mu=\sum_\lambda K_{\lambda\mu}s_\lambda[X(1-t)], \tag{4.1}$$

where s_λ denotes a Schur function. If we set

$$\tilde K_{\lambda\mu}(q,t)=t^{n(\mu)}K_{\lambda\mu}(q,t^{-1}) \tag{4.2}$$

then it follows that

$$\tilde H_\mu=\sum_\lambda \tilde K_{\lambda\mu}s_\lambda.$$

It will be convenient to extend the definition of $\tilde K_{\lambda\mu}$ to cases where λ and μ are partitions of unequal size by setting

$$\tilde K_{\lambda\mu}=\sum_{\substack{|\rho|=|\mu|\\ \rho/\lambda\in HS}}\tilde K_{\rho\mu}, \tag{4.3}$$

where the sum is over partitions ρ whose diagram is obtained by adding a horizontal strip of size $|\mu|-|\lambda|$ to that of λ. In particular, according to this definition, $\tilde K_{\lambda\mu}=0$ for $|\mu|<|\lambda|$.

Theorem 4. *For each λ there is a symmetric function k_λ, inhomogeneous of degree $|\lambda|$, such that with the extended definition above, $\tilde{K}_{\lambda\mu}(q,t) = k_\lambda[B_\mu]$ for all partitions μ, of any size.*

Proof. The extended definition (4.3) is equivalent to setting $\tilde{K}_{\lambda\mu} = \langle s_\lambda \Omega, \tilde{H}_\mu \rangle$, since the classical Pieri rule gives the product $s_\lambda h_k$ as the sum of s_ρ over all ρ obtained by adding a horizontal strip of size k to λ. This can also be written

$$\begin{aligned}\tilde{K}_{\lambda\mu} &= \langle \Omega[\frac{-X}{M}] s_\lambda[\frac{-X}{M}], \tilde{H}_\mu \rangle_* \\ &= \Gamma(s_\lambda[\frac{-X}{M}])[A_\mu],\end{aligned}$$

By Theorem 3. Hence $k_\lambda(X) = (\Gamma(s_\lambda[-X/M]))[1 - MX]$ has the desired property. □

The explicit value of k_λ in the above proof can be put into a more attractive and symmetrical form by introducting the operator

$$\Pi_A f = f[AX].$$

Then we have

Corollary 6. $k_\lambda = \Pi_{-M} \tau_1 \omega \nabla^{-1} \tau_{-1} \Pi_{-M}^{-1} s_\lambda$.

The main currently unsolved problem on Macdonald polynomials is to prove that the $\tilde{K}_{\lambda\mu}$ (or equivalently the $K_{\lambda\mu}$) are polynomials in q and t with non-negative integer coefficients. As noted in the introduction, the polynomiality part of this has only recently been established. We can give another proof.

Theorem 5. $K_{\lambda\mu}(q,t) \in \mathbf{Z}[q,t]$.

Proof. From Corollaries 4 and 6 we see that $\tilde{K}_{\lambda\mu} \in \mathbf{Z}[q,t,(qtM)^{-1}]$, that is, there are at worst factors of q, t, $1-q$, and $1-t$ in their denominators. But we already know by Properties 5 and 6, together with the symmetry $\tilde{K}_{\lambda\mu}(t,q) = \tilde{K}_{\lambda\mu'}(t,q)$ (which follows, among other ways, from Corollary 6) that there are no poles at $q=0$, $t=0$, $q=1$, or $t=1$. □

Knop and Sahi [12, 14] discovered that the Macdonald polynomials arise as the leading terms of *interpolating polynomials* characterized by a certain vanishing property. Expressed in the notation used here, their result says that for each partition λ of d there is a symmetric function $R_\lambda(X)$, inhomogeneous of degree d, uniquely characterized by the condition

$$R_\lambda[A_\mu] = \delta_{\lambda\mu}, \tag{4.4}$$

and furthermore, the degree-d component of R_λ is a suitable scalar multiple of $\tilde{H}_\lambda(X; q^{-1}, t^{-1})$.

We remark that the existence of polynomials R_λ satisfying (4.4) implies they are unique. From their existence we deduce that for $|\mu| \le d$, the linear functionals on $\Lambda^{\le d}$ given by plethystic evaluation at A_μ are independent, and hence, since their number is equal to the dimension of the space, they are a basis. Therefore any function $f \in \Lambda^{\le d}$ is determined by its values $f[A_\mu]$, $|\mu| \le d$.

To recover the results of Knop and Sahi we need an additional property of the Macdonald polynomials.

Property 8. *If we define the skew Macdonald polynomials $\tilde{H}_{\mu/\nu}$ through the expansion*

$$\tau_A \tilde{H}_\mu = \tilde{H}_\mu[X+A] = \sum_\nu \tilde{H}_{\mu/\nu}[A]\tilde{H}_\nu[X],$$

then $\tilde{H}_{\mu/\nu} = 0$ unless $\nu \subseteq \mu$, that is, the diagram of ν is contained in that of μ.

Proof. Follows from [13] (VI,7.7i) and (VI,7.9). □

Theorem 6. *Polynomials R_λ satisfying* (4.4) *are given explicitly by the formula*

$$R_\lambda = \frac{(-1)^{|\lambda|}\Gamma \tilde{H}_\lambda}{h_\lambda} = \frac{(-1)^{|\lambda|}}{h_\lambda} \sum_{\nu \subseteq \lambda} \tilde{H}_{\lambda/\nu}[-1]\tilde{H}_\nu(X; q^{-1}, t^{-1}), \tag{4.5}$$

where h_λ is given by (2.12). *In particular, the highest-degree component of R_λ is $((-1)^{|\lambda|}/h_\lambda)\tilde{H}_\lambda(X; q^{-1}, t^{-1})$. Moreover we have*

$$R_\lambda[A_\mu] = \tilde{H}_{\mu/\lambda}[1] \tag{4.6}$$

for all μ; in particular $R_\lambda[A_\mu]$ vanishes whenever $\mu \not\supseteq \lambda$.

Proof. The second expression for R_λ in (4.5) derives from the first by the definition of Γ and Property 3. To establish the vanishing property it is sufficient to show that if we define R_λ by the first expression in (4.5), then (4.6) holds.

By Property 4 and the definition of $\tilde{H}_{\mu/\lambda}$ we have

$$\tilde{H}_{\mu/\lambda}[1] = \frac{(-1)^{|\lambda|}}{h_\lambda} \langle \tilde{H}_\lambda, \tilde{H}_\mu[X+1]\rangle_* \tag{4.7}$$

$$= \frac{(-1)^{|\lambda|}}{h_\lambda} \langle \Omega[-X/M]\tilde{H}_\lambda, \tilde{H}_\mu\rangle_*, \tag{4.8}$$

and this is equal to $((-1)^{|\lambda|}/h_\lambda)\Gamma\tilde{H}_\lambda[A_\mu]$ by Theorem 3. □

Finally, let us recall one of the more mysterious properties of the Macdonald polynomials, the *reciprocity formula* [13] (VI,6.6), which in our notation reads

$$\frac{\tilde{H}_\mu[1-uA_\lambda]}{\tilde{H}_\mu[1-u]} = \frac{\tilde{H}_\lambda[1-uA_\mu]}{\tilde{H}_\lambda[1-u]}. \tag{4.9}$$

By the remark in the proof of Property 2 this can also be written

$$\tilde{H}_\mu[1-uA_\lambda]\Omega[-uB_\lambda] = \tilde{H}_\lambda[1-uA_\mu]\Omega[-uB_\mu]. \tag{4.10}$$

Theorem 7. *Identity* (4.10) *follows from Theorem 3.*

Proof. The left hand side of (4.10) is the evaluation of $\tilde{H}_\mu[X+1]\Omega[-(X+u)/M]$ at $X = -uA_\lambda$, which is equal to

$$\begin{aligned}\Omega[-u/M]\langle \tilde{H}_\mu[X+1]\Omega[-X/M], \Omega[-uXA_\lambda]\rangle &= \Omega[-u/M]\langle \nabla^{-1}\omega\Omega[-XA_\mu/M], \Omega[-uXA_\lambda]\rangle \\ &= \Omega[-u/M]\langle \nabla^{-1}\omega\Omega[-XA_\mu/M], \Omega[uXA_\lambda/M]\rangle_*.\end{aligned}$$

Since $\nabla^{-1}\omega$ and the plethystic substitution $X \mapsto -uX$ are $*$-self-adjoint, the last expression is symmetrical in λ and μ. □

5. Additional remarks

Soon after the original conjecture of formula (1.6), we discovered the following extremely simple "proof." First rewrite the formula as

$$\tilde{H}_\mu[X] = \tau_{-1}\Omega[\frac{X}{M}]\nabla^{-1}\omega\Omega[\frac{-XA_\mu}{M}].$$

Now $\tilde{H}_\mu$ is uniquely characterized up to a scalar factor as the eigenfunction of D_0 with eigenvalue A_μ, so we must verify that the right hand side above has the same property. From Corollary 2 and Lemma 4 it is easy to derive the identity

$$D_0\tau_{-1}\Omega[\frac{X}{M}]\nabla^{-1}\omega = \tau_{-1}\Omega[\frac{X}{M}]\nabla^{-1}\omega(-M\partial_1).$$

But $-M\partial_1\Omega[-XA_\mu/M] = A_\mu\Omega[-XA_\mu/M]$, and the result follows. The missing scalar factor is easily shown to be 1 by setting $X = 0$ in the original formula. For this proof we don't even need the value of the Pieri coefficients $c_{\mu\nu}$; we only need the fact that the sum $\partial_1\tilde{H}_\mu = \sum_\nu c_{\mu\nu}\tilde{H}_\nu$ is over partitions $\nu \to \mu$, which is all we used in the proof of Lemma 4.

On further reflection, however, it is clear that something must be wrong with this proof. Indeed, if A is any quantity whatsoever, it appears to show that the expression

$$\tau_{-1}\Omega[\frac{X}{M}]\nabla^{-1}\omega\Omega[\frac{-XA}{M}]$$

is an eigenfunction of D_0 with eigenvalue A, which is clearly impossible!

The problem is that the right hand side of formula (1.6),

$$\Omega[\frac{X}{M}]\nabla^{-1}\omega\Omega[\frac{-XA_\mu}{M}], \tag{5.1}$$

is, *a priori*, a formal series, containing terms of unbounded degree. One cannot apply the operator τ_{-1} to such a series, just as one cannot subsitute $x-1$ for x in a formal power series of one variable. (The reader might now wish to go back and verify that our proof in Section 3 does not use any such unjustifiable step!)

It is possible to evade this difficulty to a certain extent by proving instead that (5.1) is an eigenfunction of the operator $\tau_1 D_0 \tau_{-1}$. This makes sense because the latter operator is equal to $D_0 - D_{-1}$, which *can* be applied to a formal series. The problem this causes is that if we admit formal series as eigenfunctions the inhomogeneous operator $D_0 - D_{-1}$ no longer has a "discrete spectrum:" it has in fact infinitely many independent eigenfunctions with any given eigenvalue A. All we can say is that $\tilde{H}_\mu[X+1]$ is its unique *polynomial* eigenfunction with eigenvalue A_μ. Absent a separate demonstration that (5.1) is actually a polynomial, the proof would be incomplete.

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