Some New Methods
In The Theory
of
COHEN-MACAULAY ALGEBRAS

by

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Notation

$\mathbf{X_n} = \{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}\}$

$\mathbf{Q}[\mathbf{X_n}] \Longrightarrow$ The algebra of polynomials in $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$

$\mathbf{V}$ a graded vector space

$\mathcal{H}_{\mathbf{m}}(\mathbf{V}) \Longrightarrow$ the subspace of the homogeneous elements of degree $\mathbf{m}$ in $\mathbf{V}$.

$$\mathbf{V} = \mathcal{H}_{\mathbf{0}}(\mathbf{V}) \oplus \mathcal{H}_{\mathbf{1}}(\mathbf{V}) \oplus \mathcal{H}_{\mathbf{2}}(\mathbf{V}) \oplus \cdots \oplus \mathcal{H}_{\mathbf{m}}(\mathbf{V}) \oplus \cdots$$

The Hilbert series of $\mathbf{V}$

$$\mathbf{F_V}(\mathbf{q}) = \sum_{\mathbf{m} \geq \mathbf{0}} \mathbf{dim}\big(\mathcal{H}_{\mathbf{m}}(\mathbf{V})\big) \ \mathbf{q^m}$$

Basics

Let $\mathbf{A}$ be a Finitely Generated Graded Algebra.

$$\mathbf{A} = \bigoplus_{m \geq 0} \mathcal{H}_m(\mathbf{A})$$

Let

(1) $\mathbf{n_1}$ = the order of "**1**" as a pole of the Hilbert series $\mathbf{F_A(q)}$.

(2) $\mathbf{n_2}$ = the maximum number of algebraically independent elements in $\mathbf{A}$.

(3) $\mathbf{n_3}$ = the minimum number of elements $\mathbf{q_1, q_2, \ldots, q_n}$ such that

$$\mathbf{dim}\ \mathbf{A}/(\mathbf{q_1, q_2, \ldots, q_n})_\mathbf{A} < \infty \qquad (*)$$

Fact: $\mathbf{n_1 = n_2 = n_3 = n_A}$ = the "*Krull dimension*" of $\mathbf{A}$

If (*) holds with $\mathbf{n = n_A}$ then $\mathbf{q_1, q_2, \ldots, q_n}$ are called a "*System Of Parameters*". **"S.O.P."** in brief.

More Basics

Let $\mathbf{q_1, q_2, \ldots, q_n} \in \mathbf{A}$ be homogeneous of degrees $\mathbf{d_1, d_2, \ldots, d_n}$ and suppose they constitute a **S.O.P** for $\mathbf{A}$. Let $\mathbf{B} = \{\mathbf{f_1, f_2, \ldots, f_N}\}$ be a basis for the quotient

$$\mathbf{A}/(\mathbf{q_1, q_2, \ldots, q_n})_{\mathbf{A}} < \infty$$

Then every $\mathbf{P} \in \mathbf{A}$ has an expansion of the form

$$\mathbf{P} = \sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{f_i}\ \mathbf{Q_i}(\mathbf{q_1, q_2, \ldots, q_n})$$

with $\mathbf{Q_1, Q_2, \ldots, Q_N} \in \mathbf{Q}[\mathbf{y_1, \ldots, y_n}]$. In other words the collection

$$\left\{\mathbf{f_i\, q_1^{p_1}, q_2^{p_2}, \ldots q_n^{p_p}}\right\}_{\mathbf{1 \le i \le N}}$$

spans $\mathbf{A}$ as a vector space. In particular it follows that

$$\mathbf{F_A(q)} << \frac{\sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{q}^{\mathbf{degree(f_i)}}}{(\mathbf{1-q^{d_1}})(\mathbf{1-q^{d_2}})\cdots(\mathbf{1-q^{d_n}})}$$

The Cohen Macaulay Property

If in the expansion

$$\mathbf{P} = \sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{f_i}\ \mathbf{Q_i}(\mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_n})$$

$\mathbf{P}$ uniquely determines the coefficients $\mathbf{Q_i}(\mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_n})$ then the collection

$$\left\{\mathbf{f_i}\, \mathbf{q_1^{p_1}}, \mathbf{q_2^{p_2}}, \ldots \mathbf{q_n^{p_p}}\right\}_{\mathbf{1 \le i \le N}}$$

is a vector space basis for $\mathbf{A}$. This holds true **if and only if** we have the the equality

$$\mathbf{F_A}(\mathbf{q}) = \frac{\sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{q}^{\mathbf{degree(f_i)}}}{(\mathbf{1}-\mathbf{q^{d_1}})(\mathbf{1}-\mathbf{q^{d_2}})\cdots(\mathbf{1}-\mathbf{q^{d_n}})}$$

Then $\mathbf{A}$ is a free module over $\mathbf{Q}[\mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_n}]$ of rank $\mathbf{N}$ and $\mathbf{A}$ is said to be **“Cohen-Macaulay”**.

A useful criterion

Theorem A

Let $\mathbf{q_1, q_2, \ldots, q_n} \in \mathbf{A}$ *be homogeneous of degrees* $\mathbf{d_1, d_2, \ldots, d_n}$ *and an* **S.O.P.** *for* $\mathbf{A}$*. Let*

$$\mathbf{dim\ A/(q_1, q_2, \ldots, q_n)_A = N}$$

with basis $\mathbf{B} = \{\mathbf{f_1, f_2, \ldots, f_N}\}$ *. Then the condition*

$$\mathbf{lim_{q \to 1}(1 - q^{d_1})(1 - q^{d_2}) \cdots (1 - q^{d_n}) F_A(q) = N}$$

forces the equality

$$\mathbf{F_A(q) = \frac{\sum_{i=1}^{N} q^{degree(f_i)}}{(1 - q^{d_1})(1 - q^{d_2}) \cdots (1 - q^{d_n})}}$$

yielding that $\mathbf{A}$ *is a free module over* $\mathbf{Q[q_1, q_2, \ldots, q_n]}$ *of rank* $\mathbf{N}$ *and therefore* $\mathbf{A}$ *is a Cohen-Macaulay algebra.*

m-Quasi-Invariants

Denote by "$\mathbf{s_{i,j}}$" the transposition that interchanges $\mathbf{x_i}$ and $\mathbf{x_j}$. This given, we set

$$\mathbf{QI_m[X_n]} = \left\{ \mathbf{P(x)} \in \mathbf{Q[X_n]} : \frac{\mathbf{(1 - s_{ij})P(x)}}{\mathbf{(x_i - x_j)^{2m+1}}} \in \mathbf{Q[X_n]} \ \forall \ \mathbf{1 \le i < j \le n} \right\}$$

Note that

$$\mathbf{Q[X_n] = QI_0[X_n] \supset QI_1[X_n] \supset QI_2[X_n] \supset \cdots \supset QI_m[X_n] \supset \cdots \supset QI_\infty[X_n] = SYM[X_n]}$$

Note further that for all $1 \le i < j \le n$ we have

$$\mathbf{(1 - s_{ij})\, PQ} = \mathbf{((1 - s_{ij})\ P)Q} + \mathbf{(s_{ij}P)(1 - s_{ij})\ Q}$$

Thus each $\mathbf{QI_m[X_n]}$ is an algebra and in fact also an $\mathbf{S_n}$-module.

r-Quasi-Symmetric

Recall the Florent Hivert "local" $\mathbf{r}$-action of $\mathbf{S_n}$

$$\mathbf{s_i \;\; x_i^a\, x_{i+1}^b \; = \begin{cases} x_i^a\, x_{i+1}^b & \text{if } a, b \geq r \\ x_i^b\, x_{i+1}^a & \text{otherwise} \end{cases}} \qquad (*)$$

here $\mathbf{s_i = s_{i,i+1}}$. Since we have

$$\mathbf{s_i^2 = 1}\,, \quad \mathbf{s_i\; s_{i+1}\, s_i = s_{i+1}\, s_i\, s_{i+1}}, \quad \mathbf{s_i\, s_j = s_j\, s_i} \;\; \forall \;\; \mathbf{|i-j| \geq 2}$$

(*) defines an action of $\mathbf{S_n}$ called the **"r-action."** Florent Hivert defines

$$\mathbf{r\text{-}QSym} \;=\; \Big\{\mathbf{P(x)} \,:\, \mathbf{P(x)} \;\;\textbf{is invariant under the r-action}\Big\}$$

We have a descending chain of algebras

$$\mathbf{Q[X_n] \supset 1\text{-}QSym[X_n] \supset 2\text{-}QSym[X_n] \supset \cdots \supset r\text{-}QSym[X_n] \supset \cdots \supset Sym[X_n]}$$

Note that $\mathbf{1\text{-}QSym[X_n]}$ is the space of Gessel's **"quasi-symmetric funtions"**.

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$\mathbf{r\text{-}QSym[X_n]}$ *is a free module over* $\mathbf{Sym[X_n]}$ *for all* $\mathbf{r} \geq 1$

(Conjectured by Florent Hivert)

(*Still open inspite of some premature announcements)*

Towards common Proof

Theorem

Let $\mathbf{A}$ *be a finitely generated graded algebra such that*

(1) $\mathbf{A} \subseteq \mathbf{Q}[\mathbf{X_n}]$

(2) $\mathbf{q_1, q_2, \dots q_n} \in \mathbf{A}$ *homogeneous* **S.O.P.** *for* $\mathbf{Q}[\mathbf{X_n}]$ *of degrees* $\mathbf{d_1, d_2, \dots d_n}$

(3) *We have* $\mathbf{\Pi}(\mathbf{x}) \in \mathbf{Q}[\mathbf{q_1, q_2, \dots q_n}]$ *such that*

$$\mathbf{\Pi}(\mathbf{x})\mathbf{Q}[\mathbf{X_n}] \subseteq \mathbf{A}$$

(4) $\mathbf{dim}\ \mathbf{A}/(\mathbf{q_1, q_2, \dots q_n})_{\mathbf{A}} \ \leq\ \mathbf{d} = \mathbf{d_1 d_2 \dots d_n}$

This given, **A is free over** $\mathbf{Q}[\mathbf{q_1, q_2, \dots q_n}]$ **of rank** $\mathbf{d}$

Sketch of Proof

We begin by chosing a point $\mathbf{a} = (\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n})$ such that

(1) $\mathbf{\Pi}(\mathbf{a}) \neq \mathbf{0}$

(2) The set $[\mathbf{a}] = \big\{\mathbf{q_1}(\mathbf{x}) = \mathbf{q_1}(\mathbf{a}), \mathbf{q_2}(\mathbf{x}) = \mathbf{q_2}(\mathbf{a}), \ldots, \mathbf{q_n}(\mathbf{x}) = \mathbf{q_n}(\mathbf{a})\big\}$ has cardinality $\mathbf{d} = \mathbf{d_1}\mathbf{d_2}\cdots\mathbf{d_n}$.

This given, we define

$$\mathbf{J}_{[\mathbf{a}]} = \big(\mathbf{P} \in \mathbf{A} : \mathbf{P}(\mathbf{x}) = \mathbf{0} \quad \forall \quad \mathbf{x} \in [\mathbf{a}]\big)$$

and set

$$\mathbf{R}_{[\mathbf{a}]} = \mathbf{A}/\mathbf{J}_{[\mathbf{a}]}$$

We show next that

$$\mathbf{dim}\ \mathbf{R}_{[\mathbf{a}]} = \mathbf{d}$$

The dimension of $\mathbf{R}_{[\mathbf{a}]}$

If $[\mathbf{a}] = \{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_d}\}$ construct $\mathbf{\Phi_i}(\mathbf{x}) \in \mathbf{Q}[\mathbf{X_n}]$ so that

$$\mathbf{\Phi_i}(\mathbf{x}) = \begin{cases} \mathbf{1} & \textbf{if } \mathbf{x} = \mathbf{b_i} \\ 0 & \textbf{if } \mathbf{x} = \mathbf{b_j} \textbf{ with } \mathbf{j} \neq \mathbf{i} \end{cases}$$

Set

$$\mathbf{\Psi_i}(\mathbf{x}) = \mathbf{\Phi_i}(\mathbf{x}) \frac{\mathbf{\Pi}(\mathbf{x})}{\mathbf{\Pi}(\mathbf{a})}$$

then $\mathbf{\Psi_i}(\mathbf{x}) \in \mathbf{A}$ and

$$\mathbf{\Psi_i}(\mathbf{x}) = \begin{cases} \mathbf{1} & \textbf{if } \mathbf{x} = \mathbf{b_i} \\ 0 & \textbf{if } \mathbf{x} = \mathbf{b_j} \textbf{ with } \mathbf{j} \neq \mathbf{i} \end{cases}$$

Moreover for all $\mathbf{P}(\mathbf{x}) \in \mathbf{A}$ we have

$$\mathbf{P}(\mathbf{x}) \equiv \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{d}} \mathbf{P}(\mathbf{b_i})\, \mathbf{\Psi_i}(\mathbf{x}) \qquad (\textbf{modulo } \mathbf{J}_{[\mathbf{a}]})$$

Thus $\{\mathbf{\Psi_i}(\mathbf{x})\}_{\mathbf{i}=\mathbf{1}}^{\mathbf{d}}$ is a basis for $\mathbf{R}_{[\mathbf{a}]}$ consequently $\mathbf{dim}\, \mathbf{R}_{[\mathbf{a}]} = \mathbf{d}$

More Quotient Rings

For $\mathbf{P}(\mathbf{x}) \in \mathbf{Q}[\mathbf{X_n}]$ let $\mathbf{h}(\mathbf{P})$ denote the homogeneous component of highest degree in $\mathbf{P}(\mathbf{x})$ and set

$$\mathbf{gr\ J_{[a]}} = \big(\mathbf{h}(\mathbf{P}) : \mathbf{P} \in \mathbf{J_{[a]}}\big)_{\mathbf{A}} \quad \textbf{and} \quad \mathbf{gr\ R_{[a]}} = \mathbf{A}/\mathbf{gr\ J_{[a]}}$$

We can easily prove that

$$\mathbf{dim\ gr\ R_{[a]}} = \mathbf{dim\ R_{[a]}} = \mathbf{d}$$

Now note that

$$\mathbf{q_i}(\mathbf{x}) - \mathbf{q_i}(\mathbf{a}) \in \mathbf{J_{[a]}} \implies \mathbf{q_i}(\mathbf{x}) \in \mathbf{gr\ J_{[a]}}$$

in particular

$$\mathbf{gr\ J_{[a]}} \supset \big(\mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_n}\big)_{\mathbf{A}}$$

and thus

$$\mathbf{d} = \mathbf{dim\ A}/\mathbf{gr\ J_{[a]}} \le \mathbf{dim\ A}/\big(\mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_n}\big)_{\mathbf{A}}$$

but then hypothesis **(4)** gives the equality

$$\mathbf{dim\ A}/\big(\mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_n}\big)_{\mathbf{A}} = \mathbf{d}$$

Conclusions

Hypothesis **(3)** gives

$$\frac{q^{\text{degree }\Pi(x)}}{(1-q)^n} << F_A(q) << \frac{1}{(1-q)^n}$$

Thus

$$\lim_{q\to 1} (1-q^{d_1})(1-q^{d_2})\cdots(1-q^{d_n})F_A(q) = d_1 d_2 \cdots d_n = d$$

and the Cohen-Macauliness of **A** now follows from Theorem A.

It remains to verify that the hypothese of Theorem B are satisfied for **Quasi-Invariants** as well as **r-Quasi-Symmetric**

(1) Is trivially satisfied

(2) In both cases we take $q_i(x) = e_i(x)$

(3) For $QI_m[X_n]$ we take $\Pi(x) = \prod_{1\le i<j\le n}(x_i - x_j)^{2m}$

(3) For **r-Qsym** we take $\Pi(x) = (x_1 x_2 \cdots x_n)^r$

The Crucial Inequality

$$\textbf{(4)}\ \mathbf{dim\ A/(q_1, q_2, \ldots q_n)_A \ \le\ d = d_1 d_2 \ldots d_n}$$

For **1-Qsym** the proof took a considerable effort.

For **r-Qsym** when $\mathbf{r}>1$ it is an open problem.

For $\mathbf{QI_m[X_n]}$ it is not difficult and we can give here a brief sketch of the argument.

Recall that if we set for two polynomials $P, Q \in \mathbf{Q}[X_n]$

$$\langle P, Q \rangle = P(\partial_x)Q(x)\big|_{x=0}$$

we get a non degenerate bilinear form. Now there is an **m**-deformation of this form which is non degenerate on $\mathbf{QI_m[X_n]}$.

The m-Bilinear form

For any homogeneous polynomial $\mathbf{P}(\mathbf{x})$ of degree $\mathbf{d}$ we have

$$\mathbf{P}(\partial_{\mathbf{x}}) = \frac{1}{2^{d} d!} \sum_{k=0}^{d} \binom{d}{k} (-1)^{k} \Delta^{d-k} \mathbf{P}(\mathbf{x}) \Delta^{k} \qquad \left(\Delta = \sum_{i=1}^{n} \partial_{x_i}^{2} \right)$$

Define

$$\mathbf{L_m} = \Delta - 2m \sum_{1 \le i < j \le n} \frac{1}{x_i - x_j} (\partial_{x_i} - \partial_{x_j})$$

and for each $\mathbf{P} \in \mathbf{QI_m}[\mathbf{X_n}]$, homogeneous of degree $\mathbf{d}$ set

$$\gamma_{\mathbf{P}}(\mathbf{x}, \partial_{\mathbf{x}}) = \frac{1}{2^{d} d!} \sum_{k=0}^{d} \binom{d}{k} (-1)^{k} \mathbf{L_m}^{d-k} \mathbf{P}(\mathbf{x}) \mathbf{L_m}^{k}$$

For $\mathbf{P}, \mathbf{Q} \in \mathbf{QI_m}[\mathbf{X_n}]$ we set $\langle \mathbf{P} , \mathbf{Q} \rangle_{\mathbf{m}} = \gamma_{\mathbf{P}}(\mathbf{x}, \partial_{\mathbf{x}}) \, \mathbf{Q}(\mathbf{x}) \Big|_{\mathbf{x}=0}$

The m-Harmonics

Using the $\mathbf{m}$-bilinear form define "*the* $\mathbf{m}$*-Harmonics of* $[\mathbf{a}]$ " by

$$\mathbf{H}_{[\mathbf{a}]}(\mathbf{m}) = \big(\mathbf{gr}\,\mathbf{J}_{[\mathbf{a}]}\big)^{\perp_{\mathbf{m}}}$$

and the "*the* $\mathbf{m}$*-Harmonics of* $\mathbf{S_n}$ " by

$$\mathbf{H}_{\mathbf{S_n}}(\mathbf{m}) = \big(\mathbf{e_1},\mathbf{e_2},\ldots,\mathbf{e_n}\big)^{\perp_{\mathbf{m}}}$$

we have that $\big(\mathbf{e_1},\mathbf{e_2},\ldots,\mathbf{e_n}\big) \subseteq \mathbf{gr}\,\mathbf{J}_{[\mathbf{a}]} \quad \Longrightarrow \quad \mathbf{H}_{[\mathbf{a}]}(\mathbf{m}) \subseteq \mathbf{H}_{\mathbf{S_n}}(\mathbf{m})$

and the non degeneracy of the $\mathbf{m}$-bilinear form gives

$$\mathbf{n!} = \mathbf{dim\ grR}_{[\mathbf{a}]} = \mathbf{dim\ H}_{[\mathbf{a}]}(\mathbf{m}) \quad \leq \ \mathbf{dim\ H}_{\mathbf{S_n}}(\mathbf{m}) = \mathbf{dim\ QI_m}[\mathbf{X_n}]/(\mathbf{e_1},\mathbf{e_2},\ldots,\mathbf{e_n}\big)$$

so we only need to show that $\quad \mathbf{dim\ H}_{\mathbf{S_n}}(\mathbf{m}) \leq \mathbf{n!}$

The Dimension Bound

It can be shown that

$$\mathbf{H_{S_n}(m)} = \left\{\mathbf{Q} \in \mathbf{QI_m[X_n]} : \gamma_{\mathbf{e_i}}(\mathbf{x}, \partial_\mathbf{x})\mathbf{Q} = \mathbf{0} \quad \textbf{for} \quad \mathbf{i = 1, 2, \ldots, n}\right\}$$

Now it follows from the definition that

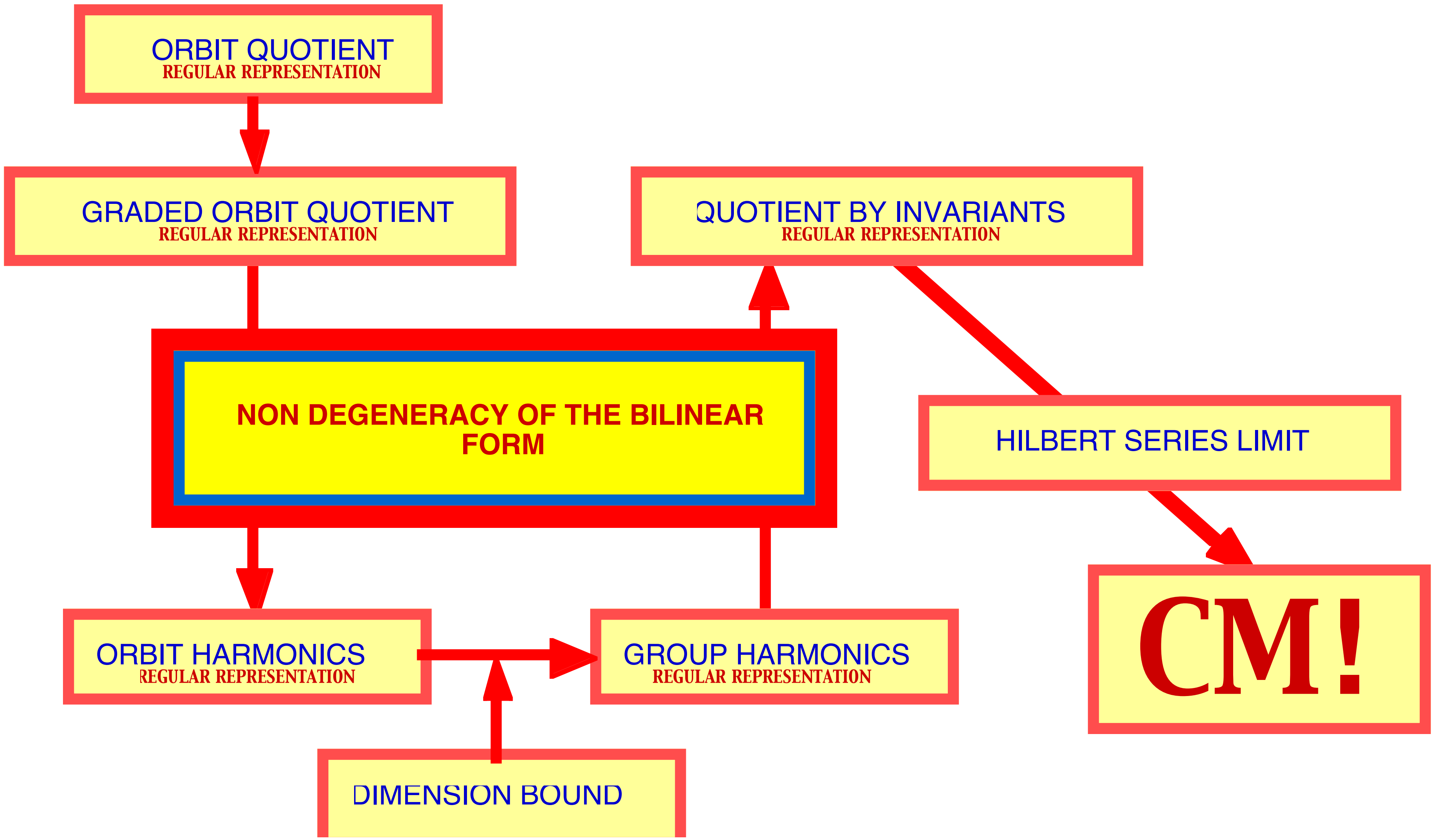
$$\gamma_{\mathbf{e_i}}(\mathbf{x}, \partial_\mathbf{x}) = \mathbf{e_i}(\partial_\mathbf{x}) + \cdots (\textbf{lower order nterms})$$

From this it follows that every derivative of an $\mathbf{m}$-Harmonic $\mathbf{Q}(\mathbf{x})$ at the point $\mathbf{a}$ espressed in terms of the $\mathbf{n}!$ derivatives

$$\partial_{\mathbf{x_1}}^{\mathbf{p_1}} \partial_{\mathbf{x_2}}^{\mathbf{p_2}} \cdots \partial_{\mathbf{x_n}}^{\mathbf{p_n}} \mathbf{Q}(\mathbf{x}) \Big|_{\mathbf{x=a}} \qquad \textbf{for} \quad \mathbf{0 \le p_i \le i - 1}$$

from which it follows that $\quad \mathbf{dim\, H_{S_n}(m) \le n!}$

THE PROOF OF COHEN MACAULINESS for m-QUASI-INVARIANTS



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