SOME NEW METHODS

IN THE THEORY

OF

m-QUASI-INVARIANTS

by

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BUT

TODAY I WILL SPEAK ON

ON

SOME OLD METHODS

IN THE THEORY

OF

m-QUASI-INVARIANTS

m-QUASI-INVARIANTS

Denote by "$\mathbf{s_{i,j}}$" the transposition that interchanges $\mathbf{x_i}$ and $\mathbf{x_j}$.

For any $\mathbf{P(x)} \in \mathbf{Q[X_n]}$ for some integer $\mathbf{r} \geq \mathbf{0}$ we have

$$\mathbf{(1 - s_{ij})P(x) \;=\; (x_i - x_j)^{2r+1} P_{ij}(x)}$$

With $\mathbf{X_n} = \{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}\}$ we set

$$\mathbf{QI_m[X_n] \;=\; \Big\{ P(x) \in Q[X_n] \,:\, (1 - s_{ij})P(x) = (x_i - x_j)^{2m+1} P_{ij}(x) \;\; \forall \;\; 1 \leq i < j \leq n \Big\}}$$

$\mathbf{QI_m[X_n]}$ is an $\mathbf{S_n}$-module

$$\mathbb{Q}\mathbf{[X_n] = QI_0[X_n] \supset QI_1[X_n] \supset QI_2[X_n] \supset \cdots \supset QI_m[X_n] \supset \cdots \supset QI_\infty[X_n] = SYM[X_n]}$$

$\mathbf{QI_m[X_n]}$ is a ring

$$\mathbf{(1 - s_{ij})\, PQ \;=\; ((1 - s_{ij})\; P)Q \;+\; (s_{ij}P)(1 - s_{ij})\; Q}$$

SOME REMARKABLE RESULTS

Theorem 1 (Etingof-Ginsburg)

$\mathbf{QI_m[X_n]}$ *is free over* $\mathbf{SYM[X_n]}$

Theorem 1 (Felder-Veselov)

The quotient

$$\mathbf{QI_m[X_n]/(e_1, e_2, \ldots, e_n)_{QI_m[X_n]}}$$

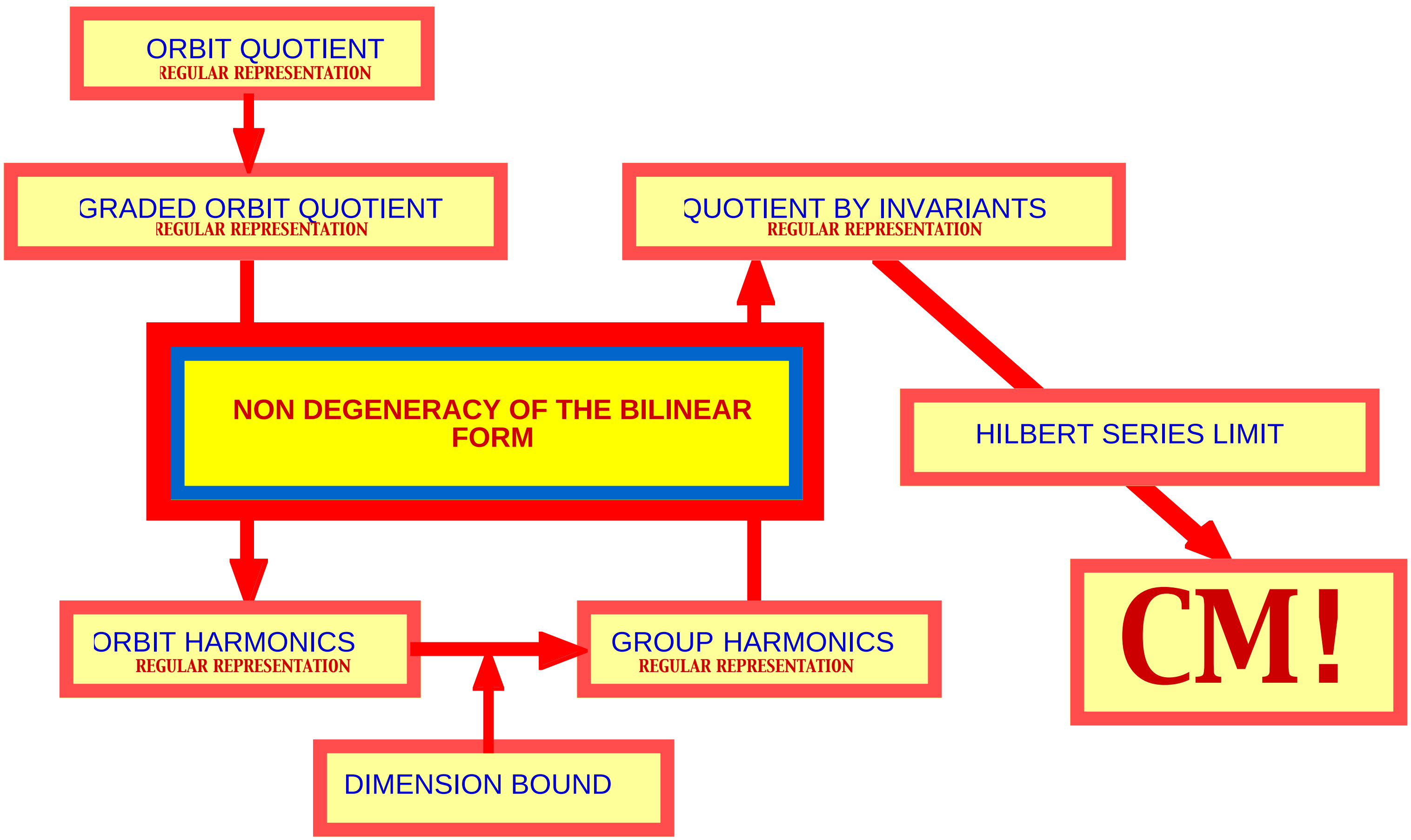
is a graded version of the Left-Regular representation of $\mathbf{S_n}$ *with Frobenius characteristic*

$$\sum_{\lambda \vdash \mathbf{n}} \mathbf{S}_\lambda(\mathbf{x}) \sum_{\mathbf{T} \in \mathbf{ST}(\lambda)} \mathbf{q}^{\mathbf{co(T)+m}\left(\binom{\mathbf{n}}{\mathbf{2}} - \mathbf{c}_\lambda\right)}$$

$$\mathbf{c}_\lambda = \sum_{(\mathbf{i},\mathbf{j}) \in \lambda} (\mathbf{j} - \mathbf{i}) = \mathbf{n}_\lambda - \mathbf{n}_{\lambda'}$$

(There are **m**-analogs of everything in sight!)

THE PROOF OF COHEN MACAULINESS for m-QUASI-INVARIANTS



THE ORBIT QUOTIENT

We select a regular point $a = (a_1, a_2, \ldots, a_n)$ and define the "*orbit of* a "

$$[\mathbf{a}] \;=\; \big\{ \mathbf{a}\sigma \;:\; \sigma \in \mathbf{S_n} \big\}$$

then define the "*ideal of the orbit of* a "

$$\mathbf{J}_{[\mathbf{a}]}(\mathbf{m}) \;=\; \Big(\mathbf{P} \in \mathbf{QI_m}[\mathbf{X_n}] \;:\; \mathbf{P}(\mathbf{b}) = \mathbf{0} \quad \forall\; \mathbf{b} \;\in [\mathbf{a}] \Big)_{\mathbf{QI_m}[\mathbf{X_n}]}$$

then define the "*Ring of the orbit of* a "

$$\mathbf{R}_{[\mathbf{a}]}(\mathbf{m}) \;=\; \mathbf{QI_m}[\mathbf{X_n}] / \mathbf{J}_{[\mathbf{a}]}(\mathbf{m})$$

Theorem(easy)

$\mathbf{R}_{[\mathbf{a}]}(\mathbf{m})$ *has dimension* $\mathbf{n}!$ *and carries the left regular representation of* $\mathbf{S_n}$.

Proof

To get a basis of $\mathbf{R}_{[\mathbf{a}]}(\mathbf{m})$, for each $\mathbf{b} \in [\mathbf{a}]$ we construct a polynomial

$$\phi_{\mathbf{b}}(\mathbf{x}) \;=\; \begin{cases} \mathbf{1} & \textbf{if } \mathbf{x} = \mathbf{b} \\ 0 & \textbf{if } \mathbf{x} \neq \mathbf{b} \end{cases} \qquad\qquad \textbf{for } \mathbf{x} \in [\mathbf{a}]$$

Then $\mathbf{S_n}$ acts on this basis precisely as it acts on itself.

THE GRADED ORBIT QUOTIENT

For a polynomial $\mathbf{P}$ define $\mathbf{h}(\mathbf{P})$ to be the homogeneous component of highest degree in $\mathbf{P}$ then set

$$\mathbf{gr}\,\mathbf{J}_{[\mathbf{a}]} \;=\; \Big(\mathbf{h}(\mathbf{P})\;:\;\mathbf{P}\in\mathbf{J}_{[\mathbf{a}]}(\mathbf{m})\Big)_{\mathbf{QI}_{\mathbf{m}}[\mathbf{X}_{\mathbf{n}}]}$$

and define "*graded Ring of the orbit of a* "

$$\mathbf{gr}\,\mathbf{R}_{[\mathbf{a}]}(\mathbf{m}) \;=\; \mathbf{QI}_{\mathbf{m}}[\mathbf{X}_{\mathbf{n}}]/\mathbf{gr}\,\mathbf{J}_{[\mathbf{a}]}(\mathbf{m})$$

Theorem(easy)

$\mathbf{gr}\,\mathbf{R}_{[\mathbf{a}]}(\mathbf{m})$ has dimension $\mathbf{n}!$ and carries the left regular representation of $\mathbf{S}_{\mathbf{n}}$.

Proof

A standard argument transfers properties from $\mathbf{R}_{[\mathbf{a}]}(\mathbf{m})$ to $\mathbf{gr}\,\mathbf{R}_{[\mathbf{a}]}(\mathbf{m})$

THE BILINEAR FORM

Proposition(easy)

For any homogeneous polynomial $\mathbf{q}(\mathbf{x})$ of degree $\mathbf{d}$ we have

$$\mathbf{q}(\partial_\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{2^d d!}} \sum_{\mathbf{k=0}}^{\mathbf{d}} \binom{\mathbf{d}}{\mathbf{k}} \boldsymbol{\Delta}^{\mathbf{d-k}} \mathbf{q}(\mathbf{x}) \boldsymbol{\Delta}^{\mathbf{k}}$$

Define

$$\mathbf{L_m} = \boldsymbol{\Delta} - \mathbf{2m} \sum_{\mathbf{1 \le i < j \le n}} \frac{\mathbf{1}}{\mathbf{x_i - x_j}} (\partial_{\mathbf{x_i}} - \partial_{\mathbf{x_j}})$$

and for each $\mathbf{q} \in \mathbf{QI_m}[\mathbf{X_n}]$, homogeneous of degree $\mathbf{d}$ set

$$\gamma_\mathbf{q}(\mathbf{x}, \partial_\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{2^d d!}} \sum_{\mathbf{k=0}}^{\mathbf{d}} \binom{\mathbf{d}}{\mathbf{k}} \mathbf{L_m^{d-k}} \mathbf{q}(\mathbf{x}) \mathbf{L_m^k}$$

then extend to all of $\mathbf{QI_m}[\mathbf{X_n}]$ by linearity.

Theorem (very hard)

The map $\mathbf{q} \to \gamma_\mathbf{q}$ *is a ring isomorphism from* $\mathbf{QI_m}[\mathbf{X_n}]$ *to the ring of operators that commute with* $\mathbf{L_m}$*. Moreover if for* $\mathbf{p}, \mathbf{q} \in \mathbf{QI_m}[\mathbf{X_n}]$ *we set*

$$\langle \mathbf{p} \, , \, \mathbf{q} \rangle_\mathbf{m} = \gamma_\mathbf{p} \, \mathbf{q} \, \Big|_{\mathbf{x=0}}$$

we get a **non-degenerate, symmetric, bilinear form** *on* $\mathbf{QI_m}[\mathbf{X_n}]$

ORBIT HARMONICS

Using the bilinear form define "*the* $\mathbf{m}$-*Harmonics of* $[\mathbf{a}]$ " by

$$\mathbf{H}_{[\mathbf{a}]}(\mathbf{m}) = \left\{\mathbf{P} \in \mathbf{QI}_{\mathbf{m}}[\mathbf{X}_{\mathbf{n}}] \,:\, \left\langle \mathbf{q}, \mathbf{P} \right\rangle_{\mathbf{m}} = \mathbf{0} \quad \forall\ \mathbf{q} \in \mathbf{gr}\, \mathbf{J}_{[\mathbf{a}]}\right\}$$

Theorem

$\mathbf{H}_{[\mathbf{a}]}(\mathbf{m})$ *has dimension* $\mathbf{n}!$ *and carries the left regular representation of* $\mathbf{S}_{\mathbf{n}}$.

Proof

Using the non degeneracy of the bilinear form we can transfer properties from $\mathbf{gr}\,\mathbf{R}_{[\mathbf{a}]}$ to $\mathbf{H}_{[\mathbf{a}]}(\mathbf{m})$

THE GROUP HARMONICS

Let $\mathbf{J}_{\mathbf{S}_{\mathbf{n}}}$ be the ideal

$$\mathbf{J}_{\mathbf{S}_{\mathbf{n}}} = \Big(\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\Big)_{\mathbf{QI}_{\mathbf{m}}[\mathbf{X}_{\mathbf{n}}]}$$

then set

$$\mathbf{H}_{\mathbf{S}_{\mathbf{n}}}(\mathbf{m}) = \left\{\mathbf{P} \in \mathbf{QI}_{\mathbf{m}}[\mathbf{X}_{\mathbf{n}}] \,:\, \left\langle \mathbf{q}, \mathbf{P} \right\rangle_{\mathbf{m}} = \mathbf{0} \quad \forall\ \mathbf{q} \in \mathbf{J}_{\mathbf{G}}\right\}$$

THE CRUCIAL EQUALITY

Proposition 1

For both $\mathbf{H_{[a]}(m)}$ *and* $\mathbf{H_{S_n}(m)}$ *we have*

$$\mathbf{H_{[a]}(m)} \;=\; \big\{\mathbf{P} \in \mathbf{QI_m[X_n]} \,:\, \gamma_{\mathbf{q}}(\mathbf{x}, \partial_{\mathbf{x}})\mathbf{P} = \mathbf{0} \quad \forall\;\; \mathbf{q} \in \mathbf{gr}\,\mathbf{J_{[a]}}\big\}$$

$$\mathbf{H_{S_n}(m)} \;=\; \big\{\mathbf{P} \in \mathbf{QI_m[X_n]} \,:\, \gamma_{\mathbf{e_i}}(\mathbf{x}, \partial_{\mathbf{x}})\mathbf{P} = \mathbf{0} \quad \forall\;\; \mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}\big\}$$

Proof

Immediate application of the non-degeneracy of the form.

Proposition 2

$$\mathbf{H_{[a]}(m)} \;\subseteq\; \mathbf{H_{S_n}(m)} \tag{$*$}$$

Proof

Since $\mathbf{e_i(x)} - \mathbf{e_i(a)} \in \mathbf{J_{[a]}}$ then $\mathbf{e_i(x)} \in \mathbf{gr}\,\mathbf{J_{[a]}}$ and (*) follows from Proposition 1.

THE ROLE OF THE DIMENSION BOUND

Theorem

$$\mathbf{dim}\,\mathbf{H_{S_n}(m)} \le \mathbf{n!} \quad\Longrightarrow\quad \mathbf{H_{[a]}(m)} = \mathbf{H_{S_n}(m)}$$

and thus $\mathbf{H_{S_n}(m)}$ *carries the left regular representation*

Two simple observations

Recall

$$\mathbf{q}(\partial_{\mathbf{x}}) \;=\; \frac{\mathbf{1}}{\mathbf{2^d d!}} \sum_{\mathbf{k=0}}^{\mathbf{d}} \binom{\mathbf{d}}{\mathbf{k}} \boldsymbol{\Delta}^{\mathbf{d-k}} \mathbf{q}(\mathbf{x}) \boldsymbol{\Delta}^{\mathbf{k}} \qquad \gamma_{\mathbf{q}}(\mathbf{x}, \partial_{\mathbf{x}}) \;=\; \frac{\mathbf{1}}{\mathbf{2^d d!}} \sum_{\mathbf{k=0}}^{\mathbf{d}} \binom{\mathbf{d}}{\mathbf{k}} \mathbf{L_m^{d-k}} \mathbf{q}(\mathbf{x}) \mathbf{L_m^{k}}$$

$$\mathbf{L_m} \;=\; \boldsymbol{\Delta} - \mathbf{2m} \sum_{\mathbf{1 \le i < j \le n}} \frac{\mathbf{1}}{\mathbf{x_i} - \mathbf{x_j}} (\partial_{\mathbf{x_i}} - \partial_{\mathbf{x_j}})$$

It follows that

$$\gamma_{\mathbf{q}}(\mathbf{x}, \partial_{\mathbf{x}}) \;=\; \mathbf{q}(\partial_{\mathbf{x}}) + \sum_{|\mathbf{p}| < \mathbf{degree(q)}} \mathbf{c_p}(\mathbf{x})\, \partial_{\mathbf{x}}^{\mathbf{p}}$$

with $\mathbf{c_p}(\mathbf{x})$ having only the factors $\mathbf{x_i} - \mathbf{x_j}$ in the denominators.

NOTE:

(1) $\mathbf{c_p}(\mathbf{x})$ **is OK as long as** $\mathbf{x}$ **is not in one of the reflecting hyperplanes**

(2) the difference $\gamma_{\mathbf{q}}(\mathbf{x}, \partial_{\mathbf{x}}) - \mathbf{q}(\partial_{\mathbf{x}})$ **is of lower order than** $\mathbf{q}(\partial_{\mathbf{x}})$

THE DIMENSION BOUND

Theorem (Feigin-Veselov)

$$\mathbf{dim\ H_{S_n}(m) \le n!}$$

Proof

Classical $(\mathbf{m=0})$ Harmonic theory gives that for every monomial $\mathbf{x^p}$ we have

$$\mathbf{x^p} = \sum_{\mathbf{0 \le \epsilon_i \le i-1}} \mathbf{x}^\epsilon \mathbf{A}_\epsilon(\mathbf{x}) \qquad (\textbf{with } \mathbf{A}_\epsilon \in \mathbf{R^{S_n}})$$

and for any $\mathbf{q} \in \mathbf{H_{S_n}(m)}$ and any $\mathbf{x^p}$ we can write

$$\partial_{\mathbf{x}}^{\mathbf{p}} \mathbf{q}(\mathbf{x}) = \sum_{\mathbf{0 \le \epsilon_i \le i-1}} \partial_{\mathbf{x}}^\epsilon \gamma_{\mathbf{A}_\epsilon}(\mathbf{x}, \partial_{\mathbf{x}}) \mathbf{q}(\mathbf{x}) + \sum_{\mathbf{0 \le \epsilon_i \le i-1}} \partial_{\mathbf{x}}^\epsilon \Big(\mathbf{A}_\epsilon(\partial_{\mathbf{x}}) - \gamma_{\mathbf{A}}(\mathbf{x}, \partial_{\mathbf{x}}) \Big) \mathbf{q}(\mathbf{x}) \qquad (*)$$

Since

$$\mathbf{A}_\epsilon(\mathbf{x}) = \mathbf{A}_\epsilon(\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n})$$

it follows that for any $\mathbf{q} \in \mathbf{H_{S_n}(m)}$ we get

$$\gamma_{\mathbf{A}_\epsilon}(\mathbf{x}, \partial_{\mathbf{x}}) \mathbf{q}(\mathbf{x}) = \mathbf{A}_\epsilon(\mathbf{0}) \mathbf{q}(\mathbf{x})$$

so (*) gives for a regular point $\mathbf{x_o}$

$$\partial_{\mathbf{x}}^{\mathbf{p}} \mathbf{q}(\mathbf{x}) \Big|_{\mathbf{x=x_o}} = \sum_{\mathbf{0 \le \epsilon_i \le i-1}} \mathbf{A}_\epsilon(\mathbf{0}) \partial_{\mathbf{x}}^\epsilon \mathbf{q}(\mathbf{x}) \Big|_{\mathbf{x=x_o}} + \cdots (\textbf{lower order derivatives of } \mathbf{q}(\mathbf{x}) \textbf{ at } \mathbf{x_o})$$

and by induction we get that $\mathbf{q}(\mathbf{x})$ is determined by $\mathbf{n}!$ derivatives at $\mathbf{x_o}$.

THE HILBERT SERIES LIMIT

Theorem (known more or less)

Let $\mathbf{A}$ *be a finitely generated graded algebra and let*

$$\mathbf{a_1, a_2, \ldots, a_k} \in \mathbf{A}$$

be homogeneous of degrees

$$\mathbf{d_1, d_2, \ldots, d_k}$$

and suppose that

$$\mathbf{dim\, A/(a_1, a_2, \ldots, a_k)_A \leq N}$$

with $\mathbf{k}$ *minimal. Then*

$$\mathbf{lim_{q \to -1}\, (1-q^{d_1})(1-q^{d_2})\cdots(1-q^{d_2})F_A = N}$$

Implies that **A is COHEN-MACAULAY**

THE m-QUASI-INVARIANT SANDWICH

Theorem

The Hilbert series of the Ring $\mathbf{QI}_m[\mathbf{X_n}]$ *of* $\mathbf{m}$*-Quasi-Invariants satisfies the coefficient-wise inequalities*

$$\frac{\mathbf{q^{2m\binom{n}{2}}}}{\mathbf{(1-q)^n}} << \mathbf{F_{QI_m[X_n]}(q)} << \frac{\mathbf{1}}{\mathbf{(1-q)^n}}$$

Thus

$$\mathbf{lim_{q\to -1}(1-q)(1-q^2)\cdots(1-q^n)F_{QI_m[X_n]}(q) \;=\; n!}$$

Proof

Every polynomial $\mathbf{P(x)} \in \mathbb{Q}[\mathbf{X_n}]$ multiplied by

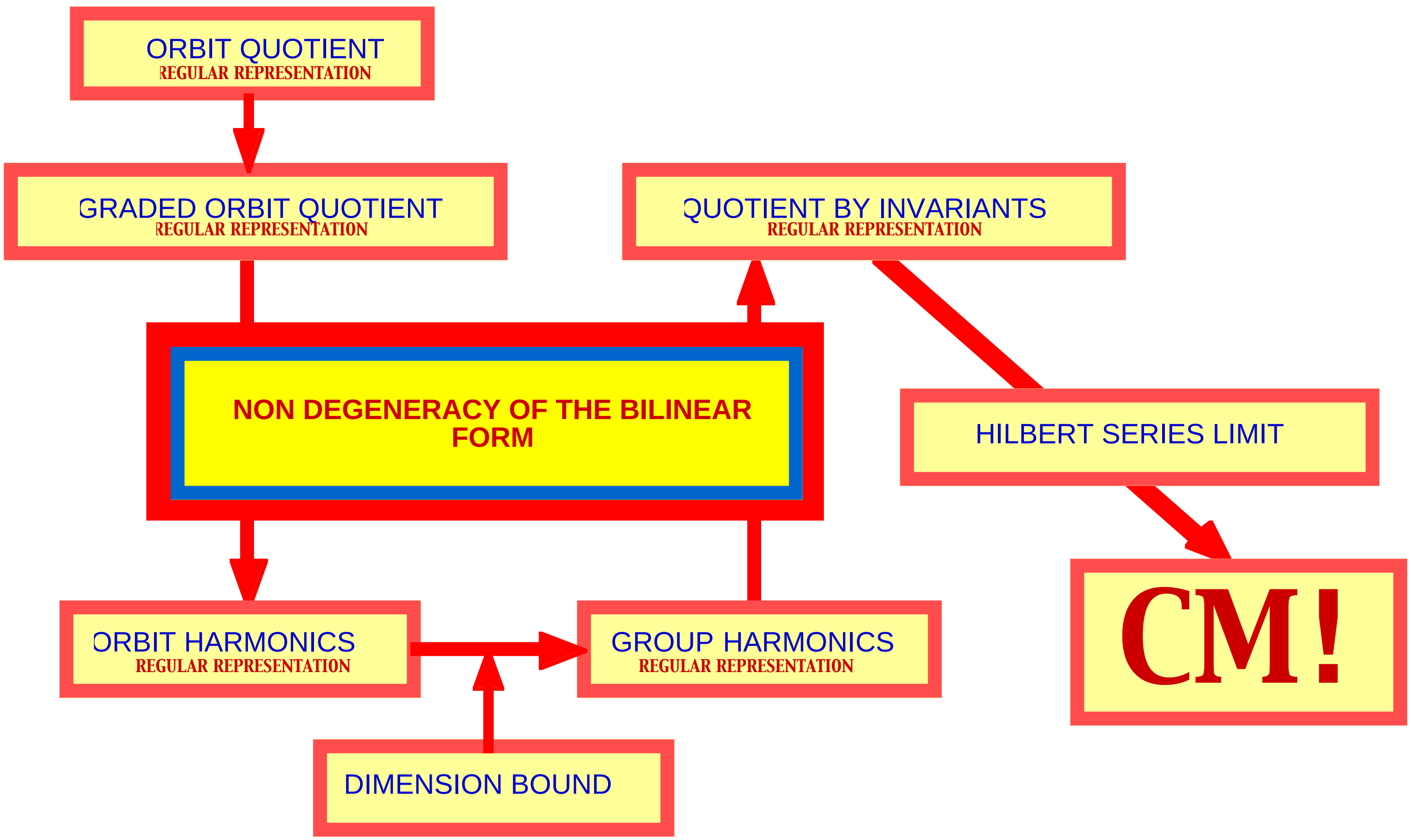
$$\mathbf{\Pi(x)^{2m}} = \prod_{\mathbf{1\le r<s\le n}} \mathbf{(x_r - x_s)^{2m}}$$

gives an $\mathbf{m}$-Quasi-Invariant. In fact for all $\mathbf{1 \le i < j \le n}$ we have

$$\mathbf{(1-s_{ij})P(x)\Pi(x)^{2m} \;=\; \Pi(x)^{2m}(1-s_{ij})P(x)}$$

QED!

THE PROOF OF COHEN MACAULINESS for m-QUASI-INVARIANTS



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