

**An Evaluation  
of the  
Dunkl-ised Vandermonde  
acting on  
the Vandermonde**

## The Dunkl operators

Let  $s_{ij}$  be the transposition that interchanges  $x_i$  and  $x_j$  then for any pair  $a, b$  we have

$$\frac{1}{(x_i - x_j)}(1 - s_{ij})x_i^a x_j^b = \begin{cases} \sum_{a \leq r \leq b-1} x_j^r x_i^{b-1-r} & \text{if } a \leq b, \\ \sum_{b \leq r \leq a-1} x_i^r x_j^{a-1-r} & \text{if } a > b. \end{cases}$$

Thus the “*divided difference*” operators

$$\delta_{ij} = \frac{1}{(x_i - x_j)}(1 - s_{ij})$$

act on polynomials.

The Dunkl operators are defined by setting for  $1 \leq i \leq n$

$$\nabla_i(\mathbf{m}) = \partial_{x_i} + \mathbf{m} \sum_{j=1}^n \binom{i}{j} \frac{1}{x_i - x_j} (1 - s_{ij})$$

**Theorem**(C. F. Dunkl)  
*They commute !!!*

$$\nabla_i(\mathbf{m})\nabla_j(\mathbf{m}) = \nabla_j(\mathbf{m})\nabla_i(\mathbf{m})$$

## The “Constant”

For a polynomial  $\mathbf{P}(x_1, x_2, \dots, x_n)$  we define “the Dunkl-ised  $\mathbf{P}(x)$ ”

$$\mathbf{P}(\nabla(\mathbf{m})) = \mathbf{P}(\nabla_1(\mathbf{m}), \nabla_2(\mathbf{m}), \dots, \nabla_n(\mathbf{m}))$$

The commutativity of the Dunkl operators makes this unambiguous.

**Theorem** (Opdam?)

$$\prod_{1 \leq i < j \leq n} (\nabla_i(\mathbf{m}) - \nabla_j(\mathbf{m})) \prod_{1 \leq i < j \leq n} (x_i - x_j) = n! \prod_{1 \leq i < j \leq n} (j\mathbf{m} + i)$$

**Easy? Hard?**

**OPEN PROBLEM:**

**Find an elementary proof**

## The Metha Integral

With

$$\Pi(\mathbf{x}) = \prod_{1 \leq i < j \leq n} (\mathbf{x}_i - \mathbf{x}_j)$$

we have

$$\frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \Pi(\mathbf{x})^{2m} e^{-|\mathbf{x}|^2/2} d_{\mathbf{x}_1} d_{\mathbf{x}_2} \cdots d_{\mathbf{x}_n} = \prod_{j=1}^n \frac{(jm)!}{m!} = c_m$$

Easy? Hard?

**OPEN PROBLEM:**

Find an elementary proof

## The Selberg Integral

$$\int_0^1 \int_0^1 \cdots \int_0^1 \left( \prod_{i=1}^n t_i^{x-1} (1-t_i)^{y-1} \right) \prod_{1 \leq i < j \leq n} (t_i - t_j)^{2k} dt_1 dt_2 \cdots dt_n$$

$$= \prod_{j=1}^n \frac{\Gamma(x + (j-1)k) \Gamma(y + (j-1)k)}{\Gamma(x + y + (n+j-2)k)} \prod_{j=1}^n \frac{\Gamma(1 + jk)}{\Gamma(1 + k)}.$$

( More or less elementary quite ingenious proof by Selberg )  
 (tricky details usually skipped)

**The Selberg Integral  $\implies$  The Metha Integral**

( A rather messy derivation )

# Enter Zeilberger

( Toward a WZ proof of the Metha integral)

For a symmetric polynomial  $\mathbf{P}$

$$\sum_{i=1}^n \partial_{x_i} \left( (\partial_{x_i} \mathbf{P}) e^{-|\mathbf{x}|^2/2} \Pi(\mathbf{x})^{2k} \right) = e^{-|\mathbf{x}|^2/2} \Pi(\mathbf{x})^{2k} \left( \mathbf{L}_m \mathbf{P} - \sum_{i=1}^n x_i \partial_{x_i} \mathbf{P} \right).$$

where

$$\mathbf{L}_m = \nabla_1(\mathbf{m})^2 + \nabla_2(\mathbf{m})^2 + \cdots + \nabla_n(\mathbf{m})^2$$

Integrating gives, for  $\mathbf{P}$  homogeneous of degree  $2\mathbf{d}$

$$\int_{\mathbb{R}_n} \mathbf{L}_m \mathbf{P} \Pi(\mathbf{x})^{2m} e^{-|\mathbf{x}|^2/2} d_{x_1} \cdots d_{x_n} = 2\mathbf{d} \int_{\mathbb{R}_n} \mathbf{P} \Pi(\mathbf{x})^{2m} e^{-|\mathbf{x}|^2/2} d_{x_1} \cdots d_{x_n}$$

and by iteration

$$\mathbf{L}_m^{\mathbf{d}} \mathbf{P} \int_{\mathbb{R}_n} \Pi(\mathbf{x})^{2m} e^{-|\mathbf{x}|^2/2} d_{x_1} \cdots d_{x_n} = 2^{\mathbf{d}} \mathbf{d}! \int_{\mathbb{R}_n} \mathbf{P} \Pi(\mathbf{x})^{2m} e^{-|\mathbf{x}|^2/2} d_{x_1} \cdots d_{x_n}$$

setting  $\mathbf{P} = \Pi(\mathbf{x})^2$  and  $\mathbf{d} = \binom{n}{2}$

$$\frac{1}{2^{\mathbf{d}} \mathbf{d}!} \mathbf{L}_m^{\mathbf{d}} \Pi(\mathbf{x})^2 c_m = c_{m+1}$$

## A DEAD END?

Zeilberger notes that the above argument shows that the Metha identity is **equivalent** to the (**simple ?**) identity

$$\frac{1}{2^d d!} L_m^d \Pi(\mathbf{x})^2 = \frac{c_{m+1}}{c_m} = n! \prod_{1 \leq i < j \leq n} (j\mathbf{m} + \mathbf{i}) \quad (*)$$

then offers 25\$ for a simple proof of (\*): I am not aware that there were any takers.

Nevertheless it turns out that (\*) is a windfall for us here

For, it develops that (\*) permits the reduction of the Dunk-lized Vandermonde identity to the Metha identity. In fact, we can prove that

$$\frac{1}{2^d d!} L_m^d \Pi(\mathbf{x})^2 = \Pi(\nabla(\mathbf{m})) \Pi(\mathbf{x})$$

## Some useful operator identities

For  $A, B, C, \dots$  linear operators on a vector space  $V$  set

$$D_A B = AB - BA$$

then

$$D_A(BC) = (D_A B)C + B D_A C$$

More generally

$$D_A^d B_1 B_2 \cdots B_k = \sum_{p_1 + p_2 + \cdots + p_k = d} \frac{d!}{p_1! p_2! \cdots p_k!} D_A^{p_1} B_1 D_A^{p_2} B_2 \cdots D_A^{p_k} B_k$$

and we also have

$$D_A^d B = \sum_{r=0}^d \binom{d}{r} (-1)^r A^{d-r} B A^r$$



# Sheer Magics

**Step 1:** for  $\mathbf{V} = \mathbb{Q}[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n]$

$$\mathbf{D}_{\mathbf{L}_m \underline{\mathbf{x}}_i} = 2\nabla_i(\mathbf{m})$$

**Step 2:**

$$\frac{1}{\mathbf{a}!2^{\mathbf{a}}} \mathbf{D}_{\mathbf{L}_m \underline{\mathbf{x}}_i}^{\mathbf{a}} = \nabla_i(\mathbf{m})^{\mathbf{a}}$$

and since  $\mathbf{L}_m = \nabla_1(\mathbf{m})^2 + \dots + \nabla_n(\mathbf{m})^2$  and  $\nabla_i(\mathbf{m})$  commute

$$\mathbf{D}_{\mathbf{L}_m \underline{\mathbf{x}}_i}^{\mathbf{b}} = \mathbf{0} \quad \forall \mathbf{b} > \mathbf{a}$$

## Theorem

For all polynomials  $\mathbf{P}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$  homogeneous of degree  $\mathbf{d}$

$$\mathbf{P}(\nabla(\mathbf{m})) = \frac{1}{2^{\mathbf{d}}\mathbf{d}!} \mathbf{D}_{\mathbf{L}_m}^{\mathbf{d}} \mathbf{P} = \sum_{\mathbf{r}=0}^{\mathbf{d}} \binom{\mathbf{d}}{\mathbf{r}} (-1)^{\mathbf{r}} \mathbf{L}_m^{\mathbf{d}-\mathbf{r}} \mathbf{P} \mathbf{L}_m^{\mathbf{r}} \quad (*)$$

## Proof

$$\mathbf{D}_{\mathbf{L}_m \underline{\mathbf{x}}_1}^{\mathbf{p}_1} \dots \mathbf{D}_{\mathbf{L}_m \underline{\mathbf{x}}_n}^{\mathbf{p}_n} = \sum_{\mathbf{a}_1 + \dots + \mathbf{a}_n = \mathbf{d}} \frac{\mathbf{d}!}{\mathbf{a}_1! \dots \mathbf{a}_n!} \mathbf{D}_{\mathbf{L}_m \underline{\mathbf{x}}_1}^{\mathbf{a}_1} \dots \mathbf{D}_{\mathbf{L}_m \underline{\mathbf{x}}_n}^{\mathbf{a}_n}$$

but (\*) and  $\mathbf{d} = \mathbf{p}_1 + \mathbf{p}_2 + \dots + \mathbf{p}_n$  gives

$$\frac{1}{2^{\mathbf{d}}\mathbf{d}!} \mathbf{D}_{\mathbf{L}_m \underline{\mathbf{x}}_1}^{\mathbf{p}_1} \dots \mathbf{D}_{\mathbf{L}_m \underline{\mathbf{x}}_n}^{\mathbf{p}_n} = \frac{1}{2^{\mathbf{p}_1}\mathbf{p}_1! \dots 2^{\mathbf{p}_n}\mathbf{p}_n!} \mathbf{D}_{\mathbf{L}_m \underline{\mathbf{x}}_1}^{\mathbf{p}_1} \dots \mathbf{D}_{\mathbf{L}_m \underline{\mathbf{x}}_n}^{\mathbf{p}_n} = \nabla_1(\mathbf{m})^{\mathbf{p}_1} \dots \nabla_n(\mathbf{m})^{\mathbf{p}_n}$$

## Punch Line

From the previous result

$$\Pi(\nabla(\mathbf{m})) = \frac{1}{2^{\mathbf{d}}\mathbf{d}!} \mathbf{D}_{\mathbf{L}_m}^{\mathbf{d}} \Pi(\underline{\mathbf{x}})$$

Thus

$$\Pi(\nabla(\mathbf{m}))\Pi(\mathbf{x}) = \frac{1}{2^{\mathbf{d}}\mathbf{d}!} \sum_{r=0}^{\mathbf{d}} \binom{\mathbf{d}}{r} (-1)^r \mathbf{L}_m^{\mathbf{d}-r} \Pi(\underline{\mathbf{x}}) \mathbf{L}_m^r \Pi(\mathbf{x}) \quad (*)$$

But for  $r > 0$

$$\mathbf{L}_m^r \Pi(\mathbf{x}) = \mathbf{0}$$

and so (\*) is none other than

$$\Pi(\nabla(\mathbf{m}))\Pi(\mathbf{x}) = \frac{1}{2^{\mathbf{d}}\mathbf{d}!} \mathbf{L}_m^{\mathbf{d}} \Pi(\underline{\mathbf{x}})\Pi(\mathbf{x})$$

**End of Story!**