Some Problems and Conjectures

in the

(manipulatorial)

Theory of Macdonald Polynomials
Notation

\[ n(\mu) = \sum_{\sigma \in \mu} l'(\sigma) = \sum_{\sigma \in \mu} l(\sigma) \]

\[ n'(\mu) = \sum_{\sigma \in \mu} a'(\sigma) - \sum_{\sigma \in \mu} a(\sigma) \]

\[ n(\mu') = \sum_{i=1}^{l(\mu)} \binom{\mu_i}{2} \]

\[ T_\mu = t^{n(\mu)} q^{n(\mu')} = \prod_{\sigma \in \mu} q^{a'_\sigma(\sigma)} t^{l'_\sigma(\sigma)} \]

\[ B_\mu(q, t) = \sum_{\sigma \in \mu} t^{l'_\sigma(\sigma)} q^{a'_\sigma(\sigma)} \]

\[ D_\mu = (1 - t)(1 - q)B_\mu(q, t) - 1 \]
Preliminaries

The Macdonald polynomials we work with here are those whose Schur function expansion is

\[ \tilde{H}_\mu(x; q, t) = \sum_\lambda S_\lambda(x) \tilde{K}_{\lambda, \mu}(q, t) \]

where

\[ \tilde{K}_{\lambda, \mu}(q, t) = t^{n(\mu)} K_{\lambda, \mu}(q, 1/t) \]

with \( K_{\lambda, \mu}(q, t) \) the Macdonald \( q, t \)-Kostka coefficient.

\[ \tilde{H}_{[3,2]}(x; q, t) = s_5 + s_{4,1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + s_{3,2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} + s_{3,1,1} q \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} + s_{2,2,1} q \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + s_{2,1,1,1} t q^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} + s_{1,1,1,1,1} t^2 q^4 \]

\( t^2 q^3 (a + bq + cq^2 + dt + etq + ft^2) \Rightarrow t^2 q^3 \begin{pmatrix} f & 0 & 0 \\ d & 0 & 0 \\ a & b & c \end{pmatrix} \) next
The Dual Pieri Rule

\[
\delta_{\mu, \nu} \tilde{H}_\mu(x; q, t) = \sum_{\nu} c_{\mu \nu}(q, t) \tilde{H}_\nu(x; q, t)
\]

with

\[
c_{\mu \nu}(q, t) = \prod_{s \in R_{\mu \nu}} \frac{t^{l_\mu(s)} - q^{o_{\mu}(s)+1}}{t^{l_\mu(s)} - q^{o_{\mu}(s)}} \prod_{s \in C_{\mu \nu}} \frac{q^{o_{\mu}(s)} - t^{l_\mu(s)+1}}{q^{o_{\mu}(s)} - t^{l_\mu(s)}}
\]

We have a combinatorial proof that

\[
\sum_{\nu \rightarrow \mu} c_{\mu \nu}(q, t) = B_\mu(q, t)
\]

We have a purely combinatorial description of the polynomial solutions of the recurrence

\[
B_k(q, t) = \sum_{\nu \rightarrow \mu} c_{\mu \nu}(q, t) B_{k-1}(q, t) \quad (B_0(q, t) = B(q, t))
\]

Problem

Give a combinatorial proof. More generally explain combinatorially why so many recurrences involving these dual Pieri coefficients have integral polynomial solutions.
$$\nabla \tilde{H}_\mu = T_\mu \tilde{H}_\mu$$

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<td>$\nabla S_4 \rightarrow$</td>
<td>0</td>
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Totally positive or totally negative

Conjecture I

For any pair of partitions $\lambda, \mu$ and for a every positive integer $m$ we have

$$(-1)^{b(\lambda')}(\nabla S_{\lambda}, S_{\mu}) \in N[q,t] \quad (*)$$

with $\langle \ , \ \rangle$ the Hall inner product and

$$b(\lambda) = \binom{1(\lambda)}{2} + \sum_{\lambda_1 < (i-1)} (i - 1 - \lambda_1)$$

The sign in $(*)$ was identified by M. Bousquet-Melou who gave a combinatorial interpretation to the left hand side of $(*)$, for $m = 1$ and $\mu = 1^n$.

Note for $\lambda = 1^n$ $(*)$ is a corollary of a general result of Mark Haiman.

Conjecture II (A. Lascoux)

For any partitions $\lambda, \mu$ we have

$$(-1)^{|\mu| - i(\mu)} \langle \nabla \bar{H}(x;0,t), S_{\lambda}(x) \rangle \in N[q,t]$$
NABLA at \( t=1 \)

\[ \tilde{\nabla} \tilde{h}_m \left[ \frac{X}{1-q} \right] = q^{(m)}(1) \left[ \frac{X}{1-q} \right] \]

\[ \tilde{\nabla} \left( F(x) \times G(x) \right) = \tilde{\nabla} F(x) \times \tilde{\nabla} G(x) \]

\[ T_\mu \bigg|_{t=1} = \prod_{i=1}^{l(\mu)} q^{(\mu, i)}(2) \]

\[ \tilde{H}_\mu(x; q, 1) = \prod_{i=1}^{l(\mu)} (1-q) \cdots (1-q^{\mu_i})h_{\mu_i} \left[ \frac{X}{1-q} \right] \]

\[ \tilde{\nabla} \tilde{H}_\mu(x; q, 1) = \prod_{i=1}^{l(\mu)} (1-q) \cdots (1-q^{\mu_i})q^{(\mu, i)}(1) h_{\mu_i} \left[ \frac{X}{1-q} \right] \]
What is known

**Theorem (M. Haiman)**

For all $\mu$

$$\langle \nabla e_\mu, S_\mu \rangle \in \mathbb{N}[q,t]$$

**Theorem (C. Lennart)**

For all $\lambda, \mu$

$$(-1)^{b(\lambda)} \langle \tilde{\nabla} S_\lambda, S_\mu \rangle \in \mathbb{N}[q,t]$$

Note Jacobi-Trudi plus multiplicativity gives

$$\tilde{\nabla} S_\lambda = \det \left\| \nabla e_{\lambda_j + j - i} \right\|_{i,j}$$

(Lennart and Bousquet-Melou use this to prove their result)

next
PROBLEM: Nabla "Lagrange inverts" what?

At $t=1$

Theorem

The formal series

$$f(z) = z \sum_{n>0} q^n \nabla e_n z^n$$

$$\left( \nabla e_n = \sum_{\mu=0} \left( \prod_{i} q^{(\mu)} h_{\mu i} \left[ \frac{x}{1-x q} \right] \right) f_{\mu} [1-q] \right)$$

is the $q$-Lagrange inverse of the series

$$F(z) = \sum_{n \geq 1} F_n z^n = \frac{z}{E(z)}$$

$$\left( E(z) = \sum_{n>0} e_n(x) z^n \right).$$

More precisely we show that

$$\sum_{n \geq 1} F_n f(z)f(zq) \cdots f(zq^{n-1}) = z.$$  

What if $t=1/q$?

$$f(z) = z \sum_{n \geq 0} q^{(\frac{n}{2})} \nabla e_n \bigg|_{t=1/q} z^n$$

$$q^{(\frac{n}{2})} \nabla e_n \bigg|_{t=1/q} = \frac{e_n \left[ X(1+q+\cdots+q^n) \right]}{1+q+\cdots+q^n} = \frac{1}{1+q+\cdots+q^n} R(z) R(zq) \cdots R(zq^n) \bigg|_{z^n}$$

$$R(z) = \sum_{n \geq 0} e_n(x) z^n$$

$$f(z) = z \sum_{n \geq 0} \frac{1}{1+q+\cdots+q^n} R(z) R(zq) \cdots R(zq^n) \bigg|_{z^n} ,$$

next
NABLA AND PARKING FUNCTIONS

\[ \hat{\nabla} e_n = \sum_{D \in \mathcal{D}_n} q^{\alpha(D)} \prod_{i=1}^{n} e_{\alpha_i(D)}(x) \]

\( \hat{\nabla}_n \) is a \( q \)-analogue of the Frobenius characteristic of a

sign twisted version of the action of \( S_n \) on parking functions

next
Two Important Operators to be further explored

\[ D_k F[X] = F[X + \frac{(1-t)(1-q)}{z}] \sum_{m \geq 0} (-z)^m e_m[X] \bigg|_{z^k} \]

\[ D_k^* F[X] = F[X - \frac{(1-1/t)(1-1/q)}{z}] \sum_{m > 0} z^m h_m[X] \bigg|_{z^k} \]

(i) \quad D_0 \tilde{H}_\mu = -D_\mu (q, t) \tilde{H}_\mu \quad , \quad (i)^* \quad D_0^* \tilde{H}_\mu = -D_\mu (1/q, 1/t) \tilde{H}_\mu 

(ii) \quad D_k e_1 - e_1 D_k = M D_{k+1} \quad , \quad (ii)^* \quad D_k^* e_1 - e_1 D_k^* = -\tilde{M} D_{k+1}^* 

(iii) \quad \nabla e_1 \nabla^{-1} = -D_1 \quad , \quad (iii)^* \quad \nabla D_1^* \nabla^{-1} = e_1 

(iv) \quad \nabla^{-1} \partial_1 \nabla = \frac{1}{M} D_{-1} \quad , \quad (iv)^* \quad \nabla^{-1} D_{-1}^* \nabla = -\tilde{M} \partial_1 

M = (1-t)(1-q) \quad , \quad \tilde{M} = (1-1/t)(1-1/q) 

For \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s, 1^a) \) with \( \lambda_1 \geq \lambda_2 \geq \cdots \lambda_s \geq 2 \) and \( a \geq 0 \) set

\[ W_\lambda[X; q, t] = e_1^a D_1^* e_1^{\lambda_1-1} D_1^* e_1^{\lambda_2-1} \cdots D_1^* e_1^{\lambda_s-1} \]

\[ U_\lambda[X; q, t] = D_1^* e_1 D_1^{\lambda_1-1} e_1 D_1^{\lambda_2-1} \cdots e_1 D_1^{\lambda_s-1} 1 \]

**Theorem**

The operator \( \nabla \) may be computed from the identity

\[ \nabla W_\lambda[X; q, t] = (-1)^{a+\sum_{i=1}^{s} (\lambda_s-1)} U_\lambda[X; q, t] \]
Lattice Polynomials

\[ \Delta_L = \det \begin{pmatrix}
  y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 \\
  y_1^3 & y_2^3 & y_3^3 & y_4^3 & y_5^3 \\
  x_1 y_1 & x_2 y_2 & x_3 y_3 & x_4 y_4 & x_5 y_5 \\
  x_1 y_1^2 & x_2 y_2^2 & x_3 y_3^2 & x_4 y_4^2 & x_5 y_5^2 \\
  x_1 y_1^3 & x_2 y_2^3 & x_3 y_3^3 & x_4 y_4^3 & x_5 y_5^3
\end{pmatrix} \]

More generally if

\[ \mathcal{L} = \{ (p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n) \} \]

\[ \Delta_L[X, Y] = \left\| x_i^{p_j} y_i^{q_j} \right\|_{i,j=1}^n \]
The Corresponding $S_n$ Modules

$$M_L = \mathcal{L} \left[ \partial_x^p \partial_y^q \Delta_J [X, Y] \right]$$

**Definition**

A Lattice diagram with $n$ cells will be called “regular” if the corresponding module is a multiple of the regular representation of $S_n$.

**Theorem (Mark Haiman)**

Every Ferrers diagram is regular and its bigraded Frobenius characteristic is

$$\tilde{H}_\mu(x; q, t)$$

**Corollary**

The diagram $\mu/00$ obtained by removing the 00 cell from a Ferrers diagram of a partition $\mu \vdash n + 1$ is regular and its bigraded Frobenius characteristic is

$$\partial_{p_1} \tilde{H}_\mu(x; q, t)$$
**Pierced Diagrams**

**Conjecture III** (Francois and Nantel)

For $\mu \vdash n + 1$ the lattice diagram $\mu/i_j$ is always regular and the module $M_{\mu/i_j}$ affords as many copies of the regular representation of $S_n$ as there are cells in the shadow of $(i,j)$.

$$\partial p_1 = c_1(q,t) + c_2(q,t) + c_3(q,t).$$

A Macdonald dual Pieri rule

$$= c_1(q,t) + c_2(q,t) + c_3(q,t).$$

The hole propagates to one of the corners in its shadow!!
The action of the operators

\[ D_x = \sum_{i=1}^{n} \partial x_i \uparrow \quad \text{THE CASE OF (4,3,1)} \]

\[ D_y = \sum_{i=1}^{n} \partial y_i \iff \]

next
THE CHARACTERISTIC OF PIERCED FERRERS MODULES

Conjecture IV

Denoting by \( C_{\mu/ij}(x; q, t) \) the characteristic of \( M_{\mu/ij} \), we have

\[
C_{\mu/ij}(x; q, t) = \sum_{\rho \rightarrow \tau} c_{\tau\rho}(q, t) \tilde{H}_{\mu - \tau + \rho}(x; q, t) \quad (\ast)
\]

where \( \tau \) denotes the partition corresponding to the shadow of \((i, j)\) and the symbol "\( \mu - \tau + \rho \)" is to represent replacing \( \tau \) by \( \rho \) in the shadow of \((i, j)\).

Theorem

Equation \((\ast)\) above is equivalent to the four term recursion

\[
C_{\mu/ij} = \frac{t^{l} - q^{a+1}}{t^{l} - q^{a}} C_{\mu/i,j+1} + \frac{t^{l+1} - q^{a}}{t^{l} - q^{a}} C_{\mu/i+1,j} - \frac{t^{l+1} - q^{a+1}}{t^{l} - q^{a}} C_{\mu/i+1,j+1},
\]

with \( l \) and \( a \) the leg an arm of \((i, j)\) in \( \mu \).
Kernels and Atoms

Let $K^x_{ij}$ denote the kernel of the operator $D_x$ as a map of $M_{ij}$ onto $M_{i+1,j}$. Similarly, let $K^y_{ij}$ be the kernel of $D_y$ as a map of $M_{ij}$ onto $M_{i,j+1}$.

Note we have

$$K^x_{i,j+1} \subseteq K^x_{ij} \quad \text{as well as} \quad K^y_{i+1,j} \subseteq K^y_{ij}$$

Set

$$A^x_{ij} = K^x_{ij} / K^x_{i,j+1} \quad \text{and} \quad A^y_{ij} = K^y_{ij} / K^y_{i+1,j}$$

and let $A^x_{ij}$ and $A^y_{ij}$ denote their respective Frobenius characteristics.

Proposition

$$K^x_{ij} = C_{\mu/ij} - t C_{\mu/i+1,j} \quad \text{and} \quad K^y_{ij} = C_{\mu/ij} - q C_{\mu/i+1}$$

and

$$A^x_{ij} = K^x_{ij} - K^x_{i,j+1} \quad \text{and} \quad A^y_{ij} = K^y_{ij} - K^y_{i+1,j}$$

In particular the four term recurrence may be rewritten in the simple form

$$t^1 A^x_{ij} = q^a A^y_{ij}$$
Computing Atoms
for the Ferrers Diagram of a $\mu + n + 1$

1. Decompose $\mu$ into rectangles since (up to a scalar) atoms do not vary within a rectangle.

2. Atoms have explicit expressions in terms of the Macdonald polynomials indexed by partitions obtained from $\mu$ by removing one of the corners in the shadow of the atom.

For the Atom $\mathbf{A}_{00}$ for $\mu = [3, 2]$ the formula is

$$\mathbf{A}_{00} = c_{32,31} \tilde{H}_{31} + c_{32,22} \tilde{H}_{22} - (c_{21,11} \tilde{H}_{22} + c_{21,2} \tilde{H}_{31} - t \tilde{H}_{31})$$

$$= \frac{1 - q}{t - q} \tilde{H}_{22} + \frac{t - 1}{t - q} \tilde{H}_{31}$$

For the Atom $\mathbf{A}_{10}$ for $\mu = [3, 2, 1]$ the formula is

$$\mathbf{A}_{10} = c_{21,11} \tilde{H}_{31} + c_{21,2} \tilde{H}_{32} - t \tilde{H}_{32} - \tilde{H}_{31}$$

$$= \frac{1 - q}{t - q} \tilde{H}_{31} + \frac{t - 1}{t - q} \tilde{H}_{32}$$

next
Gistol Polynomials

(1) A “gistol” is a lattice diagram that can be transformed to a skew diagram by row and column interchanges

(2) We postulate the existence of a family of polynomials indexed by gistols with the following basic properties:

\[
\begin{align*}
(0) & \quad G_D(x; q, t) = \tilde{H}_\mu(x; q, t) \quad \text{if} \quad D \text{ is the diagram of } \mu \\
(1) & \quad G_{D_1}(x; q, t) = G_{D_2}(x; q, t) \quad \text{if} \quad D_1 \cong D_2 \\
(2) & \quad G_{D_1}(x; q, t) = G_{D_2}(x; t, q) \quad \text{if} \quad D_2 \cong D_1' \\
(3) & \quad G_D(x; q, t) = G_{D_1}(x; q, t)G_{D_2}(x; q, t) \quad \text{if} \quad D \cong D_1 \times D_2 \\
(4) & \quad \partial_{p_1} G_D(x; q, t) = \sum_{s \in D} w_{s,D}(q, t) \ G_{D/s}(x; q, t) ,
\end{align*}
\]

Representational theoretical reasons suggest that,

in the case that \( D \) is a skew diagram,

\[
\begin{align*}
\text{a)} & \quad w_1[s, D] = t^{l_D(s)} q^{a_D(s)} \quad \text{and} \quad \text{b)} \quad w_2[s, D] = t^{l_D(s)} q^{a_D(s)}
\end{align*}
\]

Note: these properties overdetermine the family \( \{G_D(x; q, t)\}_D \),

existence is by no means guaranteed.
A Problem and a Conjecture

Problem

Construct a family of polynomials indexed by gístols that satisfies the gístol "axioms"

Conjecture

Gístol polynomials are Schur positive integral in \( q, t \).

\[
\begin{bmatrix}
    s_{3, 1} + s_{2, 2} & 2s_{2, 1, 1} & s_{1, 1, 1, 1} \\
    s_4 & 2s_{3, 1} & s_{2, 2} + s_{2, 1, 1}
\end{bmatrix}
\]

\[
s_4 + 3s_{3, 1} + 2s_{2, 2} + 3s_{2, 1, 1} + s_{1, 1, 1, 1}
\]
More on Gistols

It is a cute exercise to show that for any skew Diagram $D$ we have

$$\sum_{c \in D} t^l(c) q^{a'}(c) = \sum_{c \in D} t'^l(c) q^a(c)$$

It follows from this that Gistol Polynomials should be characteristics of left regular representations. In other words, for all gistols with $n$ cells we have

$$G_D(x; q, t)|_{t=q=1} = e^n_1$$

**Problem 1:**

*Define the bigraded modules whose Frobenius characteristics are given by Gistol polynomials*

**Problem 2:**

*Find by a combinatorial construction a family of polynomials which satisfies the Gistol ”axioms”*
A family of pistols is “complete” if it is closed under removal of cells.

Closed families are “pistols”, “ribbons”, “down-one-ribbons”, “right-one-ribbons”, etc.

Some facts:

(1) In previous work we gave a construction of a family of pistol polynomials that satisfies the pistol axioms.

(2) Jason Bandlow showed that pistol as well as down-one ribbons can be constructed by an extension of the Jim Haglund statistics.

(3) Curiously, the right-one ribbons do not carry the Haglund statistics.
Computing Gistol Polynomials

\[ \partial_{(1+t)} = (1+t) + t^2 + q + q^2 + t \]

\[ \partial_{(1+t)} = t(1+t) + q^2 + q + t + t \]

\[ \partial_{(1+t)} = \frac{t^2 - q^2}{t^2 - 1} \times \frac{q^2 - t}{t^2 - 1} \times \frac{t - 1}{t^2 - 1} \]

\[ \Box \times \Box = \frac{(1-t)(q-t^3)}{(q-t)(q^2-t^3)} + \frac{(1-t^2)(q-1)}{(q-t^2)(q-t)} + \frac{(q-1)(q^2-t^2)}{(q-t^2)(q^2-t^3)} \]

\[ \Box \times \Box = \frac{(1-t^2)(1-t)}{(q^2-t^2)(q-t^3)} + \frac{(1-t^2)(q-1)(q-t^2)}{(q-t^2)(q^2-t^3)} \]

\[ \Box = \frac{(1-t)(q-1)}{(q-t)} + \frac{(q-1)(q^2-t)}{(q-t^2)(q^2-t^3)} \]

next
Yet another surprise

$$\partial_{p_1} = (1 + t) + t^2 + q + t$$

$$\partial_{p_2} = (t^2 + t) + q + t + 1$$

$$\partial_{p_3} = \frac{q - t^2}{1 - t^2} + \frac{t - q}{1 - t^2} + \frac{1 - t}{1 - t^2}$$

$$\partial_{p_4} = \frac{(1 - t)(1 - t^2)}{(q - t)(q - t^2)} + \frac{(1 - t)(q - 1)(1 + t)^2}{(q - t)(q - t^3)} + \frac{(q - t)(q - 1)}{(q - t^2)(q - t^5)}$$

$$\partial_{p_5} = \frac{t^3 - 1}{t^3 - q} + \frac{1 - q}{t^3 - q}$$

$$\partial_{p_6} = \frac{1 - t}{q - t} + \frac{q - 1}{q - t}$$

$$\partial_{p_7} = \frac{(1 - q)}{(t - q)} + \frac{(t - 1)}{(t - q)}$$

The Atom!!!!
Everybody heard about the $n!$ conjecture

oops!!! Theorem!!!!!

But did you hear about the $n!/k$ conjecture?

Let $\mu \models n+1$ be a $k$ corner partition and 

$$\alpha_1, \alpha_2, \ldots, \alpha_k$$

be the partitions obtained by removing one of the corners of $\mu$

Conjecture:

$$\dim \left( M_{\alpha_1} \cap M_{\alpha_2} \cap \cdots \cap M_{\alpha_k} \right) = \frac{n!}{k}$$
Science Fiction?

\[ \text{Flip}_\alpha \ M = \left\{ b(\partial_x, \partial_y) \Delta_\alpha(x, y) : b(x, y) \in M \right\} \]

**Theorem**

*If \( M \) is a submodule of \( M_\alpha \) with Frobenius characteristic \( \Phi_M(x; q, t) \) then \( \text{Flip}_\alpha M \) has Frobenius characteristic*

\[ T_\alpha \omega \Phi_M(x; \frac{1}{q}, \frac{1}{t}) \]

*Let \( \alpha, \beta, \gamma \) be the three partitions obtained by removing a corner from a three corner partition.*

*As a result the Frobenius characteristics of each of these intersection submodules can be explicitly expressed in terms of the Macdonald polynomials*

\[ \tilde{H}_\alpha(x; q, t), \tilde{H}_\beta(x; q, t), \tilde{H}_\gamma(x; q, t) \]