Hilbert series of Invariants
and
Constant Term Identities

Garsia -Musiker-Wallach-Xin-Zabrocki
The action of a matrix on a polynomial

For $A = \|a_{i,j}\|_{i,j=1}^n$ and $P(x) = P(x_1, x_2, \ldots, x_n)$ we set

$$T_A P(x) = P(xA)$$

Example

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$$

$$T_A x_1^{p_1} x_2^{p_2} = (x_1 a_{1,1} + x_2 a_{2,1})^{p_1} (x_1 a_{1,2} + x_2 a_{2,2})^{p_2}$$

```plaintext
> expand(x[1]^2*x[2]^3);  
(x_1 a_{1,1} + x_2 a_{2,1})^2 (x_1 a_{1,2} + x_2 a_{2,2})^3

> expand(" ");  

x_1^5 a_{1,1}^2 a_{1,2}^3 + 3 x_1^4 a_{1,1} a_{1,2}^2 x_2 a_{2,2} + 3 x_1^3 a_{1,1} a_{1,2} x_2^2 a_{2,2} + x_1^2 a_{1,1} x_2^3 a_{2,2}^3 + 2 x_1^4 a_{1,1} x_2 a_{2,1} a_{1,2}^3 + 6 x_1^3 a_{1,1} x_2^2 a_{2,1} a_{1,2}^2 a_{2,2} + 6 x_1^2 a_{1,1} x_2^3 a_{2,1} a_{1,2} a_{2,2}^2 + 2 x_1^4 a_{1,1} x_2^4 a_{2,1} a_{1,2}^3 + x_2^3 a_{2,1}^3 x_1 a_{1,2} a_{2,2} + 3 x_2^3 a_{2,1}^2 x_1^2 a_{1,2}^2 a_{2,2} + 3 x_2^4 a_{2,1}^2 x_1 a_{1,2} a_{2,2}^2 + x_2^5 a_{2,1}^2 a_{2,2}^3

next
Hilbert series of Invariants

Let $G$ be a group of $n \times n$ matrices.

Recall that for $P(x) = P(x_1, x_2, \ldots, x_n)$ a polynomial in $R = \mathbb{C}[x_1, x_2, \ldots, x_n]$ and $M \in G$ we set

$$T_M P(x) = P(xM)$$

then

$$R^G = \left\{ P \in R : T_M P(x) = P(x) \quad \forall \quad M \in G \right\}$$

We let $\mathcal{H}_m(R^G)$ denote the subspace of $G$-invariants that are homogeneous of degree $m$ and set

$$F_{RG}(q) = \sum_{m \geq 0} q^m \dim \mathcal{H}_m(R^G)$$
A basic result of invariant Theory

Theorem

For any group $G$ with a finite invariant measure in particular for any finite group we can always construct homogeneous polynomials $\theta_1, \theta_2, \ldots, \theta_n; \phi_1, \phi_2, \ldots, \phi_k \in \mathbb{R}^G$

such that every invariant $P$ has an expansion of the form

$$P = \sum_{i=1}^{k} \Phi_i Q_i(\theta_1, \theta_2, \ldots, \theta_n) \quad \text{ (with} \quad Q_i(y_1, y_2, \ldots, y_n) \in \mathbb{C}[y_1, y_2, \ldots, y_n]$$

where the polynomials $Q_i(y_1, y_2, \ldots, y_n)$ are **uniquely** determined by $P$.

Equivalently the collection

$$\mathcal{B} = \{ \Phi_i \theta_1^{p_1} \theta_2^{p_2} \cdots \theta_n^{p_n} \}_{1 \leq i \leq k; p_i \geq 0}$$

is a basis for $\mathbb{R}^G$. We call such a collection a "basic set for $\mathbb{R}^G$$"

Note then it follows that

$$\dim \mathcal{H}_m(\mathbb{R}^G) = \sum_{i=1}^{k} \sum_{p_1 \geq 0} \sum_{p_2 \geq 0} \cdots \sum_{p_n \geq 0} q^{\deg(\Phi_i \theta_1^{p_1} \theta_2^{p_2} \cdots \theta_n^{p_n})}$$

equivalently

$$F_{\mathbb{R}^G}(q) = \frac{\sum_{i=1}^{k} q^{\deg(\Phi_i)}}{\prod_{i=1}^{n}(1 - q^{\deg \theta_i})}$$
When do we have a Basic set?

Note if the elements of a collection

$$\mathcal{B} = \{\Phi_i \theta_1^{p_1} \theta_2^{p_2} \cdots \theta_n^{p_n}\}_{1 \leq i \leq k; p_i \geq 0}$$

span $\mathcal{R}^G$ then

$$\dim \mathcal{H}_m(\mathcal{R}^G) \leq \frac{\sum_{i=1}^{k} q^{\deg(\Phi_i)}}{\prod_{i=1}^{n} (1 - q^{\deg(\theta_i)})} \bigg|_{q^m}$$

alternatively if the elements of $\mathcal{B}$ are independent then

$$\frac{\sum_{i=1}^{k} q^{\deg(\Phi_i)}}{\prod_{i=1}^{n} (1 - q^{\deg(\theta_i)})} \bigg|_{q^m} \leq \dim \mathcal{H}_m(\mathcal{R}^G)$$

Thus if we know that

$$F_{RG}(q) = \frac{\sum_{i=1}^{k} q^{\deg(\Phi_i)}}{\prod_{i=1}^{n} (1 - q^{\deg(\theta_i)})}$$

Then to show that $\mathcal{B}$ is basic we need only to show one of these two properties
How do we compute the Hilbert series?

**Moliens Theorem**

For $G$ a finite group we have

$$F_{RC}(q) = \frac{1}{|G|} \sum_{M \in G} \frac{1}{\det(1 - qM)}$$

More generally for groups with a finite $G$-invariant measure

$$F_{RC}(q) = \frac{1}{\omega(G)} \int_{M \in G} \frac{1}{\det(1 - qM)} \, d\omega$$

Such an integral often reduces to taking the constant term of the integrand
(including the density of the measure)
Combining actions

Action of a matrix $g \in \text{SL}(n; \mathbb{C})$ on a row vector

$$U = (u_1, u_2, \ldots, u_n) \rightarrow Ug^{-1}$$

Action of a matrix $g \in \text{SL}(n; \mathbb{C})$ on a column vector

$$V = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \rightarrow gV$$

Action of a matrix $g \in \text{SL}(n; \mathbb{C})$ on an $n \times n$ matrix

$$X = \|x_{i,j}\|_{i,j=1}^n \rightarrow gXg^{-1}$$

Action of a matrix $g \in \text{SL}(n; \mathbb{C})$ on a polynomial in the variables $\{u_i, v_j, x_{i,j}\}_{i,j=1}^n$

$$T_g P(U, V, X) = P(Ug^{-1}, gV, gXg^{-1})$$
Some Invariants under this action

\[ \Pi_k = tr ace X^k \quad \Theta_k = tr ace U X^{k-1} V \]

Note
\[ T_g tr ace X^k = tr ace (gXg^{-1})^k = tr ace gX^k g^{-1} = tr ace X^k \]
\[ T_g U X^k V = U g^{-1} (gXg^{-1})^k g V = U g^{-1} gX^k g^{-1} g V = U X^k V \]

\[ X := \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix} \]

\[ tr ac e (m u l t i p l y (X, X)); \]
\[ x_{1,1}^2 + 2x_{1,2}x_{2,1} + 2x_{1,3}x_{3,1} + x_{2,2}^2 + 2x_{2,3}x_{3,2} + x_{3,3}^2 \]

\[ > e x p a n d (t r a c e (m u l t i p l y (X, X, X))); \]
\[ 3x_{1,1}x_{1,3}x_{3,1} + 3x_{1,1}x_{1,2}x_{2,1} + 3x_{3,1}x_{1,3}x_{3,2} + 3x_{3,1}x_{1,2}x_{2,3} + 3x_{2,1}x_{1,3}x_{3,2} + 3x_{2,1}x_{1,2}x_{2,2} + 3x_{2,2}x_{2,3}x_{3,2} + 3x_{3,2}x_{2,3}x_{3,3} + x_{1,1}^3 + x_{2,2}^3 + x_{3,3}^3 \]

\[ > e x p a n d (m u l t i p l y (U, X, V) [1, 1]); \]
\[ v_1 u_1 x_{1,1} + v_1 u_2 x_{2,1} + v_1 u_3 x_{3,1} + v_2 u_1 x_{1,2} + v_2 u_2 x_{2,2} + v_2 u_3 x_{3,2} + v_3 u_1 x_{1,3} + v_3 u_2 x_{2,3} + v_3 u_3 x_{3,3} \]
Some not so obvious Invariants

\[ \Phi(U, X) = \det \begin{vmatrix} U & UX & UX^2 & \cdots & UX^{n-1} \\ \end{vmatrix} \quad \Psi(V, X) = \det \begin{vmatrix} V, XV, X^2V, \cdots, X^{n-1}V \\ \end{vmatrix} \]

\[ T_g \det \begin{pmatrix} U \\ UX \\ UX^2 \\ \vdots \\ UX^{n-1} \end{pmatrix} = \det \begin{pmatrix} U g^{-1} \\ U g^{-1} gXg^{-1} \\ U g^{-1} gX^2g^{-1} \\ \vdots \\ U g^{-1} gX^{n-1}g^{-1} \end{pmatrix} = \det \begin{pmatrix} U g^{-1} \\ UXg^{-1} \\ UX^2g^{-1} \\ \vdots \\ UX^{n-1}g^{-1} \end{pmatrix} = \det \begin{pmatrix} U \\ UX \\ UX^2 \\ \vdots \\ UX^{n-1} \end{pmatrix} \]

\[ T_g \det (V \ XV \ X^2V \ \cdots \ X^{n-1}V) = \det (gV \ gXV \ gX^2V \ \cdots \ gX^{n-1}V) \]

\[ = \det g \times \det (V \ XV \ X^2V \ \cdots \ X^{n-1}V) \]
The Constant Term Problem

Theorem

The Hilbert series of the polynomials $P(U, V, X)$ invariants under this action is

$$F_{UVX}(q) = \frac{1}{(1-q)^n} \prod_{i=1}^{n} \frac{1}{(1-qx_i)(1-q/x_i)} \prod_{1 \leq i < j \leq n} \frac{1-x_i/x_j}{(1-qx_i/x_j)(1-qx_j/x_i)} x_1 x_2 \cdots x_n = 1 x_1^0 x_2^0 \cdots x_n^0.$$ 

Proof

Passing from $SL_n[c]$ to $SU[n]$ and using Molien's Theorem, we derive that

$$F_{UVX}(q) = \int_{T_n} \frac{1}{\text{det}[1-qD(g)]} \, d\omega(g)$$

with $D(g)$ giving the action of $T_g$ on the alphabet $\{u_i, v^j, x_{i,j}\}_{i,j=1}^n$ with $d\omega(g)$ the Haar measure.

The integral need only be carried out over the Torus $T_n$ of diagonal matrices

$$g = \text{diag}\{e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}\} \quad \text{with} \quad e^{i\theta_1} e^{i\theta_2} \cdots e^{i\theta_n} = 1,$$

Thus

$$T_g \{u_r, v_s, x_{r,s}\}_{r,s=1}^n = \{u_r e^{-i\theta_r}, e^{i\theta_s} v_s, e^{i\theta_s} x_{r,s}, e^{-i\theta_r}\}_{r,s=1}^n.$$

gives

$$D(g) = \text{diag}\{e^{-i\theta_1}, \ldots, e^{-i\theta_n}; e^{i\theta_1}, \ldots, e^{i\theta_n}; e^{-i\theta_s} e^{i\theta_s} : 1 \leq r, s \leq n\}.$$ 

Thus

$$\text{det}[1-qD(g)] = \prod_{r=1}^{\tilde{n}} (1-q/e^{i\theta_r}) (1-q/e^{i\theta_r}) \prod_{r,s=1}^{\tilde{n}} (1-q e^{i\theta_r} e^{i\theta_s})$$

and (*) becomes

$$F_{UVX}(q) = \frac{1}{(1-q)^n} \int_{T_n} \prod_{r=1}^{n} \frac{1}{(1-q/e^{i\theta_r}) (1-q/e^{i\theta_r})} \prod_{1 < r < s < n} \frac{(1-e^{i\theta_s}/e^{i\theta_r})}{(1-q e^{i\theta_r} e^{i\theta_s}) (1-q e^{i\theta_s} e^{i\theta_r})} \frac{d\theta_1 d\theta_2 \cdots d\theta_{n-1}}{(2\pi)^{n-1}}.$$
Our Constant term can has an explicit evaluation!

Theorem

\[
\frac{1}{(1 - q)^n} \prod_{i=1}^{n} \frac{1}{(1 - qx_i)(1 - q/x_i)} \prod_{1 \leq i < j \leq n} \frac{1 - x_i/x_j}{(1 - qx_i/x_j)(1 - qx_j/x_i)} \bigg|_{x_1 x_2 \cdots x_n = 1} \bigg|_{x_1^0 x_2^0 \cdots x_n^0}
\]

\[= \frac{1 + q^{\frac{n+1}{2}}}{(1 - q) \prod_{i=1}^{n} (1 - q^i)^2 (1 - q^{n+1}) (1 - q^{\frac{n+1}{2}})}
\]

How do we show this?
A Remarkable Identity

Theorem

\[ \sum_{\sigma \in S_n} \text{sign}(\sigma) \sigma \left( \prod_{1 \leq i < j \leq n} (x_i - qx_j) \right) = ? \]

\[ = \prod_{i=1}^{n} \frac{1 - q^i}{1 - q} \prod_{1 \leq i < j \leq n} (x_i - x_j) \]

\[ \sum_{\sigma \in S_n} \text{sign}(\sigma) \sigma \left( \prod_{1 \leq i < j \leq n} (x_i - x_j) \right) = n! \prod_{1 \leq i < j \leq n} (x_i - x_j) \]

Proof (by induction)
The first reduction

Setting

\[ G(x; q) = \frac{1}{(1-q)^n} \prod_{1 \leq i < j \leq n} \frac{1}{(1-qx_i/x_j)(1-qx_j/x_i)} \]

Proposition 2

For any \( n \geq 2 \) we have

\[ G(x; q) = \frac{1}{\prod_{i=1}^{n}(1-q^i) \Delta(x)} \sum_{\sigma \in S_n} \text{sign}(\sigma)\sigma \prod_{i=1}^{n} x_i^{n-i} \left( \prod_{1 \leq i < j \leq n} \frac{1}{1-qx_i/x_j} \right) \]

where \( \Delta(x) \) denotes the Vandermonde determinant in \( x_1, x_2, \ldots, x_n \).

Proof

The previous identity can be rewritten in the form

\[ \frac{1}{\prod_{i=1}^{n}(1-q^i) \Delta(x)} \sum_{\sigma \in S_n} \text{sign}(\sigma)\sigma \left( \prod_{i=1}^{n} x_i^{n-i} \prod_{1 \leq i < j \leq n} (1-qx_j/x_i) \right) = \frac{1}{(1-q)^n}. \]

Divide both sides by the rational function \( \prod_{1 \leq i < j \leq n} \frac{1}{(1-qx_i/x_j)(1-qx_j/x_i)}. \)
The next reduction

**Proposition 3**

For \( n \geq 2 \) and for any symmetric rational function \( A(x) \) we have

\[
\frac{1}{(1-q)^n} A(x) \prod_{1 \leq i < j \leq n} \frac{1 - x_i/x_j}{(1-q x_i/x_j)(1-q x_j/x_i)} \bigg|_{x_1 x_2 \cdots x_n = 1} x_1^0 x_2^0 \cdots x_n^0 =
\]

\[
\frac{1}{\prod_{i=1}^{n}(1-q^i)} A(x) \prod_{1 \leq i < j \leq n} \frac{1 - x_i/x_j}{(1-q x_i/x_j)} \bigg|_{x_1 x_2 \cdots x_n = 1} x_1^0 x_2^0 \cdots x_n^0.
\]

(proved by manipulations using the previous identity)

If we use this with

\[
A(x) = \prod_{i=1}^{n} \frac{1}{(1-u/x_i)(1-v x_i)}
\]

we get the “tri-graded” Hilbert series

\[
F_{UVX}(u, v, q) = \frac{1}{(1-q)^n} \prod_{i=1}^{n} \frac{1}{(1-u/x_i)(1-v x_i)} \prod_{1 \leq i < j \leq n} \frac{1 - x_i/x_j}{(1-q x_i/x_j)(1-q x_j/x_i)} \bigg|_{x_1 x_2 \cdots x_n = 1} x_1^0 x_2^0 \cdots x_n^0
\]

\[
= \frac{1}{\prod_{i=1}^{n}(1-q^i)} \prod_{i=1}^{n} \frac{1}{(1-u/x_i)(1-v x_i)} \prod_{1 \leq i < j \leq n} \frac{1 - x_i/x_j}{(1-q x_i/x_j)} \bigg|_{x_1 x_2 \cdots x_n = 1} x_1^0 x_2^0 \cdots x_n^0.
\]

which keeps track of the degrees of the variables \( u_i, v_j \) and \( x_{i,j} \) separately

next
Computing Constant terms by partial fraction

We choose a total order for the variables, say $x_1, x_2, \ldots, x_n$.

We then work with the field of iterated formal Laurent series (IFLS): $\mathbb{K}((x_1))((x_2)) \cdots ((x_n))$
recursively defined as the field of formal Laurent series in $x_n$ with coefficients in $\mathbb{K}((x_1))((x_2)) \cdots ((x_{n-1}))$.
Here
\[
\mathbb{K}((x_1)) = \left\{ \sum_{m>M} a_m x_1^m : a_m \in \mathbb{K} \right\}
\]
This total order allows us to imbed the field of rational functions $\mathbb{K}(x_1, x_2, \ldots, x_n)$ as a subfield
of $\mathbb{K}((x_1))((x_2)) \cdots ((x_n))$.
Under this imbedding all the identities in $\mathbb{K}(x_1, x_2, \ldots, x_n)$ become identities in $\mathbb{K}((x_1))((x_2)) \cdots ((x_n))$.
The rational functions we will work with here may all be written in the form
\[
F = \frac{P}{(1-m_1)(1-m_2) \cdots (1-m_n)}
\]
with $P$ a Laurent polynomial and $m_1, m_2, \ldots, m_k$ Laurent monomials.

To convert $F$ we must decide whether a given factor $\frac{1}{1-m_i}$ should be converted to
\[
a) \sum_{s \geq 0} m_i^s \quad \text{or} \quad b) \sum_{s \geq 1} \frac{1}{m_i^s} \quad \left(= \frac{1}{1-\frac{1}{m_i}}\right)
\]

The total order forces one of the two "formal" inequalities $m_i < 1$ or $m_i > 1$ to be true.
In the first case we choose a) and in the second case we choose b).
To decide which we scan the monomial $m_i$ if the smallest variable has positive exponent
\[
\text{then } m_i < 1 \text{ if it has negative exponent then } m_i > 1.
\]
Eliminating a Variable

To avoid using summations we rewrite $F$ in the form

$$F = P \times \left( \prod_{m_i < 1} \frac{1}{1 - m_i} \right) \times \left( \prod_{m_j > 1} \frac{-1}{1 - \frac{1}{m_j}} \right)$$

This is called the "proper form" of $F$. To compute $F|_{x=a_1a_2...a_k}$ by Xin algorithm, say we want to eliminate "x". First we rewrite our rational function in the form

$$F = Q(x) + \frac{R(x)}{(1 - xU_1) \cdots (1 - xU_h)(x - V_1) \cdots (x - V_k)}$$

with $Q(x)$ a Laurent polynomial, $R(x)$ a polynomial of degree less than $h + k$ and $U_1, U_2, \ldots, U_h$ as well as $V_1, V_2, \ldots, V_k$ are monomials not containing $x$. With

$$xU_i < 1 \quad \text{for } 1 \leq i \leq h \quad \text{and} \quad V_j/x < 1 \quad \text{for } 1 \leq j \leq k$$

Then write

$$F(x) = Q(x) + \sum_{i=1}^{h} \frac{A_i}{(1 - xU_i)} + \sum_{j=1}^{k} \frac{B_j}{(x - V_j)}$$

with

$$A_i = (1 - xU_i)(F(x) - Q(x)) \bigg|_{x = 1/U_i} \quad \text{and} \quad B_j = (x - V_j)(F(x) - Q(x)) \bigg|_{x = V_j}.$$

This immediately yields the equalities

$$F\bigg|_{x = 0} = Q\bigg|_{x = 0} + \sum_{i=1}^{h} A_i = (F(0) + \sum_{i=1}^{h} B_i/V_j) \quad \text{if } P(x) \text{ is a polynomial}.$$

The reason for this is that for each term $\frac{B_j}{x - V_j}$ we have $V_j/x < 1$,

so that the proper form of the last summation is

$$\sum_{j=1}^{k} \frac{B_j/x}{(1 - V_j/x)}$$
Here we will only use a particular case

Proposition

Suppose that our kernel is of the form

\[
F(x) = \frac{1}{1-Ux} \frac{P\left(\frac{1}{x}\right)}{\prod_{i=1}^{k} (1 - m_i/x^{a_i})}
\]

with \( P \) a polynomial, and \( U, m_1, m_2, \ldots, m_k \) monomials not containing \( x \), \( a_i \geq 1 \) and

\[U < 1, \quad m_i/x^{a_i} < 1 \quad \text{for} \quad 1 \leq i \leq k\]

then

\[
F(x) \bigg|_{x^0} = \frac{P\left(\frac{1}{x}\right)}{\prod_{i=1}^{k} (1 - m_i/x^{a_i})} \bigg|_{x=1/U} = \frac{P(U)}{\prod_{i=1}^{k} (1 - m_iU^{a_i})}
\]

Proof

We have

\[
F(x) = \frac{A}{1-Ux} + \cdot
\]

Only the denominator factor \((1-Ux)\) contributes so

\[
F(x) \bigg|_{x^0} = A = (1-Ux)F(x) \bigg|_{x=1/U}
\]
Let us now compute a Constant term

Set

\[ R_n(u, v, w) = \frac{1}{1 - w/x_1 \cdots x_n} \prod_{i=1}^{n} \frac{1}{(1 - ux_i)(1 - v/x_i)} \prod_{1 \leq i < j \leq n} \frac{1 - x_j/x_i}{(1 - qx_j/x_i)}, \]

Lemma 1

\[ R_n(u, v, w)\bigg|_{x_1 = \cdots = x_n} = \frac{1}{1 - uv} R_{n-1}(uq, v, w)\bigg|_{x_1 = \cdots = x_{n-1}}. \]

Proof

\[ R_n(u, v, w) = \frac{1}{(1 - ux_1)(1 - v/x_1)(1 - w/x_1 \cdots x_n)} \prod_{2 \leq i \leq n} \frac{1 - x_j/x_1}{(1 - qx_j/x_1)}, \]

\[ \times \prod_{i=2}^{n} \frac{1}{(1 - ux_i)(1 - v/x_i)} \prod_{2 \leq i < j \leq n} \frac{1 - x_j/x_i}{(1 - qx_j/x_i)}, \]

We first eliminate \( x_1 \). Only the factor \( 1 - ux_1 \) contributes. Thus

\[ R_n(u, v, w)\bigg|_{x_1 = \cdots = x_n} = \frac{1}{(1 - v/x_1)(1 - w/x_1 \cdots x_n)} \prod_{2 \leq i \leq n} \frac{1 - x_j/x_1}{(1 - qx_j/x_1)}\bigg|_{x_1 = 1/u}, \]

\[ \times \prod_{i=2}^{n} \frac{1}{(1 - ux_i)(1 - v/x_i)} \prod_{2 \leq i < j \leq n} \frac{1 - x_j/x_i}{(1 - qx_j/x_i)}, \]

\[ = \frac{1}{(1 - uv)(1 - \frac{nw}{x_2 \cdots x_n})} \prod_{i=2}^{n} \frac{1}{(1 - x_iu)(1 - v/x_i)} \prod_{2 \leq i < j \leq n} \frac{1 - x_j/x_i}{(1 - qx_j/x_i)} \prod_{2 \leq i < j \leq n} \frac{1 - ux_j}{1 - uq}, \]

\[ = \frac{1}{(1 - uv)(1 - \frac{nw}{x_2 \cdots x_n})} \prod_{i=2}^{n} \frac{1}{(1 - x_iuq)(1 - v/x_i)} \prod_{2 \leq i < j \leq n} \frac{1 - x_j/x_i}{1 - qx_j/x_i}. \]

This is exactly \( \frac{1}{1 - uv} R_{n-1}(uq, v, uw) \) if we rename \( x_1 \) by \( x_{i-1} \).
To evaluate our constant term we also need

\[ F_{UVX}(u, v, q) = \frac{1}{\prod_{i=1}^{n}(1-q^i)(1-uvq^{i-1})(1-wu^nq^{(n/2)})} \]

\[ Q_n^*(u, v, w) = \prod_{i=1}^{n} \frac{1}{(1-ux_i)(1-v/x_i)} \prod_{1 \leq i < j \leq n} \frac{1-x_j/x_i}{(1-qx_j/x_i)} |_{x_n = 1/wx_1 \cdots x_{n-1}}. \]

\[ R_n(u, v, w) = \frac{1}{1-w/x_1 \cdots x_n} \prod_{i=1}^{n} \frac{1}{(1-ux_i)(1-v/x_i)} \prod_{1 \leq i < j \leq n} \frac{1-x_j/x_i}{(1-qx_j/x_i)}, \]

\[ R_n^*(u, v, w) = \frac{1}{1-1/wx_1 \cdots x_n} \prod_{i=1}^{n} \frac{1}{(1-ux_i)(1-v/x_i)} \prod_{1 \leq i < j \leq n} \frac{1-x_j/x_i}{(1-qx_j/x_i)}. \]

Combining these identities we get

\[ F_{UVX}(u, v, q) = \frac{1}{\prod_{i=1}^{n}(1-q^i)(1-uvq^{i-1})(1-wu^nq^{(n/2)})} + \frac{wv^nq^{(n/2)}}{\prod_{i=1}^{n}(1-q^i)(1-uvq^{i-1})(1-wv^nq^{(n/2)})}. \]
Conclusion

The UVX invariants have the tri-graded basis

\[ \mathcal{G}^{ab} = \left\{ \Phi^{r+1} \Pi_1^{p_1} \cdots \Pi_n^{p_n} \theta_1^{q_1} \cdots \theta_n^{q_n}; \Psi^s \Pi_1^{p_1} \cdots \Pi_n^{p_n} \theta_1^{q_1} \cdots \theta_n^{q_n} : r, s \geq 0, p_i \geq 0, q_i \geq 0 \right\} \]

Proof

Since we know the Hilbert series, it is sufficient to prove independence.

Suppose we had a vanishing linear combination \( P \).

We can assume that \( P \) is tri-homogeneous.

(1) The monomial \( \Phi^{r+1} \Pi_1^{p_1} \Pi_2^{p_2} \cdots \theta_n^{q_1} \theta_2^{q_2} \cdots \theta_n^{q_n} \) has tri-degree

\[ (0, (r + 1)n, (r + 1)\binom{n}{2}) + (0, 0, (\sum_i i p_i)) + (\sum_i q_i, \sum_i q_i, (\sum_i i q_i)) \]

(2) The monomial \( \Psi^s \Pi_1^{p_1} \Pi_2^{p_2} \cdots \theta_n^{q_1} \theta_2^{q_2} \cdots \theta_n^{q_n} \) has tri-degree

\[ (sn, 0, s\binom{n}{2}) + (0, 0, (\sum_i i p_i)) + (\sum_i q_i, \sum_i q_i, (\sum_i i q_i)) \]

This immediately shows that any tri-homogenous linear combination \( P \)
cannot contain both \( \Phi \) and \( \Psi \).

From this, independence easily follows.
THE END