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Hilbert series of Invariants and Constant Term Identities

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The action of a matrix on a polynomial

For $\mathbf{A} = \|a_{i,j}\|_{i,j=1}^{n}$ and $P(x) = P(x_1, x_2, \ldots, x_n)$ we set

$$T_A P(x) \;=\; P(xA)$$

Example

$$A \;=\; \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$$

$$T_A x_1^{p_1} x_2^{p_2} \;=\; (x_1 a_{1,1} + x_2 a_{2,1})^{p_1} (x_1 a_{1,2} + x_2 a_{2,2})^{p_2}$$

```
> expo(x[1]^2*x[2]^3);
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$$(x_1 a_{1,1} + x_2 a_{2,1})^2 (x_1 a_{1,2} + x_2 a_{2,2})^3$$

```
> expand(");
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$$x_1^5 a_{1,1}^2 a_{1,2}^3 + 3 x_1^4 a_{1,1}^2 a_{1,2}^2 x_2 a_{2,2} + 3 x_1^3 a_{1,1}^2 a_{1,2} x_2^2 a_{2,2}^2 + x_1^2 a_{1,1}^2 x_2^3 a_{2,2}^3$$
$$+ 2 x_1^4 a_{1,1} x_2 a_{2,1} a_{1,2}^3 + 6 x_1^3 a_{1,1} x_2^2 a_{2,1} a_{1,2}^2 a_{2,2} + 6 x_1^2 a_{1,1} x_2^3 a_{2,1} a_{1,2} a_{2,2}^2$$
$$+ 2 x_1 a_{1,1} x_2^4 a_{2,1} a_{2,2}^3 + x_2^2 a_{2,1}^2 x_1^3 a_{1,2}^3 + 3 x_2^3 a_{2,1}^2 x_1^2 a_{1,2}^2 a_{2,2}$$
$$+ 3 x_2^4 a_{2,1}^2 x_1 a_{1,2} a_{2,2}^2 + x_2^5 a_{2,1}^2 a_{2,2}^3$$

Hilbert series of Invariants

Let G be a group of $n \times n$ matrices.

Recall that for $P(x) = P(x_1, x_2, \ldots, x_n)$

a polynomial in $R = \mathbb{C}[x_1, x_2, \ldots, x_n]$ and $M \in G$ we set

$$T_M P(x) \;=\; P(xM)$$

then

$$R^G \;=\; \left\{ P \in R \,:\, T_M P(x) = P(x) \quad \forall \quad M \in G \right\}$$

We let $\mathcal{H}_m(R^G)$ denote the subspace of G-invariants

that are homogeneous of degree m and set

$$F_{R^G}(q) \;=\; \sum_{m \geq 0} q^m \dim \mathcal{H}_m(R^G)$$

next

A basic result of invariant Theory

Theorem

For any group $\mathbf{G}$ *with a finite invariant measure in particular for any finite group we can always construct homogeneous polynomials* $\theta_1, \theta_2, \ldots, \theta_n; \phi_1, \phi_2, \ldots, \phi_k \in \mathbf{R}^{\mathbf{G}}$ *such that every invariant* $\mathbf{P}$ *has an expansion of the form*

$$\mathbf{P} = \sum_{\mathbf{i}=1}^{\mathbf{k}} \Phi_{\mathbf{i}}\, \mathbf{Q}_{\mathbf{i}}(\theta_1\, \theta_2, \ldots, \theta_{\mathbf{n}}) \qquad (\textit{with} \quad \mathbf{Q}_{\mathbf{i}}(\mathbf{y}_1\, \mathbf{y}_2, \ldots, \mathbf{y}_{\mathbf{n}}) \in \mathbb{C}[\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{\mathbf{n}}])$$

where the polynomials $\mathbf{Q}_{\mathbf{i}}(\mathbf{y}_1\, \mathbf{y}_2, \ldots, \mathbf{y}_{\mathbf{n}})$ *are* **uniquely** *determined by* $\mathbf{P}$.

Equivalently the collection

$$\mathcal{B} = \left\{ \Phi_{\mathbf{i}}\, \theta_1^{\mathbf{p}_1} \theta_2^{\mathbf{p}_2} \cdots \theta_{\mathbf{n}}^{\mathbf{p}_{\mathbf{n}}} \right\}_{1 \le \mathbf{i} \le \mathbf{k}\,;\, \mathbf{p}_{\mathbf{i}} \ge 0}$$

is a basis for $\mathbf{R}^{\mathbf{G}}$. *We call such a collection a '***basic set for** $\mathbf{R}^{\mathbf{G}}$*"*

Note then it follows that

$$\mathbf{dim}\mathcal{H}_{\mathbf{m}}(\mathbf{R}^{\mathbf{G}}) = \sum_{\mathbf{i}=1}^{\mathbf{k}} \sum_{\mathbf{p}_1 \ge 0} \sum_{\mathbf{p}_2 \ge 0} \cdots \sum_{\mathbf{p}_{\mathbf{n}} \ge 0} \mathbf{q}^{\mathbf{deg}(\Phi_{\mathbf{i}} \theta_1^{\mathbf{p}_1} \theta_2^{\mathbf{p}_2} \cdots \theta_{\mathbf{n}}^{\mathbf{p}_{\mathbf{n}}})} \Big|_{\mathbf{q}^{\mathbf{m}}}$$

equivalently

$$\mathbf{F}_{\mathbf{R}^{\mathbf{G}}}(\mathbf{q}) \quad = \quad \frac{\sum_{\mathbf{i}=1}^{\mathbf{k}} \mathbf{q}^{\mathbf{deg}(\Phi_{\mathbf{i}})}}{\prod_{\mathbf{i}=1}^{\mathbf{n}} (1 - \mathbf{q}^{\mathbf{deg}\, \theta_{\mathbf{i}}})}$$

next

When do we have we have a Basic set ?

Note if the elements of a collection

$$\mathcal{B} = \left\{ \Phi_i \, \theta_1^{p_1} \theta_2^{p_2} \cdots \theta_n^{p_n} \right\}_{1 \le i \le k \,;\, p_i \ge 0}$$

span R^G then

$$\dim \mathcal{H}_m(R^G) \le \left. \frac{\sum_{i=1}^{k} q^{\deg(\Phi_i)}}{\prod_{i=1}^{n} (1 - q^{\deg \theta_i})} \right|_{q^m}$$

alternatively if the elements of $\mathcal{B}$ are independent then

$$\left. \frac{\sum_{i=1}^{k} q^{\deg(\Phi_i)}}{\prod_{i=1}^{n} (1 - q^{\deg \theta_i})} \right|_{q^m} \le \dim \mathcal{H}_m(R^G)$$

Thus if we know that

$$F_{R^G}(q) = \frac{\sum_{i=1}^{k} q^{\deg(\Phi_i)}}{\prod_{i=1}^{n} (1 - q^{\deg \theta_i})}$$

Then to show that $\mathcal{B}$ is basic we need only to show one of these two properties

next

How do we compute the Hilbert series?

Moliens Theorem

For G a finite group we have

$$F_{R^G}(q) = \frac{1}{|G|} \sum_{M \in G} \frac{1}{\det(1 - qM)}$$

More generally for groups with a finite G-invariant measure

$$F_{R^G}(q) = \frac{1}{\omega(G)} \int_{M \in G} \frac{1}{\det(1 - qM)} d\omega$$

Such an integral often reduces to taking the constant term of the integrand (including the density of the measure)

next

Combining actions

Action of a matrix $g \in SL(n;\mathbb{C})$ on a row vector

$$U = (u_1, u_2, \ldots, u_n) \longrightarrow Ug^{-1}$$

Action of a matrix $g \in SL(n;\mathbb{C})$ on a column vector

$$V = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \longrightarrow gV$$

Action of a matrix $g \in SL(n;\mathbb{C})$ on an $n \times n$ matrix

$$X = \|x_{i,j}\|_{i,j=1}^{n} \longrightarrow gXg^{-1}$$

Action of a matrix $g \in SL(n;\mathbb{C})$ on a polynomial in the variables $\{u_i, v_j, x_{i,j}\}_{i,j=1}^{n}$

$$T_g P(U, V, X) = P\left(Ug^{-1}, gV, gXg^{-1}\right)$$

Some Invariants under this action

$$\Pi_k = trace\, X^k \qquad \Theta_k = trace\, U X^{k-1} V$$

Note

$$\mathrm{T_g\, trace\, X^k = trace\,(gXg^{-1})^k = trace\, gX^kg^{-1} = trace\, X^k}$$

$$\mathrm{T_g\, U\, X^k V = Ug^{-1}\left(gXg^{-1}\right)^k gV = Ug^{-1}gX^kg^{-1}gV = U\, X^k V}$$

$$X := \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix}$$

```
trace(multiply(X,X));
```

$$x_{1,1}^2 + 2\, x_{1,2}\, x_{2,1} + 2\, x_{1,3}\, x_{3,1} + x_{2,2}^2 + 2\, x_{2,3}\, x_{3,2} + x_{3,3}^2$$

```
> expand(trace(multiply(X,X,X)));
```

$$3\, x_{1,1}\, x_{1,3}\, x_{3,1} + 3\, x_{1,1}\, x_{1,2}\, x_{2,1} + 3\, x_{3,1}\, x_{1,3}\, x_{3,3} + 3\, x_{3,1}\, x_{1,2}\, x_{2,3} + 3\, x_{2,1}\, x_{1,3}\, x_{3,2}$$
$$+ 3\, x_{2,1}\, x_{1,2}\, x_{2,2} + 3\, x_{2,2}\, x_{2,3}\, x_{3,2} + 3\, x_{3,2}\, x_{2,3}\, x_{3,3} + x_{1,1}^3 + x_{2,2}^3 + x_{3,3}^3$$

```
> expand(multiply(U,X,V)[1,1]);
```

$$v_1\, u_1\, x_{1,1} + v_1\, u_2\, x_{2,1} + v_1\, u_3\, x_{3,1} + v_2\, u_1\, x_{1,2} + v_2\, u_2\, x_{2,2} + v_2\, u_3\, x_{3,2} + v_3\, u_1\, x_{1,3}$$
$$+ v_3\, u_2\, x_{2,3} + v_3\, u_3\, x_{3,3}$$

next

Some not so obvious Invariants

$$\Phi(U,X) = \det \left\| \begin{matrix} U \\ UX \\ UX^2 \\ \vdots \\ UX^{n-1} \end{matrix} \right\| \qquad \Psi(V,X) = \det \left\| V, XV, X^2V, \ldots, X^{n-1}V \right\|$$

$$T_g \det \begin{pmatrix} U \\ UX \\ UX^2 \\ \vdots \\ UX^{n-1} \end{pmatrix} = \det \begin{pmatrix} Ug^{-1} \\ Ug^{-1}gXg^{-1} \\ Ug^{-1}gX^2g^{-1} \\ \vdots \\ Ug^{-1}gX^{n-1}g^{-1} \end{pmatrix} = \det \begin{pmatrix} Ug^{-1} \\ UXg^{-1} \\ UX^2g^{-1} \\ \vdots \\ UX^{n-1}g^{-1} \end{pmatrix} = \det \begin{pmatrix} U \\ UX \\ UX^2 \\ \vdots \\ UX^{n-1} \end{pmatrix}$$

$$\begin{aligned} T_g \det \begin{pmatrix} V & XV & X^2V & \cdots & X^{n-1}V \end{pmatrix} &= \det \begin{pmatrix} gV & gXV & gX^2V & \cdots & gX^{n-1}V \end{pmatrix} \\ &= \det g \times \det \begin{pmatrix} V & XV & X^2V & \cdots & X^{n-1}V \end{pmatrix} \end{aligned}$$

next

The Constant Term Problem

Theorem

The Hilbert series of the polynomials $P(U,V,X)$ *invariants under this action is*

$$F_{UVX}(q) = \frac{1}{(1-q)^n}\prod_{i=1}^{n}\frac{1}{(1-qx_i)(1-q/x_i)} \prod_{1\leq i<j\leq n}\frac{1-x_i/x_j}{(1-qx_i/x_j)(1-qx_j/x_i)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}$$

Proof

Passing from $SL_n[\mathbb{C}]$ to $SU[n]$ and using Moliens Theorem, we derive that

$$F_{UVX}(q) = \int_{T_n}\frac{1}{\det|1-qD(g)|}d\omega(g)$$

with $D(g)$ giving the action of T_g on the on the alphabet $\{u_i,v_j,x_{i,j}\}_{i,j=1}^n$ with $dw(g)$ the Haar measure.

The integral need only be carried out over the Thorus T_n of diagonal matrices

$$g = \mathrm{diag}\{e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}\} \quad \text{with} \quad e^{i\theta_1}e^{i\theta_2}\cdots e^{i\theta_n}=1,$$

Thus

$$T_g\{u_r,v_s,x_{r,s}\}_{r,s=1}^n = \{u_re^{-i\theta_r}, e^{i\theta_s}v_s, e^{i\theta_s}x_{r,s}e^{-i\theta_r}\}_{r,s=1}^n.$$

gives

$$D(g) = \mathrm{diag}\{e^{-i\theta_1},\ldots,e^{-i\theta_n};\ e^{i\theta_1},\ldots,e^{i\theta_n};\ \ e^{-i\theta_r}e^{i\theta_s}\ :\ 1\leq r,s\leq n\}$$

Thus

$$\det|1-qD(g)| = \prod_{r=1}^{n}(1-q/e^{i\theta_r})(1-qe^{i\theta_r})\prod_{r,s=1}^{n}(1-qe^{i\theta_s}/e^{i\theta_r})$$

and (*) becomes

$$F_{UVX}(q) = \frac{1}{(1-q)^n}\int_{T_n}\prod_{r=1}^{n}\frac{1}{(1-q/e^{i\theta_r})(1-qe^{i\theta_r})}\prod_{1<r<s<n}\frac{(1-e^{i\theta_s}/e^{i\theta_r})}{(1-qe^{i\theta_r}/e^{i\theta_s})(1-qe^{i\theta_s}/e^{i\theta_r})}\frac{d\theta_1 d\theta_2\cdots d\theta_{n-1}}{(2\pi)^{n-1}}.$$

next

Our Constant term can has an explicit evaluation!

Theorem

$$\frac{1}{(1-q)^n}\prod_{i=1}^{n}\frac{1}{(1-qx_i)(1-q/x_i)}\prod_{1\le i<j\le n}\frac{1-x_i/x_j}{(1-qx_i/x_j)(1-qx_j/x_i)}\Bigg|_{x_1x_2\cdots x_n=1}\Bigg|_{x_1^0x_2^0\cdots x_n^0}$$

$$=\frac{1+q^{\binom{n+1}{2}}}{(1-q)\prod_{i=1}^{n}(1-q^i)^2\left(1-q^{n+1}\right)\left(1-q^{\binom{n+1}{2}}\right)}$$

How do we show this?

next

A Remarkable Identity

Theorem

$$\sum_{\sigma \in S_n} sign(\sigma)\sigma\Big(\prod_{1\le i<j\le n}(x_i - qx_j)\Big) = \quad ?$$

$$= \prod_{i=1}^{n} \frac{1-q^i}{1-q} \prod_{1\le i<j\le n}(x_i - x_j)$$

$$\sum_{\sigma \in S_n} sign(\sigma)\sigma\Big(\prod_{1\le i<j\le n}(x_i - x_j)\Big) = \mathbf{n!} \prod_{1\le i<j\le n}(x_i - x_j)$$

Proof *(by induction)*

next

The first reduction

Setting

$$G(x;q) = \frac{1}{(1-q)^n} \prod_{1\le i<j\le n} \frac{1}{(1-qx_i/x_j)(1-qx_j/x_i)}$$

Proposisition 2

For any $n \ge 2$ *we have*

$$G(x;q) = \frac{1}{\prod_{i=1}^n (1-q^i)} \frac{1}{\Delta(x)} \sum_{\sigma\in S_n} \mathrm{sign}(\sigma)\sigma \prod_{i-1}^{n} x_i^{n-i} \Big(\prod_{1\le i<j\le n} \frac{1}{1-qx_i/x_j} \Big)$$

where $\Delta(x)$ *denotes the Vandermonde determinant in* $x_1, x_2, \ldots, x_n$.

Proof

The previous identity can be rewritten in the form

$$\frac{1}{\prod_{i=1}^n (1-q^i)} \frac{1}{\Delta(x)} \sum_{\sigma\in S_n} \mathrm{sign}(\sigma)\sigma \Big(\prod_{i-1}^{n} x_i^{n-i} \prod_{1\le i<j\le n} (1-qx_j/x_i) \Big) = \frac{1}{(1-q)^n}.$$

Divide both sides by the rational function $\displaystyle \prod_{1\le i<j\le n} \frac{1}{(1-qx_i/x_j)(1-qx_j/x_i)}$.

next

The next reduction

Proposition 3

For $n \geq 2$ *and for any symmetric rational funtion* $A(x)$ *we have*

$$\frac{1}{(1-q)^n} A(x) \prod_{1\leq i<j\leq n} \frac{1-x_i/x_j}{(1-qx_i/x_j)(1-qx_j/x_i)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0} =$$

$$\frac{1}{\prod_{i=1}^n(1-q^i)} A(x) \prod_{1\leq i<j\leq n} \frac{1-x_i/x_j}{(1-qx_i/x_j)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}.$$

(proved by manipulations using the previous identity)

If we use this with

$$A(x) = \prod_{i=1}^n \frac{1}{(1-u/x_i)(1-vx_i)}$$

we get the "tri-graded" Hilbert series

$$F_{UVX}(u,v,q) = \frac{1}{(1-q)^n} \prod_{i=1}^n \frac{1}{(1-u/x_i)(1-vx_i)} \prod_{1\leq i<j\leq n} \frac{1-x_i/x_j}{(1-qx_i/x_j)(1-qx_j/x_i)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}$$

$$= \frac{1}{\prod_{i=1}^n(1-q^i)} \prod_{i=1}^n \frac{1}{(1-u/x_i)(1-vx_i)} \prod_{1\leq i<j\leq n} \frac{1-x_i/x_j}{(1-qx_i/x_j)}\Big|_{x_1x_2\cdots x_n=1}\Big|_{x_1^0x_2^0\cdots x_n^0}.$$

which keeps track of the degrees of the variables u_i, v_j and $x_{i,j}$ separately

next

Computing Constant terms by partial fraction

We choose a total order for the variables, say $x_1, x_2, \ldots, x_n$.

We then work with the field of iterated formal Laurent series (IFLS): $K((x_1))((x_2))\cdots((x_n))$

recursively defined as the field of formal Laurent series in x_n with coefficients in $K((x_1))((x_2))\cdots((x_{n-1}))$.

Here

$$K((x_1)) = \left\{ \sum_{m>M} a_m x_1^m \;:\; a_m \in K \right\}$$

This total order allows us to imbed the field of rational functions $K(x_1, x_2, \ldots, x_n)$ as a subfield of $K((x_1))((x_2))\cdots((x_n))$.

Under this imbedding all the identities in $K(x_1, x_2, \ldots, x_n)$ become identities in $K((x_1))((x_2))\cdots((x_n))$.

The rational functions we will work with here may all be written in the form

$$F = \frac{P}{(1-m_1)(1-m_2)\cdots(1-m_n)}$$

with P a Laurent polynomial and $m_1, m_2, \ldots, m_k$ Laurent monomials.

To convert F we must decide whether a given factor $\frac{1}{1-m_i}$ should be converted to

$$\text{a)} \quad \sum_{s\geq 0} m_i^s \quad \text{or} \quad \text{b)} \quad -\sum_{s\geq 1} \frac{1}{m_i^s} \; \left(= \frac{-\frac{1}{m_1}}{1-\frac{1}{m_1}} \right)$$

The total order forces one of the two "*formal*" inequalities $m_i < 1$ or $m_i > 1$ to be true

In the first case we choose a) and in the second case we choose b) .

To decide which we scan the monomial m_i if the smallest variable has positive exponent then $m_i < 1$ if it has negative exponent then $m_i > 1$.

next

Eliminating a Variable

To avoid using summations we rewrite F in the form

$$F = P \times \left(\prod_{m_i<1} \frac{1}{1-m_i} \right) \times \left(\prod_{m_j>1} \frac{-\frac{1}{m_j}}{1-\frac{1}{m_j}} \right)$$

This is called the "*proper form*" of F. To compute $F\big|_{a_1^0 a_2^0 \cdots a_k^0}$ by Xin algorithm,

Say we want to eliminate "x". First we rewrite our rational function in the form

$$F = Q(x) + \frac{R(x)}{(1-xU_1)\cdots(1-xU_h)(x-V_1)\cdots(x-V_k)}$$

with $Q(x)$ a Laurent polynomial, $R(x)$ a polynomial of degree less than $h+k$ and $U_1, U_2, \ldots, U_h$

as well as $V_1, V_2, \ldots, V_k$ are monomials not containing x. With

$$xU_i < 1 \quad \text{for } 1 \le i \le h \qquad \text{and} \qquad V_j/x < 1 \quad \text{for } 1 \le j \le k.$$

Then write

$$F(x) = Q(x) + \sum_{i=1}^{h} \frac{A_i}{(1-xU_i)} + \sum_{j=1}^{k} \frac{B_j}{(x-V_j)}$$

with

$$A_i = (1-xU_i)(F(x)-Q(x))\Big|_{x=1/U_1} \qquad \text{and} \qquad B_j = (x-V_j)(F(x)-Q(x))\Big|_{x=V_j}.$$

This immediately yields the equalities

$$F\Big|_{x^0} = Q\Big|_{x^0} + \sum_{i=1}^{h} A_i = \Big(F(0) + \sum_{i=1}^{h} B_i/V_j \quad \text{if } P(x) \text{ is a polynomial}\Big)$$

The reason for this is that for each term $\frac{B_j}{x-V_j}$ we have $V_j/x < 1$,

so that the proper form of the last summation is $\displaystyle\sum_{j=1}^{k} \frac{B_j/x}{(1-V_j/x)}$

next

Here we will only use a particular case

Proposition

Suppose that our kernel is of the form

$$\mathbf{F(x)} \;=\; \frac{\mathbf{1}}{\mathbf{1-Ux}}\,\frac{\mathbf{P(\frac{1}{x})}}{\prod_{\mathbf{i=1}}^{\mathbf{k}}(\mathbf{1-m_i/x^{a_i}})}$$

with $\mathbf{P}$ *a polynomial, and* $\mathbf{U, m_1, m_2, \ldots, m_k}$ *monomials not containing* $\mathbf{x}$*,* $\mathbf{a_i \geq 1}$ *and*

$$\mathbf{Ux < 1}\ , \qquad \mathbf{m_i/x^{a_i} < 1} \quad \textbf{for} \quad \mathbf{1 \leq i \leq k}$$

then

$$\mathbf{F(x)}\Big|_{\mathbf{x^0}} = \frac{\mathbf{P(\frac{1}{x})}}{\prod_{\mathbf{i=1}}^{\mathbf{k}}(\mathbf{1-m_i/x^{a_i}})}\Bigg|_{\mathbf{x=1/U}} \;=\; \frac{\mathbf{P(U)}}{\prod_{\mathbf{i=1}}^{\mathbf{k}}(\mathbf{1-m_iU^{a_i}})}$$

Proof

We have

$$F(x) \;=\; \frac{A}{1-Ux} + \cdot$$

Only the denominator factor $(\mathbf{1-Ux})$ contributes so

$$\mathbf{F(x)}\Big|_{\mathbf{x^0}} \;=\; \mathbf{A} \;=\; (\mathbf{1-Ux})\mathbf{F(x)}\Big|_{\mathbf{x=1/U}}$$

next

Let us now compute a Constant term

Set
$$R_n(u,v,w) = \frac{1}{1-w/x_1\cdots x_n}\prod_{i=1}^{n}\frac{1}{(1-ux_i)(1-v/x_i)}\prod_{1\le i<j\le n}\frac{1-x_j/x_i}{(1-qx_j/x_i)},$$

Lemma 1
$$R_n(u,v,w)\big|_{x_1^0\cdots x_n^0} = \frac{1}{1-uv}\,R_{n-1}(uq,v,w\,u)\big|_{x_1^0\cdots x_{n-1}^0}.$$

Proof
$$R_n(u,v,w) = \frac{1}{(1-ux_1)(1-v/x_1)}\frac{1}{(1-w/x_1\cdots x_n)}\prod_{2\le j\le n}\frac{1-x_j/x_1}{(1-qx_j/x_1)},$$
$$\times\prod_{i=2}^{n}\frac{1}{(1-ux_i)(1-v/x_i)}\prod_{2\le i<j\le n}\frac{1-x_j/x_i}{(1-qx_j/x_i)},$$

We first eliminate x_1. Only the factor $1-ux_1$ contributes. Thus
$$R_n(u,v,w)\Big|_{x_1^0} = \frac{1}{(1-v/x_1)}\frac{1}{(1-w/x_1\cdots x_n)}\prod_{2\le j\le n}\frac{1-x_j/x_1}{(1-qx_j/x_1)}\Big|_{x_1=1/u},$$
$$\times\prod_{i=2}^{n}\frac{1}{(1-ux_i)(1-v/x_i)}\prod_{2\le i<j\le n}\frac{1-x_j/x_i}{(1-qx_j/x_i)},$$
$$= \frac{1}{(1-uv)\left(1-\frac{uw}{x_2\cdots x_n}\right)}\prod_{i=2}^{n}\frac{1}{(1-x_iu)(1-v/x_i)}\prod_{2\le i<j\le n}\frac{1-x_j/x_i}{1-qx_j/x_i}\prod_{2\le j\le n}\frac{1-ux_j}{1-ux_jq},$$
$$= \frac{1}{(1-uv)\left(1-\frac{uw}{x_2\cdots x_n}\right)}\prod_{i=2}^{n}\frac{1}{(1-x_iuq)(1-v/x_i)}\prod_{2\le i<j\le n}\frac{1-x_j/x_i}{1-qx_j/x_i}.$$

This is exactly $\frac{1}{1-uv}R_{n-1}(uq,v,uw)$ if we rename x_i by x_{i-1}.

next

To evaluate our constant term we also need

$$F_{UVX}(u,v,q) = \frac{1}{\prod_{i=1}^{n}(1-q^i)} Q_n^*(u,v,1)\Big|_{x_2^0\cdots x_n^0}$$

$$Q_n^*(u,v,w) = \prod_{i=1} \frac{1}{(1-ux_i)(1-v/x_i)} \prod_{1\le i<j\le n} \frac{1-x_j/x_i}{(1-qx_j/x_i)}\Big|_{x_n=1/wx_1\cdots x_{n-1}}.$$

$$R_n(u,v,w) = \frac{1}{1-w/x_1\cdots x_n} \prod_{i=1}^{n} \frac{1}{(1-ux_i)(1-v/x_i)} \prod_{1\le i<j\le n} \frac{1-x_j/x_i}{(1-qx_j/x_i)},$$

$$R_n^*(u,v,w) = \frac{1}{1-1/wx_1\cdots x_n} \prod_{i=1}^{n} \frac{1}{(1-ux_i)(1-v/x_i)} \prod_{1\le i<j\le n} \frac{1-x_j/x_i}{(1-qx_j/x_i)}.$$

$$R_n(u,v,w)|_{x_1^0\cdots x_n^0} = \frac{1}{(1-uv)(1-uvq)\cdots(1-uvq^{n-1})(1-wu^n q^{\binom{n}{2}})}.$$

$$R_n^*(u,v,w)\Big|_{x_1^0\cdots x_n^0} = -\frac{wv^n q^{\binom{n}{2}}}{(1-uv)(1-uvq)\cdots(1-uvq^{n-1})(1-wv^n q^{\binom{n}{2}})}.$$

$$R_n(u,v,w)\Big|_{x_1^0\cdots x_n^0} = Q_n(u,v,w)\Big|_{x_1^0\cdots x_{n-1}^0} + \frac{1}{1-uv} R_{n-1}(u,vq,w/v)\Big|_{x_1^0\cdots x_{n-1}^0}.$$

Combining these identities we get

$$F_{UVX}(u,v,q) = \frac{1}{\prod_{i=1}^{n}(1-q^i)(1-uvq^{i-1})(1-wu^n q^{\binom{n}{2}})} + \frac{wv^n q^{\binom{n}{2}})}{\prod_{i=1}^{n}(1-q^i)(1-uvq^{i-1})(1-wv^n q^{\binom{n}{2}})}$$

next

Conclusion

Theorem

The UVX invariants have the tri-graded basis

$$\mathcal{B}^{ab} = \left\{ \Phi^{r+1}\Pi_1^{p_1}\cdots\Pi_n^{p_n}\theta_1^{q_1}\cdots\theta_n^{q_n}\,;\ \Psi^s\Pi_1^{p_1}\cdots\Pi_n^{p_n}\theta_1^{q_1}\cdots\theta_n^{q_n} : r, s \geq 0, p_i \geq 0, q_i \geq 0 \right\}$$

Proof

Since we know the Hilbert series, it is sufficient to prove independence.

Suppose we had a vanishing linear combination P.

We can assume that P is tri-homogeneous.

(1) The monomial $\Phi^{r+1}\Pi_1^{p_1}\Pi_2^{p_2}\cdots\theta_n^{p_n}\theta_1^{q_1}\theta_2^{q_2}\cdots\theta_n^{q_n}$ has tri-degree

$$(0,(r+1)n,(r+1)\binom{n}{2}) + (0,0,(\sum_i ip_i)) + (\sum_i q_i, \sum_i q_i, (\sum_i iq_i))$$

(2) The monomial $\Psi^s\Pi_1^{p_1}\Pi_2^{p_2}\cdots\theta_n^{p_n}\theta_1^{q_1}\theta_2^{q_2}\cdots\theta_n^{q_n}$ has tri-degree

$$(sn, 0, s\binom{n}{2}) + (0,0,(\sum_i ip_i)) + (\sum_i q_i, \sum_i q_i, (\sum_i iq_i))$$

This immediately shows that any tri-homogenous linear combination P

cannot contain both Φ and Ψ.

From this, independence easily follows .

next

THE END