Reward

Seeking for n! Derivatives

$1,000$
A remarkable determinant

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General definition

\[
\Delta_{\mu}(X,Y) = \det \left| \begin{array}{c}
    \mu_1 \\
    \mu_2 \\
    \vdots \\
    \mu_n
\end{array} \right|
\]

If \((p_1,q_1),(p_2,q_2),\ldots,(p_n,q_n)\) are the cells of the Ferrers diagram of \(\mu\), then

\[
\Delta_{\mu}(X,Y) = \det \left| \begin{array}{cccc}
    x_{p_1} & y_{q_1} & \cdots & x_{p_n} \\
    y_{q_1} & x_{p_1} & \cdots & y_{q_n} \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{p_n} & y_{q_n} & \cdots & x_{p_1}
\end{array} \right|
\]
Could it be $\prod$?

6 independent derivatives!

\[ \begin{align*}
\vec{x} - \delta_{r_1} x_1 \delta_{s_1} y_1 \\
\vec{x} + \delta_{r_2} x_2 \delta_{s_2} y_2 \\
\vdots \\
\end{align*} \]

\[ \Delta \mu (X,Y) := \prod \delta_{rr} x_r \delta_{ss} y_s \]

As a starter......
The \( n! \) Conjecture

\[ \dim [X^n, Y^n] = n! \]

for an "elementary" proof

"Elementary" means:

By calculus or and combinatorics

Give an algorithm

that produces a "triangular" set of \( n! \) derivatives

Reward

\$1,000$

Proved by Mark Haiman using algebraic geometry

\( \dim [X^n, Y^n] = n! \)

For \( n \)

The \( n! \) Conjecture

(1990 - 2000)
The basic construction

Polynomials vanishing at all these tableau points:

\((\frac{1}{2} - \frac{1}{3})(\frac{1}{2} - \frac{1}{3})(\frac{1}{2} - \frac{1}{3})\)

Some polynomials vanishing at this orbit point:

\(\left(\frac{1}{4} - \frac{1}{5}\right)\left(\frac{1}{4} - \frac{1}{5}\right)\left(\frac{1}{4} - \frac{1}{5}\right)\)

From a tableau to a point in 2n dimensional space
An Elementary Method

An Independent Set with 6 ≠ 3 Elements

Kicking for the Shape

\[ \Delta_{21}[X,Y] \partial x^3 \Delta_{21}[X,Y] \partial y^1 \Delta_{21}[X,Y] \partial x^2 \Delta_{21}[X,Y] \partial y^3 \]

\[ x^2 - \alpha^1 y^1 - \beta^1 x^2 - \alpha^1 y^3 - \beta^2 x^3 - \alpha^1 \]

\[ t^2 + t^1 \]

\[ \lambda' X \]
The symmetry condition is satisfied

\[ \frac{1}{4} + \frac{6}{4} + 10 \frac{1}{4} + \frac{1}{4} \]
\( (x_2 - \alpha_1) (x_3 - \alpha_2) (y_3 - \beta_2) (y_1 - \beta_1) (x_4 - \alpha_1) (y_3 - \beta_2) \)
**Only Algebraic Combinatorics**

Δμ(X,Y) = det \[ \begin{vmatrix} x_{1} & y_{1} \\ x_{2} & y_{2} \end{vmatrix} \] = \[ \text{det} X' X'' \]

If the kicking statistic are symmetric then the top components of the kicking polynomials

Theorem

Compute the kicking statistics.

Step 4

Order the orbit points and construct the kicking polynomials.

Step 3

Construct the corresponding orbit points

Step 2

In this case

5040

Fill the Ferrers diagram of \(\lambda\) with 1, 2, \ldots until possible ways to get to the nil tableau.

Step 1

The Algorithm
The polynomials in a triangular collection are necessarily independent.

\[ \text{ldm}(P_1) > \text{ldm}(P_2) > \text{ldm}(P_m) \]

is said to be "triangular" if

\[ \{P_1, P_2, \ldots, P_m\} = \{ \text{ldm}(P_1), \text{ldm}(P_2), \ldots, \text{ldm}(P_m) \} \]

is the \text{ldm} highest monomial in \( P \).

The leading monomial "in a polynomial \( P \)" is a polynomial in \( P \) and

\[ \sum_{i=1}^{n} c_i x_i^{d_i} \]

where we have \( d_i \) for some \( i \) and only if for some \( i \) and

\[ d_i \]

and

\[ \text{degree}(x_d) > \text{degree}(x_e) \]

The degree-lexicographic order

A typical term order

Triangularity
A basic tool

\[
\begin{pmatrix}
\delta x & \frac{\partial}{\partial x} & 1 & 0 \\
\delta \tau & \frac{\partial}{\partial \tau} & 0 & 1 \\
\delta \theta & \frac{\partial}{\partial \theta} & 0 & 0 \\
1 & 1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
\delta p \\
\delta \tau \\
\delta \theta \\
1
\end{pmatrix}
= \begin{pmatrix} [x]^{\delta f} \nabla \end{pmatrix}
\]

This proves the nil conjecture for \( n = 1 \) in

\[ e_{-\delta x} x^{\delta f} \geq e_{-\delta p} x^{\delta f} \iff e_{\delta x} x^{\delta f} \geq e_{\delta p} x^{\delta f} \]

Note: because

\[
\begin{align*}
\ldots & = \tau - u x \cdots \delta \tau - u x \delta \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \\
& = (u x, \ldots, \tau x, \nabla) \frac{\tau - u x}{\delta x} \cdots \frac{\tau - u x}{\delta x} \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \\
\end{align*}
\]

then

\[
I \geq I - u a \geq 0, \ldots, \delta I \geq 0, I - u a \geq 0, I - u a \geq 0, I - u a \geq 0, I - u a \geq 0
\]

thus if

\[
\ldots \frac{\tau - u x}{\delta x} \cdots \frac{\tau - u x}{\delta x} \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \\
= (u x, \ldots, \tau x, \nabla) \frac{\tau - u x}{\delta x} \cdots \frac{\tau - u x}{\delta x} \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \tau - u x \\
\]

Proof

The Vandermonde determinant of \((u x, \ldots, \tau x, \nabla)\) has \( n \) independent derivatives.

Theorem

A collection of polynomials with distinct leading monomials

\[ \nabla \phi \]

then \( \dim \phi = \dim \bigwedge \phi \) if and only if we can find \( \phi = \bigwedge \phi \). Let \( \Lambda \) be a vector space of polynomials and suppose that \( \dim \phi = \dim \bigwedge \phi \).
Here and after we set that are homogeneous of degree \( m \) in \( X_1, x_2, \ldots, x_n \) and degree \( j \) in \( \lambda x_1, \lambda x_2, \ldots, \lambda x_n \) and denote \( S \) the linear span of derivatives of \( \Lambda \) with \( n = \text{dim } \mathsf{H}^n(\Lambda, X) \).

Our spaces \( \mathsf{H}^n(\Lambda, X) \) are bidualized, that is, we have the double decomposition

\[
\frac{\text{dim } \mathsf{H}^n(\Lambda, X)}{1} = (\Lambda)^{\text{dim } \mathsf{H}^n(\Lambda, X)} = \bigoplus_{0 \leq \nu < m} (\Lambda)^{\nu} H \bigoplus \bigoplus_{\nu = m} (\Lambda)^{\nu} H.
\]

In this case

\[
[m = \nu d + \cdots + \nu d + r d : a d x \cdots z d x, 1 x] = (\Lambda)^{\text{dim } \mathsf{H}^n(\Lambda, X)}
\]

for instance for \( H = \mathsf{H}^n(\Lambda, X) \) we have \( [\nu x, \cdots, \nu x] = (\Lambda)^{\nu} H \).

If \( \text{dim } \mathsf{H}^n(\Lambda, X) = \infty \), for all \( m \), we set

\[
(\Lambda)^{\nu} H \bigoplus \cdots \bigoplus (\Lambda)^{\nu} H \bigoplus (\Lambda)^{\nu} H = \Lambda
\]
Some Hilbert Series

Using Maple

Let's use Maple to explore these series.
This explains the symmetry

\[(\neut / \mathfrak{H} / \mathfrak{I}) \wedge \Lambda \Sigma_u \Lambda x = (\mathfrak{H} / \mathfrak{I}) \wedge \Lambda \]

In this case

\((\mathfrak{H} / \mathfrak{I}) \wedge \Lambda \Sigma_u \Lambda x = (\mathfrak{H} / \mathfrak{I}) \wedge \Lambda \)

Note that this implies that \(\Lambda \) is a cone with summit a bilinear homogeneous polynomial

\[(\Lambda, x) \nabla \]

The same is true if \(\Lambda \) is a cone with summit a bilinear homogeneous polynomial.

\[(\Lambda, x) \nabla \]

In this case

\[0 = (x) \Lambda \]

Thus

\[0 = (x) \Lambda (x) \nabla \]

and

\[0 = (x) \nabla (x) \nabla \]

Note this is a non-singular linear map of \(\Lambda \) onto \(\Lambda \) and

\[(x) \nabla (x) \nabla \]

We say that a vector space \(\Lambda \) of polynomials is a "cone" if it is the linear span

\[(x) \nabla \]

**Definition**
Now the miracles
More miracles
Some facts about ideals of polynomials

Gordan:

We say that \( x' \) contains \( x' \) if and only if \( p_1 > q_1, p_2 > q_2, \ldots, p_n > q_n \).
Orthogonal complements of Ideals

The assertion in (3) holds because the elements of \( I \) can be taken as representables.

and the assertion in (1) follows from Taylor's Theorem.

\[
0 = \frac{\partial^{u_x \cdot \cdots \cdot \partial_x} (u_x, \cdots, \partial_x) \phi}{\partial^u \phi \cdots \partial_x \phi}
\]

For all \( p \in J \) and all \( p_1, p_2, \ldots, p_n \) we have

\[
\{ f \in p : 0 = (u_x, \cdots, \partial_x) \phi \} = \langle \phi, d \rangle = \| \phi \|
\]

\( f \in p \)

\( \{ f \in p : 0 = \langle \phi, d \rangle \} = \| f \|
\]

For homogeneous Theorem

For any homogeneous ideal \( J \) set

\[
\left| \frac{\partial^{u_x \cdot \cdots \cdot \partial_x} (u_x, \cdots, \partial_x) \phi}{\partial^u \phi \cdots \partial_x \phi} \right| = \langle \phi, d \rangle
\]

(3) For \( p \in \{ u_x, \cdots, \partial_x \} \) set

(1) An ideal is called "homogeneous" if it is generated by homogeneous polynomials.

Definitions:
Next,

\[ |\mathcal{CG}|/|\mathcal{C}| = \mathcal{C} \subseteq H = \mathcal{C} \subseteq \mathcal{H} = \mathcal{C} \subseteq \mathcal{H} \] 

Now it follows that

and call the elements of this space the "orbit harmonics".

\[ \mathcal{R}[\mathcal{C}] \subseteq H \]

We also set

\[ \mathcal{R}[\mathcal{C}] \subseteq \mathcal{R} \]

and

\[ \mathcal{R}[\mathcal{C}] \subseteq \mathcal{R} \]

In any case we set

\[ \mathcal{R}[\mathcal{C}] \subseteq \mathcal{R} \]

In general we have

\[ \mathcal{R}[\mathcal{C}] \subseteq \mathcal{R} \]

If \( \mathcal{C} \) is trivial then \( a \) is called "regular" and the orbit \( [a] \) has cardinality \( |\mathcal{C}| \).

If \( \mathcal{C} \) is trivial then \( a \) is called "regular" and the orbit \( [a] \) has cardinality \( |\mathcal{C}| \).

is called the "stabilizer" of \( a \).

\[ \{ a \in \mathcal{C} : \mathcal{C} \ni a \} = \mathcal{C} \]

and call it the "orbit" of \( a \) under \( \mathcal{C} \). The subgroup

\[ \{ a \in \mathcal{C} : \mathcal{C} \ni a \} = \mathcal{C} \]

This given, we let

\[ (a_{1}, a_{2}, \ldots, a_{n}) = a \]

and a vector \( a = (a_{1}, a_{2}, \ldots, a_{n}) \) we set

Let \( G \) be a group of permutations of \( \{1, 2, \ldots, n\} \) given an element

\[ (\nu, \nu', \ldots, \nu') \in G \]
next

Theorem

(!!)

\[ H^b \mid \mathcal{P} \parallel q, \frac{a}{b} \sum_{\mathcal{P} \parallel q} \mathcal{P}^b \hat{H} = (b) \mathcal{P}^b \hat{H} = (b) \mathcal{P}^b \hat{H} \parallel \mathcal{P} \parallel q \]

(!!)

\[ \{ \mathcal{P} \subseteq \mathcal{P} : (a) \mathcal{P} \parallel \mathcal{P} \parallel q \} = (b) \mathcal{P} \parallel \mathcal{P} \parallel q \]

(!!)

\[ \frac{d}{d^\omega} \mid \mathcal{P} \parallel q, l = 0, l \]

Then the equalities

(!!)

\[ \frac{d}{d^\omega} + \cdots + \frac{d}{d^\omega} = \mid \mathcal{P} \parallel q \mid \text{ and } B_{\mathcal{P} \parallel q} \]

Suppose that \( \mathcal{P} \) is a basis for \( R^L \) with

(!!)

\[ \omega = \max \deg \mathcal{P} \]

Let \( a \) be a regular point.

Theorem

(!!)

(!!)

(!!)

Conical Orbit Harmonics
where $\mu_{\{X;Y\}}$ comes from $[\lambda^\dagger X]^n \bigvee$. With equality if and only if $H_{\{q,a\}} \subseteq [\lambda^\dagger X]^n M$

This implies

$\mu_{\{q,a\}}^S H \subseteq [\lambda^\dagger X]^n M$

and since $H_{\{q,a\}}$ is closed under differentiation it follows that $\mu_{\{q,a\}}^S H \in [\lambda^\dagger X]^n \bigvee$

$\left\{ \mu_{\{q,a\}}^S \ni \phi : (\mu_{\{q,a\}}^S \phi q, \ldots, \mu_{\{q,a\}}^S \phi q, \mu_{\{q,a\}}^S \phi q, \mu_{\{q,a\}}^S \phi q, \mu_{\{q,a\}}^S \phi q) \right\} = \mu_{\{q,a\}}^S [(q,a)]$

Then

Theorem

Construct the corresponding orbit points

Step 2

Fill the Ferrers diagrams of $n$ with $1,2,\ldots,n$ in all possible ways to get the $n!$ tableaux.

Step 1

Where $\mu_{\{X;Y\}}$ comes from $[\lambda^\dagger X]^n \bigvee$.
The Algorithm

Step 1

Step 2

Construct the corresponding orbit points

2.40

In this case, only Algebraic Combinatorics

No MAGICs?

Step 3

Order the orbit points and construct the kicking polynomials.

Step 4

Compute the kicking statistics.

\[ \Delta \mu (X, Y) = \det \left( \begin{array}{cc} x_p & y_q \\ x_p & y_q \end{array} \right) \]

\[ \left( \begin{array}{cccc} 1 & 2 & 5 \\ 6 & 0 & 8 \\ 9 & 1 & 3 \\ 12 & 7 & 5 \end{array} \right) \]

Fill the Fierro diagram of \( \lambda \) with 1, 2, \ldots , \lambda in all possible ways to get the nil tableaux.
Can you guess?

It is interesting to know when equality holds

\[ \dim G = |G| \]

and thus in full generality we must have

Now if \( a \) is regular we have \( \text{dim} H \leq \text{dim} \mathcal{H} \)

and taking orthogonal complements gives

Thus

\[
\begin{align*}
\mathcal{G} \subseteq & \dim \mathcal{H} \\
(\mathcal{G} \oplus \mathcal{H}) & \subseteq \dim \mathcal{H} \\
& \subseteq \dim \mathcal{H}
\end{align*}
\]

Note that if \( \mathcal{G} \) is any homogeneous \( G \)-invariant polynomial then for any point \( a \in \mathcal{G} \) we have

It is customary to call the elements of \( \mathcal{H} \) "\( G \)-Harmonics".

Let \( \mathcal{H} \) be the ideal generated by the homogeneous \( G \)-invariant polynomials and set

The Group Harmonics
Groups Generated by Reflections

The space $H^n_s$ (Harmonics of $S^n$) is the linear span of derivatives of

The classical example

$G$ is generated by reflections

$H_G$ is a cone with summit the product of the reflecting hyperplanes,

$\dim H_G = |G|$

$H_{H_G} = H_G$ for all regular points,

any of the statements below implies the remaining ones.

If $G$ is a finite group of $n \times n$ matrices then

Theorem
THE END