Seeking for n! Derivatives

$1,000$

Reward

next

A remarkable determinant

$$\begin{array}{|c|c|}\hline (1,0) & (1,1) \\ \hline (0,0) & (0,1) \\ \hline \end{array} \Longrightarrow \Delta_{2,2}(X,Y) = \det \begin{pmatrix} x_1^0y_1^0 & x_2^0y_2^0 & x_3^0y_3^0 & x_4^0y_4^0 \\ x_1^0y_1^1 & x_2^0y_2^1 & x_3^0y_3^1 & x_4^0y_4^1 \\ x_1^1y_1^0 & x_2^1y_2^0 & x_3^1y_3^0 & x_4^1y_4^0 \\ x_1^1y_1^1 & x_2^1y_2^1 & x_3^1y_3^1 & x_4^1y_4^1 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 \end{pmatrix}$$

$$\begin{array}{|c|c|c|}\hline (2,0) & \multicolumn{2}{c}{} \\ \hline (1,0) & (1,1) & (1,2) \\ \hline (0,0) & (0,1) & (0,2) \\ \hline \end{array} \Longrightarrow \Delta_{331}(X,Y) = \det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_5 & y_7 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & y_6^2 & y_7^2 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6^2 & x_7^2 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_5y_5 & x_6y_6 & x_7y_7 \\ x_1y_1^2 & x_2y_2^2 & x_3y_3^2 & x_4y_4^2 & x_5y_5^2 & x_6y_6^2 & x_7y_7^2 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 & x_7^2 \end{pmatrix}$$

General definition

If $(p_1,q_1),(p_2,q_2),\ldots,(p_n,q_n)$ are the cells of the Ferrers diagram of $\mu \vdash n$ then

$$\Delta_\mu(X,Y) = \det \left\| x_j^{p_i} y_j^{q_i} \right\|_{i,j=1}^{n}$$

next

As a starter .....

Theorem (easy)

For any $\mu \vdash n$ *the dimension of the linear span of the derivatives of* $\Delta_\mu(X,Y)$ *is at most* $\mathbf{n!}$

In symbols

$$\dim \mathbf{M}_\mu[\mathbf{X},\mathbf{Y}] \le \mathbf{n!}$$

where

$$M_\mu[X,Y] \;=\; L\left[\delta_{x_1}^{r_1}\delta_{y_1}^{s_1}\delta_{x_2}^{r_2}\delta_{y_2}^{s_2}\cdots\delta_{x_n}^{r_n}\delta_{y_n}^{s_n}\Delta_\mu(X,Y)\;:\;r_i,s_i\ge 0\right]$$

For example (using MAPLE)

```
DDmu([2,1]);
```

$$\begin{bmatrix} 1 & 1 & 1 \\ y1 & y2 & y3 \\ x1 & x2 & x3 \end{bmatrix}$$

```
D21:=det(");
```

$$D21 := \boxed{y2\,x3 - y3\,x2 - y1\,x3 + y1\,x2 + x1\,y3 - x1\,y2}$$

```
diff(D21,x1);
```

$$\boxed{y3 - y2}$$

```
diff(D21,x3);
```

$$\boxed{y2 - y1}$$

```
diff(D21,y1);
```

$$\boxed{-x3 + x2}$$

```
diff(D21,y3);
```

$$\boxed{-x2 + x1}$$

```
diff(D21,x3,y2);
```

$$\boxed{1}$$

6 independent derivatives!

Could it be ?????

next

The n! Conjecture

(1990 - 2000)

For $\mu \vdash \mathbf{n}$

$$\dim\, M_\mu[X,Y] \;=\; n!$$

Proved by Mark Haiman using algebraic geometry

1,000$
Reward

for an “elementary” proof

“Elementary” means:

By calculus or/and combinatorics

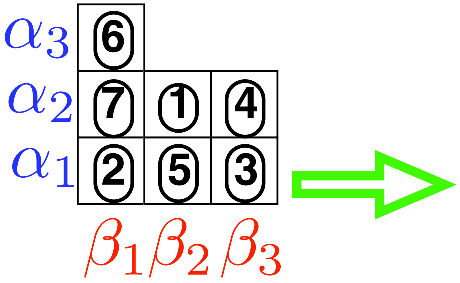
give an algorithm

that produces a “triangular” set of n! derivatives

next

The basic construction

from a tableau to a point in 2n dimensional space

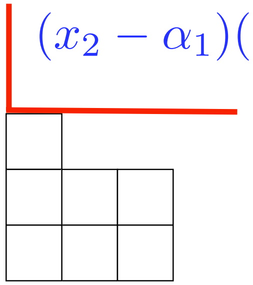


$$\left(\frac{\alpha_2\ \alpha_1\ \alpha_1\ \alpha_2\ \alpha_1\ \alpha_3\ \alpha_2\ \beta_2\ \beta_1\ \beta_3\ \beta_3\ \beta_2\ \beta_1\ \beta_1}{x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ y_1\ y_2\ y_3\ y_4\ y_5\ y_6\ y_7} \right)$$

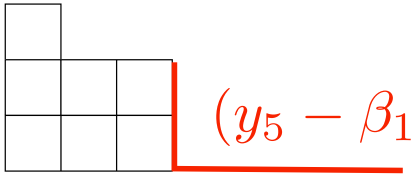
some polynomials vanishing at this orbit point

$$x_1 - \alpha_2 \qquad x_4 - \alpha_2 \qquad x_7 - \alpha_2 \qquad y_2 - \beta_1 \qquad y_5 - \beta_2$$

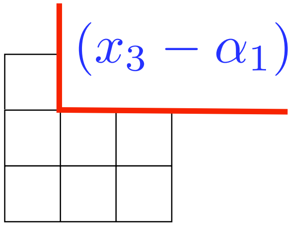
Polynomials vanishing at all these tableau points:



$(x_2 - \alpha_1)(x_2 - \alpha_2)(x_2 - \alpha_3)$



$(y_5 - \beta_1)(y_5 - \beta_2)(y_5 - \beta_3)$



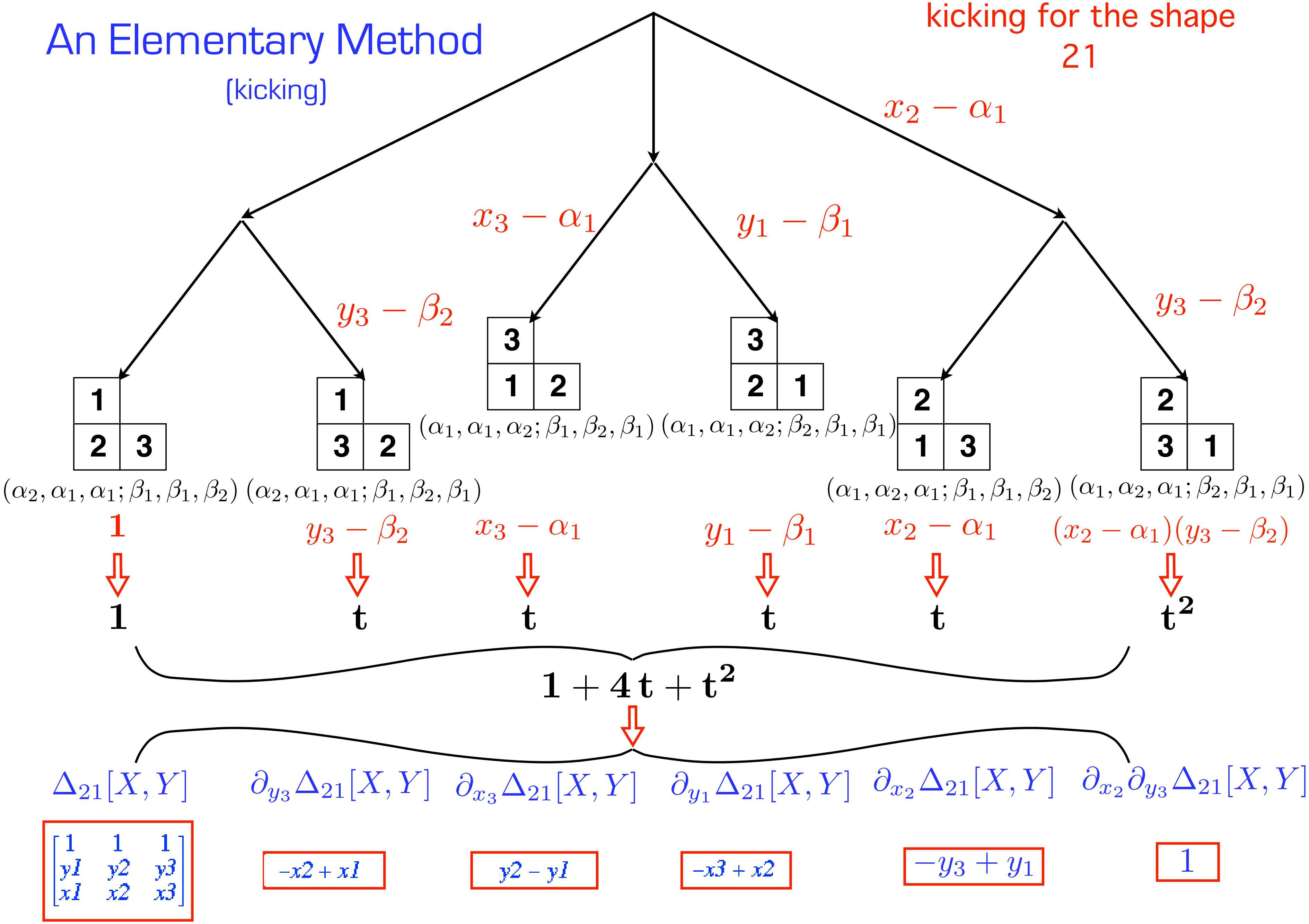
$(x_3 - \alpha_1)(x_3 - \alpha_2)(y_3 - \beta_1)$

next

An Elementary Method

(kicking)

kicking for the shape 21



$(\alpha_2,\alpha_1,\alpha_1;\beta_1,\beta_1,\beta_2)$ $(\alpha_2,\alpha_1,\alpha_1;\beta_1,\beta_2,\beta_1)$ $(\alpha_1,\alpha_1,\alpha_2;\beta_1,\beta_2,\beta_1)$ $(\alpha_1,\alpha_1,\alpha_2;\beta_2,\beta_1,\beta_1)$ $(\alpha_1,\alpha_2,\alpha_1;\beta_1,\beta_1,\beta_2)$ $(\alpha_1,\alpha_2,\alpha_1;\beta_2,\beta_1,\beta_1)$

1 $y_3-\beta_2$ $x_3-\alpha_1$ $y_1-\beta_1$ $x_2-\alpha_1$ $(x_2-\alpha_1)(y_3-\beta_2)$

$\Downarrow$

$\mathbf{1}$ $\mathbf{t}$ $\mathbf{t}$ $\mathbf{t}$ $\mathbf{t}$ $\mathbf{t^2}$

$$\mathbf{1+4\,t+t^2}$$

$\Downarrow$

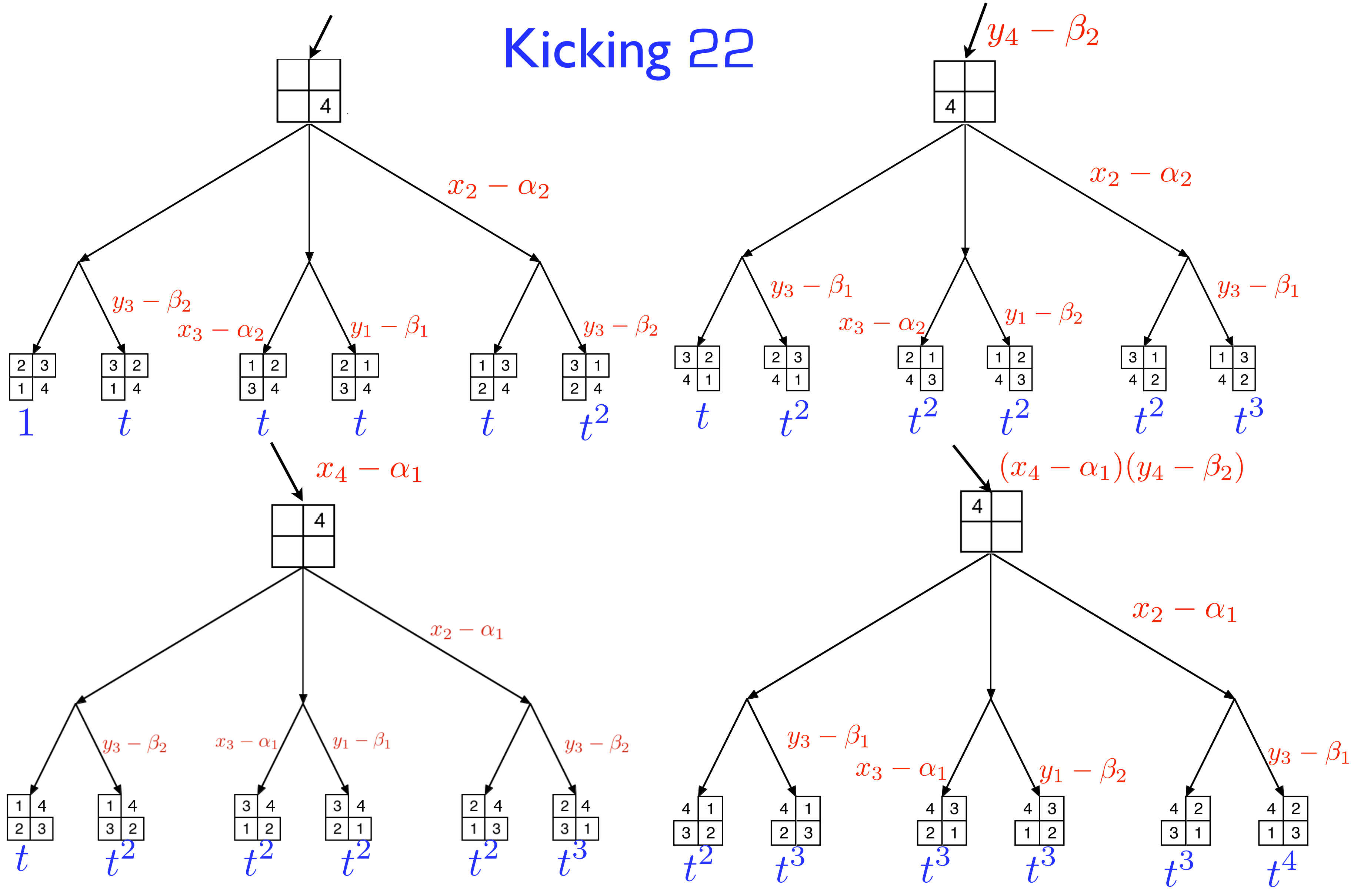
$\Delta_{21}[X,Y]$ $\partial_{y_3}\Delta_{21}[X,Y]$ $\partial_{x_3}\Delta_{21}[X,Y]$ $\partial_{y_1}\Delta_{21}[X,Y]$ $\partial_{x_2}\Delta_{21}[X,Y]$ $\partial_{x_2}\partial_{y_3}\Delta_{21}[X,Y]$

$\boxed{\begin{bmatrix} 1 & 1 & 1 \\ y1 & y2 & y3 \\ x1 & x2 & x3 \end{bmatrix}}$ $\boxed{-x2+x1}$ $\boxed{y2-y1}$ $\boxed{-x3+x2}$ $\boxed{-y_3+y_1}$ $\boxed{1}$

AN INDEPENDENT SET WITH 6=3! ELEMENTS

next

Kicking 22

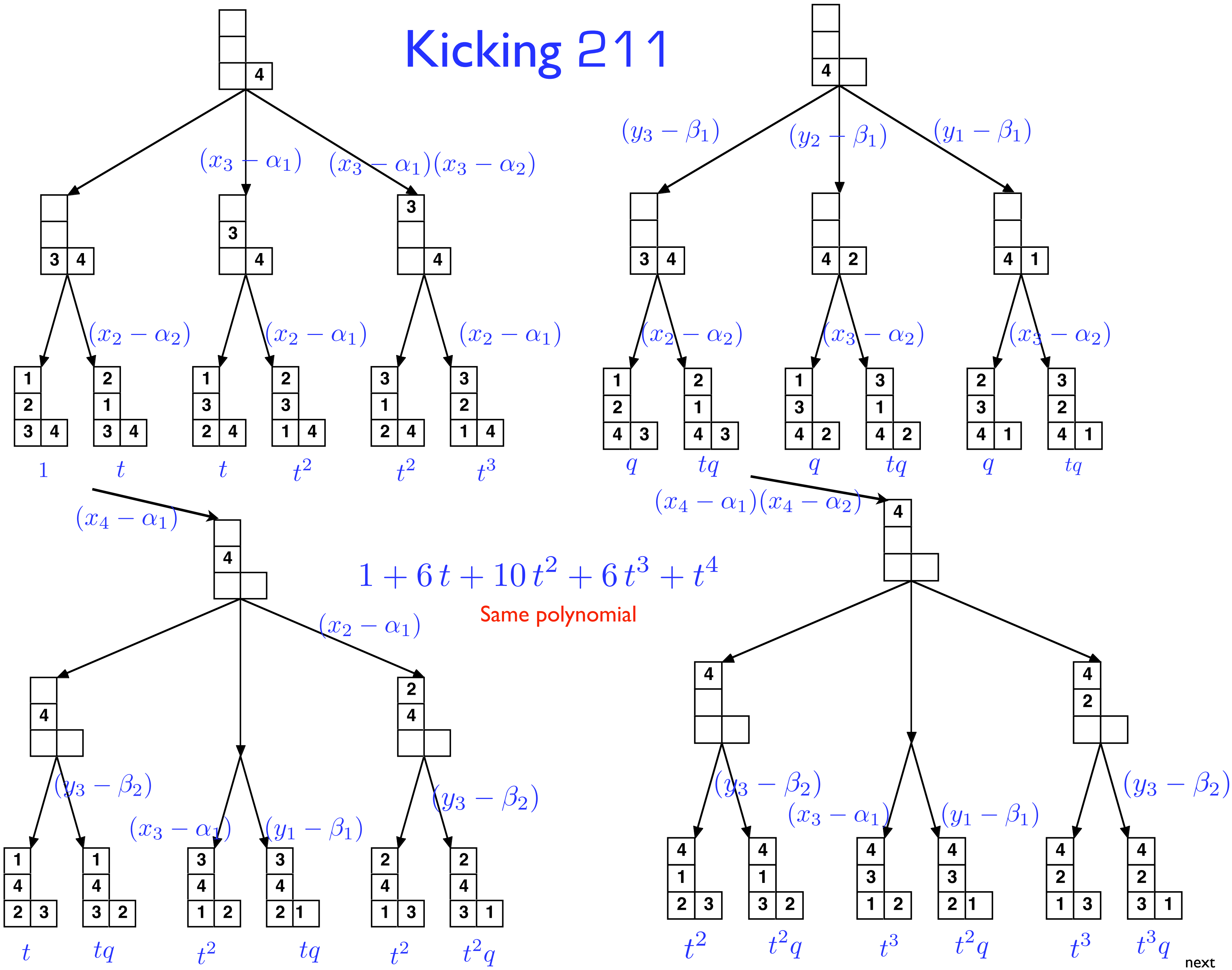


$$1 + 6\,t + 10\,t^2 + 6\,t^3 + t^4$$

The symmetry condition is satisfied

next

Kicking 211



$$1+6\,t+10\,t^2+6\,t^3+t^4$$

Same polynomial

next

The Algorithm

Step 1

Fill the Ferrers diagram of μ *with* $1, 2, \ldots, n$ *in all possible ways to get the* $n!$ *tableaux.*

$$\begin{bmatrix} 1 \\ 12 \\ 75 \\ 306 \\ 807 \\ 1319 \\ 1319 \\ 807 \\ 306 \\ 75 \\ 12 \\ 1 \end{bmatrix}$$

Step 2

Construct the corresponding orbit points

5040

In this case

3		
6	1	2
4	7	5

Step 3

Order the orbit points and construct the kicking polynomials.

Step 4

Compute the kicking statistics.

Theorem

If the kicking statistic are symmetric then the top components of the kicking polynomials give the $n!$ *independent derivatives of* $\Delta_\mu(X,Y) = \det \|x_j^{p_i} y_j^{q_i}\|_{i,j=1}^{n}$

MAGICS?

No

Only Algebraic Combinatorics

next

Triangularity

A typical term order

The degree-Lexicographic order

$$x^p = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} \ <_{dlex} \ x^q = x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n}$$

$$\Updownarrow$$

$$\text{degree}(x^p) < \text{degree}(x^q) \quad \text{or} \quad \text{degree}(x^p) = \text{degree}(x^q) \quad \text{and} \quad p <_{lex} q$$

Where we have $p <_{lex} q$ if and only if for some $1 \leq i \leq n-1$

$$p_1 = q_1\ ,\ p_2 = q_2\ ,\ \ldots,\ ,\ p_i = q_i\ , \quad \text{and} \quad p_{i+1} < q_{i+1}$$

The "*Leading monomial*" in a polynomial $P = \sum_p c_p x^p$ is the dlex highest monomial in P

Definition

In a vector space V *of polynomials a collection* $\mathcal{C} = \{P_1, P_2, \ldots, P_m\}$ *is said to be "triangular" if*

$$\text{ldm}(P_1) <_{dlex} \text{ldm}(P_2) <_{dlex} \cdots <_{dlex} \text{ldm}(P_m)$$

Note:

The polynomials in a triangular collection are necessarily independent!

next

A basic tool

Let V be a vector space of polynomials and suppose that $\boxed{\textit{dim } V \leq d}$

then $\boxed{\textit{dim } V = d}$ *if and only if we can find in V*

a collection of $\boxed{d}$ *polynomials with* $\boxed{\textit{distinct}}$ *leading monomials.*

Theorem

The vandermonde determinant $\Delta_n(x_1, x_2, \ldots, x_n)$ *has* $n!$ *independent derivatives*

Proof

$$\Delta_n(x_1, x_2, \ldots, x_n) \;=\; \det\|x_i^{n-j}\|_{i,j=1}^{n} \;=\; x_1^{n-1}x_2^{n-2}x_3^{n-3}\cdots x_n^{n-n} + \geq_{dlex} \cdots$$

thus if

$$0 \leq a_1 \leq n-1\,,\quad 0 \leq a_2 \leq n-2\,,\; 0 \leq a_3 \leq n-3\,,\; \cdots\,,\; 0 \leq a_{n-1} \leq 1$$

then

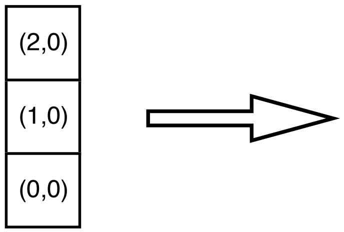
$$\partial_{x_1}^{a_1}\partial_{x_2}^{a_2}\partial_{x_3}^{a_3}\cdots\partial_{x_{n-1}}^{a_{n-1}}\Delta_n(x_1, x_2, \ldots, x_n) \;=\; x_1^{n-1-a_1}x_2^{n-2-a_2}x_3^{n-3-a_3}\cdots x_{n-1}^{1-a_{n-1}} + \geq_{dlex} \cdots$$

because

$$\boxed{x^p \leq_{dlex} x^q \quad\Longrightarrow\quad x^{p-a} \leq_{dlex} x^{q-a}}$$

Note:

This proves the $n!$ conjecture for $\lambda = 1^n$



$$\Delta_{1^3}[X] \;=\; \det\begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{pmatrix}$$

next

Hilbert series

A vector space $\mathbf{V}$ is called "*graded* " if and only if

$$\mathbf{V} \;=\; \mathbf{H_o(V)} \oplus \mathbf{H_1(V)} \oplus \mathbf{H_2(V)} \oplus \cdots \oplus \mathbf{H_m(V)} \oplus \cdots$$

The subspace "$\mathbf{H_m(V)}$ " is called the "$\mathbf{m^{th}}$ *homogeneous component*" of $\mathbf{V}$.

its elements are called homogeneous of degree m

If $\dim \mathbf{H_m(V)} < \infty$ for all m, we set

$$\mathbf{F_V(t)} \;=\; \sum_{\mathbf{m}>0} \mathbf{t^m} \dim \mathbf{H_m(V)}$$

the "*Hilbert series* " of V

For instance for $\mathbf{R} = \mathbb{Q}[\mathbf{x_1, x_2, \ldots, x_n}]$ we have $R = H_0(R) \oplus H_1(R) \oplus H_2(R) \oplus \cdots$ with

$$\mathbf{H_m(R)} \;=\; \mathbf{L}[\mathbf{x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n} \;:\; p_1 + p_2 + \cdots + p_n = m}]$$

In this case

$$\dim \mathbf{H_m(R)} \;=\; \binom{\mathbf{m+n-1}}{\mathbf{n-1}} \qquad \text{and} \qquad \mathbf{F_R(t)} \;=\; \frac{1}{(1-\mathbf{t})^{\mathbf{n}}}$$

Our spaces $\mathbf{M_\mu[X,Y]}$ are "*bigraded* that is we have the double decomposition

$$\boxed{\mathbf{M_\mu[X,Y]} \;=\; \bigoplus_{\mathbf{r}=0}^{\mathbf{n}(\mu)} \bigoplus_{\mathbf{s}=0}^{\mathbf{n}(\mu')} \mathbf{H_{r,s}}(\mathbf{M_\mu[X,Y]})}$$

With $\mathbf{H_{r,s}}(\mathbf{M_\mu[X,Y]})$ the linear span of derivatives of $\mathbf{\Delta_\mu(x,y)}$

that are homogeneous of degree ***r*** in $\mathbf{x_1, x_2, \ldots, x_n}$ and degree ***S*** in $\mathbf{y_1, y_2, \ldots, y_n}$

Here and after we set

$$\boxed{\mathbf{F_\mu(q,t)} \;=\; \sum_{\mathbf{r}=0}^{\mathbf{n}(\mu)} \sum_{\mathbf{s}=0}^{\mathbf{n}(\mu')} \mathbf{t^r q^s} \dim \mathbf{H_{r,s}}(\mathbf{M_\mu[X,Y]})}$$

next

Some Hilbert series

Theorem

If V is graded and B is a homogeneous basis of V then

$$F_V(t) = \sum_{b\in B} t^{\mathrm{degree}(b)}$$

Using this we get for the linear span of derivatives of the Vandermonde

$$F_{1^n}(t) = (1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{n-1}) = [n]_t!$$

Using Maple

```
hilb([3,2]);
```

$$q^4t^2+4q^4t+4q^3t^2+5q^4+\boxed{15q^3t}+9q^2t^2+11q^3+22q^2t+\boxed{11qt^2}+9q^2+15qt+5t^2+4q+4t+1$$

The dimension of $H_{2,1}(M_{3,2}[X,Y])$

$$\begin{array}{c|ccccc} 2\longrightarrow & 5 & \boxed{11} & 9 & 4 & 1\\ 1\longrightarrow & 4 & 15 & 22 & \boxed{15} & 4\\ 0\longrightarrow & 1 & 4 & 9 & 11 & 5\\ \hline & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow\\ & 0 & 1 & 2 & 3 & 4 \end{array}$$

The dimension of $H_{1,3}(M_{3,2}[X,Y])$

The determinant $\begin{bmatrix} 1 & 1 & 1 & 1 & 1\\ y1 & y2 & y3 & y4 & y5\\ y1^2 & y2^2 & y3^2 & y4^2 & y5^2\\ x1 & x2 & x3 & x4 & x5\\ x1\,y1 & x2\,y2 & x3\,y3 & x4\,y4 & x5\,y5 \end{bmatrix}$ has 15 independent bihomogeneous derivatives of degree $\boxed{1}$ in $x_1, x_2, \ldots, x_5$ and degree $\boxed{3}$ in $y_1, y_2, \ldots, y_5$.

```
hilb([2,2,1]);
```

$$q^2t^4+4q^2t^3+4qt^4+9q^2t^2+15qt^3+5t^4+11q^2t+22qt^2+11t^3+5q^2+15qt+9t^2+4q+4t+1$$

$$\begin{bmatrix} 5 & 4 & 1\\ 11 & 15 & 4\\ 9 & 22 & 9\\ 4 & 15 & 11\\ 1 & 4 & 5 \end{bmatrix}$$

next

Flip

Definition

We say that a vector space $\mathbf{V}$ *of polynomials is a "cone " if it is the linear span of derivatives of a single homogeneous polynomial* $\mathbf{\Delta}(x)$.

In this case $\mathbf{V}$ *has an automorphism " Flip " defined by*

$$\text{flip}\, P(x) \;=\; P(\partial_x)\mathbf{\Delta}(x)$$

Note this is a non-singular linear map of $\mathbf{V}$ onto $\mathbf{V}$ since if

$$P(x) \;=\; Q(\partial_x)\mathbf{\Delta}(x) \qquad \text{and} \qquad P(\partial_x)\mathbf{\Delta}(x) = 0$$

then

$$P(\partial_x)P(x) \;= P(\partial_x)Q(\partial_x)\mathbf{\Delta}(x) \;= Q(\partial_x)P(\partial_x)\mathbf{\Delta}(x) \;= 0$$

Thus

$$\text{flip}\, P(x) = 0 \;\Longrightarrow\; P(\partial_x)P(x) = 0 \;\Longrightarrow\; P(x) = 0$$

Note that this implies that if degree $\mathbf{\Delta}(x) = n_o$ then

$$F_V(t) \;=\; t^{no}\, F_V(1/t)$$

The same is true if $\mathbf{V}$ is a cone with summit a bihomogeneous polynomial $\mathbf{\Delta}(x, y)$

of bi-degree n_x, n_y In this case

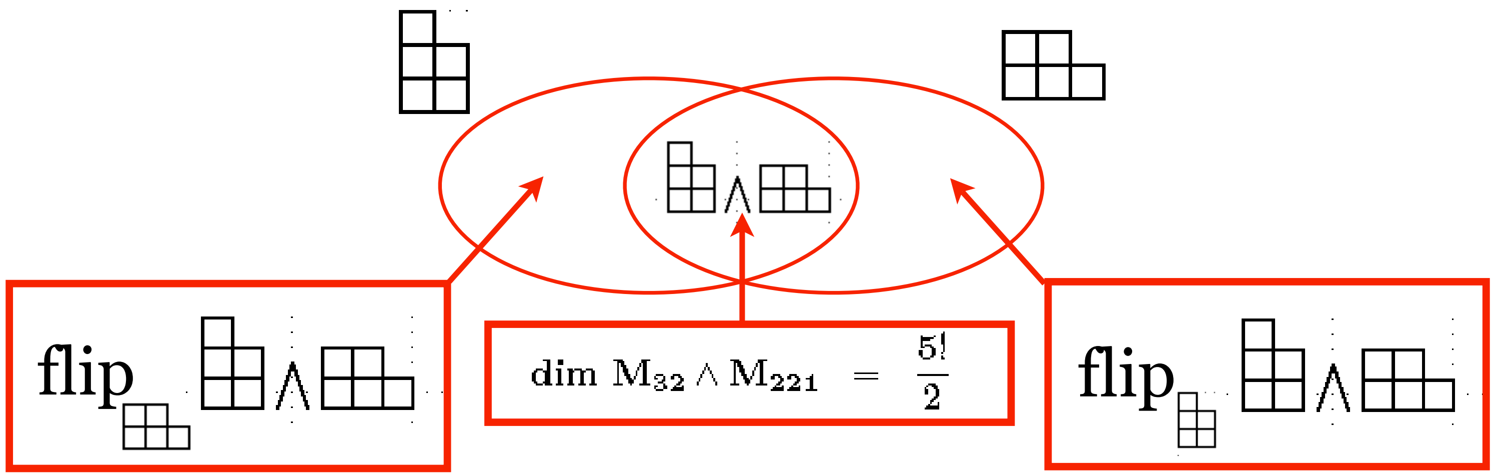
$$\boxed{F_V(q, t) \;=\; t^{n_x} q^{n_y} F_V(1/q, 1/t)}$$

This explains the symmetry

$$F_{32}(q, t) \;=\; \begin{bmatrix} \boxed{5} & 11 & \boxed{9} & \boxed{4} & 1 \\ 4 & 15 & 22 & 15 & 4 \\ 1 & \boxed{4} & \boxed{9} & 11 & \boxed{5} \end{bmatrix}$$

next

Now the miracles



$$\dim M_{32} \wedge M_{221} = \frac{5!}{2}$$

$$M_{221} = M_{32} \wedge M_{221} \oplus \mathrm{flip}_{221} M_{32} \wedge M_{221} \qquad M_{32} = M_{221} \wedge M_{32} \oplus \mathrm{flip}_{32} M_{221} \wedge M_{32}$$

$\frac{n!}{2}$ **Conjecture** (still open!)

If $\alpha, \beta \vdash n$ *differ only by the position of one corner cell then* $\dim M_\alpha \wedge M_\beta = \frac{n!}{2}$

SF Conjecture (still open!)

If $\alpha, \beta \vdash n$ *differ only by the position of one corner cell then*

$$M_\alpha = M_\alpha \wedge M_\beta \oplus \mathrm{flip}_\alpha M_\alpha \wedge M_\beta \quad , \qquad M_\beta = M_\alpha \wedge M_\beta \oplus \mathrm{flip}_\beta M_\alpha \wedge M_\beta$$

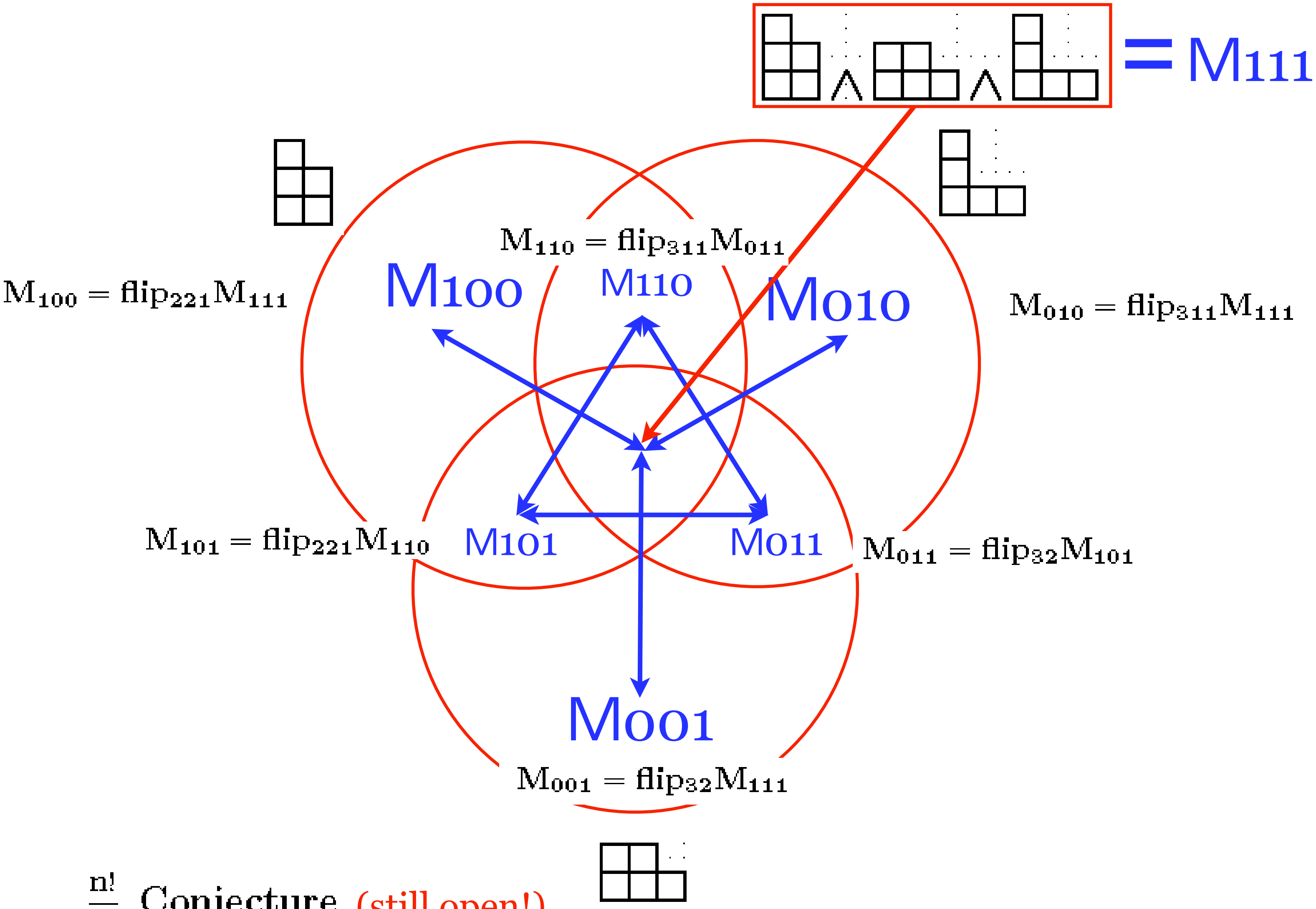
This implies

$$F_{M_\alpha \wedge M_\beta}(q,t) = \frac{T_\beta F_{M_\alpha}(q,t) - T_\alpha F_{M_\beta}(q,t)}{T_\alpha - T_\beta}$$

with $T_\mu = t^{n(\mu)} q^{n(\mu')}$ *and* $n(\mu) = \sum_i (i-1)\mu_i$

next

More miracles



$\wedge \wedge = M_{111}$

$M_{100} = \mathrm{flip}_{221} M_{111}$

$M_{110} = \mathrm{flip}_{311} M_{011}$

$M_{010} = \mathrm{flip}_{311} M_{111}$

M100

M110

M010

M101

M011

$M_{101} = \mathrm{flip}_{221} M_{110}$

$M_{011} = \mathrm{flip}_{32} M_{101}$

M001

$M_{001} = \mathrm{flip}_{32} M_{111}$

$\frac{n!}{k!}$ **Conjecture** (still open!)

If $\alpha_1, \alpha_2, \ldots, \alpha_k \vdash n$ *differ only by one cell from each other then*

$$\boxed{\dim M_{\alpha_1} \wedge M_{\alpha_2} \wedge \cdots \wedge M_{\alpha_k} = \frac{n!}{k}}$$

Etc Etc Etc

next

Some facts about ideals of polynomials

Definition

We say that x^p *contains* x^q *if and only if* $p_1 \geq q_1, p_2 \geq q_1, \dots, p_n \geq q_n$

Gordan: *Under the containement partial order every collection of monomials in* $x_1, x_2, \dots, x_n$ *has a finite number of minimal elements*

Let $J \subseteq \mathbb{Q}[x_1, x_2, \dots, x_n]$ be an ideal and let $M(J)$ the collection of all th $f_1, f_2, \dots, f_N \in J$ be such that ; of J

let $m_1, m_2, \dots, m_N$ be the minimal elements of $M(J)$ and

$$\mathrm{leadmon}(f_i) = m_i$$

then

$$\boxed{J = (f_1, f_2, \dots, f_N)}$$

Let B_J be the collection of monomials that do not contain any of $m_1, m_2, \dots, m_N$

then B_J is a basis for the quotient

$$\boxed{R_J = \mathbb{Q}[x_1, x_2, \dots, x_n]/J}$$

For a polynomial $P(x)$ we will denote by $h(P)$ the homogeneous component of P of highest degree.

Define

$$\boxed{\mathrm{gr}\, J = \big(h(P) : P \in J\big)} \quad \text{and} \quad \boxed{\mathrm{gr}\, R_J = \mathbb{Q}[x_1, x_2, \dots, x_n]/\mathrm{gr}\, J}$$

we have:

$$\boxed{B_J \text{ is also basis for } \mathrm{gr}\, R_J.}$$

$$\boxed{\text{Indeed } M(J) = M(\mathrm{gr}\, J)} \text{ and } \boxed{\mathrm{gr}\, J = \big(h(f_1), h(f_2), \dots, h(f_N)\big)}$$

In particular

$$\boxed{\dim R_J = \dim \mathrm{gr}\, R_J}$$

next

Orthogonal complements of Ideals

Definitions:

(1) *An ideal is called "homogeneous" if it is generated by homogeneous polynomials*

(2) *For* $P, Q \in \mathbb{Q}[x_1, x_2, \ldots, x_n]$ *set*

$$\langle P, Q \rangle \;=\; P(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}) Q(x_1, x_2, \ldots, x_n)\Big|_{x_1, x_2, \ldots, x_n = 0}$$

(3) *For any homogeneous ideal* J *set*

$$J^{\perp} \;=\; \{ Q : \langle P, Q \rangle = 0 \;\forall\; P \in J \}$$

Theorem

For J *homogeneous*

1) $J^{\perp} = \{ Q : P(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}) Q(x_1, x_2, \ldots, x_n) = 0 \;\forall\; P \in J \}$

2) $(J^{\perp})^{\perp} \;=\; J$

3) $\mathbb{Q}[x_1, x_2, \ldots, x_n]/J$ *and* $J^{\perp}$ *have the same Hilbert series*

Note:

For all $P \in J$ and all $p_1, p_2, \ldots, p_n \geq 0$ we have

$$\partial_{x_1}^{p_1} \partial_{x_2}^{p_2} \cdots \partial_{x_n}^{p_n} P(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}) Q(x_1, x_2, \ldots, x_n)\Big|_{x_1, x_2, \ldots, x_n = 0} = 0$$

and the assertion in 1) follows from Taylor's Theorem.

The assertion in 3) holds because the elements of $J^{\perp}$ can be taken as representatives of the elements of $\mathbb{Q}[x_1, x_2, \ldots, x_n]/J$.

next

ORBITS

Let G be a group of permutations of $1, 2, \ldots, n$. Given an element $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n) \in G$ and a vector $a = (a_1, a_2, \ldots, a_n)$ we set

$$\sigma\, a \;=\; (a_{\sigma_1}, a_{\sigma_2}, \ldots, a_{\sigma_n})$$

This given, we let

$$[a]_G \;=\; \{ b = \sigma a \;:\; \sigma \in G \}$$

and call it the "*orbit*" of a under G. The subgroup

$$G_a \;=\; \{ \sigma \in G \;:\; \sigma a = a \}$$

is called the "*stabilizer* " of a.

If G_a is trivial then a is called "*regular*" and the orbit $[a]_G$ has cardinality $|G|$.

In general we have

$$|[a]_G| \;=\; |G|/|G_a|$$

In any case we set

$$J_{[a]_G} \;=\; \big(P \;:\; P(b) = 0 \quad \forall \quad b \in [a]_G \big)$$

and

$$R_{[a]_G} = \mathbb{Q}[x_1, x_2, \ldots, x_n]/J_{[a]_G}\,, \qquad \mathrm{gr}\, R_{[a]_G} = \mathbb{Q}[x_1, x_2, \ldots, x_n]/\mathrm{gr}\, J_{[a]_G}$$

We also set

$$H_{[a]_G} \;=\; (\mathrm{gr}\, J_{[a]_G})^{\perp}$$

and call the elements of this space the "*orbit harmonics*".

Now it follows that

$$\dim R_{[a]_G} = \dim \mathrm{gr}\, R_{[a]_G} = \dim H_{[a]_G} \;=\; |G|/|G_a|$$

next

Conical Orbit Harmonics

(This is why symmetry proves the n! result)

Theorem

Let a *be a regular point. Set*

$$n_o = \mathrm{maxdegree}\ R_{[a]_G}$$

Suppose that $\mathcal{B}$ *is a basis for* $R_{[a]_G}$ *with*

$$\mathcal{B}^{\le n_o} = \mathcal{B} \qquad \text{and} \qquad |\mathcal{B}^{\le i}| = d_o + d_1 + \cdots + d_i \qquad (\, i = 0, 1, \ldots, n_o \,)$$

Then the equalities

$$d_i \;=\; d_{n_o - i} \qquad (\, i = 0, 1, \ldots, n_o \,)$$

imply

(i) $h(\mathcal{B}) \;=\; \{\, h(b) \,:\, b \in \mathcal{B} \,\}$ is a basis for $\mathrm{gr}\, R_{[a]}$

(ii) $F_{\mathrm{gr} R_{[a]_G}}(q) \;=\; F_{H_{[a]_G}}(q) \;=\; \sum_{i=0}^{n_o} d_i\, q^i$,

(iii) $H_{[a]_G}$ is a cone.

next

Where $\Delta_\mu[X;Y]$ comes from

Step 1

Fill the Ferrers diagram of μ *with* $1, 2, \ldots, n$ *in all possible ways to get the* $n!$ *tableaux.*

Step 2

Construct the corresponding orbit points

Theorem

Call $(a, b) = (a_1, a_2, \cdots, a_n; b_1, b_2, \ldots, b_n)$ *the first point so constructed.*

Then

(1) (a, b) *together with all the other points are none other than the orbit*

$$[(\mathbf{a}, \mathbf{b})]_{S_n} = \left\{(\mathbf{a}_{\sigma_1}, \mathbf{a}_{\sigma_2}, \ldots, \mathbf{a}_{\sigma_n}; \mathbf{b}_{\sigma_1}, \mathbf{b}_{\sigma_2}, \ldots, \mathbf{b}_{\sigma_n}) : \sigma \in \mathbf{S}_\mathbf{n}\right\}$$

(2) $\Delta_\mu[X, Y] \in H_{[(a,b)]_{S_n}}$

(3) and since $H_{[(a,b)]_{S_n}}$ *is closed under differentiation it follows that*

$$\mathbf{M}_\mu[\mathbf{X}; \mathbf{Y}] \subseteq \mathbf{H}_{[a,b)]_{S_n}}$$

This implies

$$\mathbf{dim}\, \mathbf{M}_\mu[\mathbf{X}; \mathbf{Y}] \leq \mathbf{n}!$$

with equality if and only if $H_{[(a,b)]_{S_n}}$ *is a cone.*

next

The Algorithm

Step 1

Fill the Ferrers diagram of μ with $1, 2, \ldots, n$ in all possible ways to get the $n!$ tableaux.

Step 2

Construct the corresponding orbit points

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In this case

Step 3

Order of the orbit points and construct the kicking polynomials.

Step 4

Compute the kicking statistics.

3		
6	1	2
4	7	5

$$\begin{bmatrix} 1 \\ 12 \\ 75 \\ 306 \\ 807 \\ 1319 \\ 1319 \\ 807 \\ 306 \\ 75 \\ 12 \\ 1 \end{bmatrix}$$

Theorem

If the kicking statistic are symmetric then the top components of the kicking polynomials give the $n!$ independent derivatives of $\Delta_\mu(X,Y) = \det \|x_j^{p_i} y_j^{q_i}\|_{i,j=1}^{n}$

MAGICS?

No

Only Algebraic Combinatorics

The Group Harmonics

Let J_G be the ideal generated by the homogeneous G-invariant polynomials and set

It is customary to call the elements of H_G "*G-Harmonics*" .

Note that if $P(x)$ is any homogeneous G -invariant polynomial then for any point $a = (a_1, a_2, \ldots, a_n)$ we have

$$P(\sigma a) - P(a) = 0 \quad (\forall \quad \sigma \in G) \implies P(x) - P(a) \in J_{[a]_G} \implies P(x) \in \mathrm{gr}\, J_{[a]_G}$$

Thus

$$J_G \subseteq \mathrm{gr}\, J_G \qquad (2)$$

and taking orthogonal complements gives

$$H_{[a]_G} \subseteq H_G \qquad (3)$$

Now if a is regular we have $|G| = |[a]_G| = \dim H_{[a]_G}$ and thus in full generality we must have

$$\boxed{|G| \leq \dim H_G}$$

It is interesting to know when equality holds

Can you guess?

next

Groups Generated by Reflections

Theorem

If G *is a finite group of* $n \times n$ *matrices then any of the statements below implies the remaining ones.*

(A) $H_{[a]_G} = H_G$ *for all regular points,*

(B) $\dim H_G = |G|$,

(C) H_G *is a cone with summit the product of the reflecting hyperplanes,*

(D) G *is generated by reflexions*

The classical example

The space H_{S_n} (Harmonics of S_n) is the linear span of derivatives of

$$DD_{1^n}(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

next

THE END