Kronecker Products and Combinatorics

of some

Remarkable Diophantine System

Joint work of
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The Kronecker coefficients

For k irreducible S_n characters $\chi^{\lambda^{(1)}}\chi^{\lambda^{(2)}}\cdots\chi^{\lambda^{(k)}}$ we set

$$c_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(k)}} = \sum_{b_1+2b_2+\cdots+nb_n=n} \frac{\chi_b^{\lambda^{(1)}}\chi_b^{\lambda^{(2)}}\cdots\chi_b^{\lambda^{(k)}}}{z_b}$$

where for $b = 1^{b_1}2^{b_2}\cdots n^{b_n}$

$$z_b = 1^{b_1}2^{b_2}\cdots n^{b_n}\, b_1!\, b_2!\cdots b_n!$$

Fundamental Problems:

1 *Give efficient algorithms*

2 *Find combinatorial interpretation (such as the LR rule)*

3 *Find generating functions for families of Kronecker coefficients*

The Kronecker Product

The Kronecker Product of the Schur functions

$$S_{\lambda^{(1)}}, S_{\lambda^{(2)}}, \ldots, S_{\lambda^{(k)}}, \qquad (\lambda^{(i)} \vdash n)$$

is defined as

$$S_{\lambda^{(1)}} * S_{\lambda^{(2)}} * \cdots * S_{\lambda^{(k)}} = \sum_{b_1+2b_2+\cdots+nb_n=n} \frac{\chi_b^{\lambda^{(1)}} \chi_b^{\lambda^{(2)}} \cdots \chi_b^{\lambda^{(k)}}}{z_b} p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n}$$

It follows from the orthogonality of the power sums that for all $1 \leq i \leq k$

$$c_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(k)}} = \left\langle S_{\lambda^{(1)}} * \cdots * S_{\lambda^{(i)}} \, , \, S_{\lambda^{(i+1)}} * \cdots * S_{\lambda^{(k)}} \right\rangle$$

as well as

$$c_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(k)}} = \left\langle S_{\lambda^{(1)}} * \cdots * S_{\lambda^{(k)}} \, , \, S_{(n)} \right\rangle$$

A challenging question

(Wallach) From quantum computing:

Construct the generating function

$$\mathbf{W_k(q) \;=\; \sum_{d\geq 0} q^d \big\langle S_{d,d} * S_{d,d} * \cdots * S_{d,d}\;,\; S_{2d} \big\rangle} \qquad \mathbf{k \geq 2}$$

(the term in the scalar product has $\mathbf{k}$ *factors)*

Easy answers

$$\mathbf{W_2(q) \;=\; \sum_{d\geq 0} q^d \big\langle S_{d,d}\;,\; S_{d,d} \big\rangle \;=\; \sum_{d\geq 0} q^d \;=\; \frac{1}{1-q}}$$

$$\mathbf{W_3(q) \;=\; \sum_{d\geq 0} q^d \big\langle S_{d,d} * S_{d,d}\;,\; S_{d,d} \big\rangle \;=\; \sum_{d\,\text{even}} q^d \;=\; \frac{1}{1-q^2}}$$

A first non-trivial result

Mike Zabrocki using symmetric function identities and Jeff Remmel using combinatorial trickery were able to show that

Theorem

$$\mathbf{S_{d,d} * S_{d,d}(x)} = \sum_{\substack{\alpha \vdash \mathbf{2d} \\ \alpha \text{ with 4 even parts } \geq 0}} \mathbf{S}_\alpha(\mathbf{x}) + \sum_{\substack{\beta \vdash \mathbf{2d} \\ \beta \text{ with 4 odd parts}}} \mathbf{S}_\beta(\mathbf{x}) \qquad (*)$$

Since

$$\left\langle \mathbf{S_{d,d} * S_{d,d} * S_{d,d} * S_{d,d}}\,,\ \mathbf{S_{2d}} \right\rangle = \left\langle \mathbf{S_{d,d} * S_{d,d}}\,,\ \mathbf{S_{d,d} * S_{d,d}} \right\rangle$$

from (*) we derive that

$$\mathbf{W_4(q)} = \sum_{\substack{\alpha \vdash \mathbf{2d} \\ \alpha \text{ with 4 even parts } \geq 0}} \mathbf{q}^{|\alpha|/2} + \sum_{\substack{\beta \vdash \mathbf{2d} \\ \beta \text{ with 4 odd parts}}}$$

This gives

$$\mathbf{W_4(q)} = \frac{\mathbf{1+q^2}}{\mathbf{(1-q)(1-q^2)(1-q^3)(1-q^4)}} = \frac{\mathbf{1}}{\mathbf{(1-q)(1-q^2)^2(1-q^3)}}.$$

The next challenge: Computing $\mathbf{W_5(q)}$

any guesses ?

Wait and see

It is a fascinating story

Guessing the answer

If $\chi^{d,d}$ denotes the Young irreducible character of the symmetric group $\mathcal{S}_{2d}$, then for $k \geq 2$ factors we have

$$\left\langle S_{d,d} * S_{d,d} * \cdots * S_{d,d} \;,\; S_{2d} \right\rangle \;=\; c_d(k) = \frac{1}{(2d)!} \sum_{\alpha \in S_{2d}} \chi^{d,d}(\alpha)^k \qquad (*)$$

We want

$$W_5(q) \;=\; 1 + \sum_{d \geq 1} c_d(5) q^d .$$

Symmetric function machinery quickly yields the first few terms of this series.

But note that for $d = 20$ the definition requires working with $\mathcal{S}_{40}$ and since

$$40! \;=\; 815915283247897734345611269596115894272000000000$$

Some trickery is necessary here.

Proceeding by the naive approach

Using available symmetric function (MAPLE) packages we get

$$\mathbf{W_5(q) = 1 + 5q^2 + q^3 + 36q^4 + 15q^5 + 228q^6 + 231q^7 + 1313q^8 + 1939q^9 + 6971q^{10} + 11899q^{11} + 33118q^{12} + 59543q^{13} + 140620q^{14} + 254476q^{15} + 538042q^{16} + 959028q^{17} + 1871808q^{18} + 3258512q^{19} + 5981444q^{20} + \cdots}$$

Wallach: “*not enough* ”

Using new formulas for $\mathcal{S}_n$ characters (from joint work with Alain Goupil) we obtained 7 more coefficients

$$\mathbf{\cdots + 10140360q^{21} + 17726166q^{22} + 29257848q^{23} + 49127549q^{24} + 79032258q^{25} + 128267727q^{26} + 201437596q^{27} + \cdots}$$

Wallach: “*Very impressive but not enough, we need 52 coefficients to predict* $\mathbf{W_5(q)}$*!* ”

Me: “*forget it!* ”

$$\mathbf{104! = 1029901674514562762384858386476504428305377245499907218232549177688787173 2475287174542709871683888003235965704141638377695179741979175588724736000000000000000000000000}$$

The Miracle

One morning Wallach wakes up to the realization that using fancy Lie Group trickery he could compute coefficients of $W_5(q)$ way beyond $d = 27$. After several hours of computer time the needed 52 coefficients were obtained and Wallacch was able to construct a rational expression for $W_5(q)$.

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$$W_5(q) = \frac{\begin{aligned}\Big(q^{54} + q^{52} + 16q^{50} + 9q^{49} + 98q^{48} + 154q^{47} + 465q^{46} + 915q^{45} + 2042q^{44} + 3794q^{43} \\ + 7263q^{42} + 12688q^{41} + 21198q^{40} + 34323q^{39} + 52205q^{38} + 77068q^{37} + 108458q^{36} \\ + 147423q^{35} + 191794q^{34} + 241863q^{33} + 292689q^{32} + 342207q^{31} + 386980q^{30} + 421057q^{29} \\ + 443990q^{28} + 451398q^{27} + 443990q^{26} + 421057q^{25} + 386980q^{24} + 342207q^{23} + 292689q^{22} \\ + 241863q^{21} + 191794q^{20} + 147423q^{19} + 108458q^{18} + 77068q^{17} + 52205q^{16} + 34323q^{15} \\ + 21198q^{14} + 12688q^{13} + 7263q^{12} + 3794q^{11} + 2042q^{10} + 915q^{9} + 465q^{8} + 154q^{7} + \\ + 98q^{6} + 9q^{5} + 16q^{4} + q^{2} + 1\Big)\end{aligned}}{(1-q^2)^4(1-q^3)(1-q^4)^6(1-q^5)(1-q^6)^5}$$

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Combinatorial interpretation?

Working Towards a Proof

A truly surprising identity

Theorem

If

$$\begin{aligned} F(a_1, a_2, \ldots, a_k; q) &= \frac{\prod_{i=1}^{k}(1-a_i)}{\prod_{S\subseteq[1,k]}\left(1 - q\prod_{i\in S} a_i\right)} \\ &= \sum_{r_1\geq 0}\sum_{r_2\geq 0}\cdots\sum_{r_k\geq 0}\sum_{m\geq 0} c_{m;r_1,r_2,\ldots,r_k}\, q^m\, a_1^{r_1} a_2^{r_2}\cdots a_k^{r_k} \end{aligned}$$

then

$$c_{m;r_1,r_2,\ldots,r_k} = \left\langle S_{m-r_1,r_1} * S_{m-r_2,r_2} * \cdots * S_{m-r_k,r_k}\ ,\ S_m \right\rangle \qquad \forall \quad m \geq 2\max(r_1, r_2, \ldots, r_k)$$

In particular

$$c_{2d;d,d,\ldots,d} = \left\langle S_{d,d} * S_{d,d} * \cdots * S_{d,d}\ ,\ S_{2d} \right\rangle$$

Thus the desired series $W_k(q)$ is obtained by selecting the terms where

$$m = 2d\,, \quad r_1 = d\,, \quad r_2 = d\,, \quad \ldots, r_k = d\,,$$

Selecting a subseries

Proposition

If

$$\mathbf{F}(a_1, a_2, \ldots a_k; q) = \sum_{r_1 \geq 0} \sum_{r_2 \geq 0} \cdots \sum_{r_k \geq 0} \sum_{m \geq 0} c_{m;r_1,r_2,\ldots,r_k} \, q^m \, a_1^{r_1} a_2^{r_2} \cdots a_k^{r_k}$$

then

$$W_k(q^2) = \sum_{d \geq 0} c_{2d;d,d,\ldots d} \; q^{2d} \;=\; F(a_1^{-2}, a_2^{-2}, \cdots, a_k^{-2}; q \; a_1 a_2 \cdots a_k)\Big|_{a_1^0, a_2^0, \ldots, a_2^0,}$$

Proof

$$F(a_1^{-2}, a_2^{-2}, \cdots, a_k^{-2}; q \; a_1 a_2 \cdots a_k) \;=$$

$$= \sum_{r_1 \geq 0} \sum_{r_2 \geq 0} \cdots \sum_{r_k \geq 0} c_{m;r_1,r_2,\ldots,r_k} \, q^m \, a_1^{m-2r_1} a_2^{m-2r_2} \cdots a_k^{m-2r_k}$$

QED

Kronecker Products and Constant terms identities

Recalling that

$$\mathbf{F}(a_1, a_2, \ldots, a_k; q) = \frac{\prod_{i=1}^{k} (1 - a_i)}{\prod_{S \subseteq [1,k]} \left(1 - q \prod_{i \in S} a_i\right)}$$

we derive that

$$\mathbf{F}(a_1^{-2}, a_2^{-2}, \cdots, a_k^{-2}; q\ \ a_1 a_2 \cdots a_k) = \frac{\prod_{i=1}^{k} \left(1 - \frac{1}{a_i^2}\right)}{\prod_{S \subseteq [1,k]} \left(1 - \frac{q\, a_1 a_2 \cdots a_k}{\prod_{i \in S} a_i^2}\right)} = \frac{\prod_{i=1}^{k} \left(1 - \frac{1}{a_i^2}\right)}{\prod_{S \subseteq [1,k]} \left(1 - q \frac{\prod_{i \notin S} a_i}{\prod_{i \in S} a_i}\right)}$$

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We can thus state

Theorem

$$W_k(q^2) = \sum_{d \geq 0} \left\langle S_{d,d} * S_{d,d} * \cdots * S_{d,d}\ ,\ S_{2d} \right\rangle q^{2d} = \left. \frac{\prod_{i=1}^{k} \left(1 - \frac{1}{a_i^2}\right)}{\prod_{S \subseteq [1,k]} \left(1 - q \frac{\prod_{i \notin S} a_i}{\prod_{i \in S} a_i}\right)} \right|_{a_1^0 a_2^0 \cdots a_k^0}$$

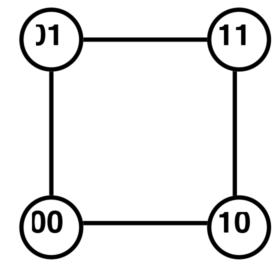
Some Remarkable Diophantine Systems

A general Problem

Determine all possible ways to assign ≥ 0 integral weights to the vertices of the k-dimensional hypercube so that each hyperface has the same weight.

Example $k = 2$ (see attached figure) we must have

$$\begin{aligned} p_{00} + p_{01} &= p_{10} + p_{11} \\ p_{00} + p_{10} &= p_{01} + p_{11} \end{aligned}$$



In simpler notation we want all compositions $p = (p_1, p_2, p_3, p_4)$ which are solutions of

$$\mathcal{S}_2 = \left\| \begin{aligned} p_1 + p_2 - p_3 - p_4 &= 0 \\ p_1 - p_2 + p_3 - p_4 &= 0 \end{aligned} \right.$$

Here "all solutions" means the generating function

$$GF_{\mathcal{S}_2}(x_1, x_2, x_3, x_4) = \sum_{p \in \mathcal{S}_2} x_1^{p_1} x_2^{p_2} x_3^{p_3} x_4^{p_4}$$

MacMahon "Partition Analysis"

To solve the system

$$\mathcal{S}_2 = \left\| \begin{array}{l} \mathbf{p_1 + p_2 - p_3 - p_4 = 0} \\ \mathbf{p_1 - p_2 + p_3 - p_4 = 0} \end{array} \right. \qquad (\textbf{with} \quad \mathbf{p_i \geq 0})$$

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we construct the corresponding "Omega Function"

$$\begin{aligned} \Omega_{\mathcal{S}_2} &= \sum_{p_1 \geq 0} \sum_{p_2 \geq 0} \sum_{p_3 \geq 0} \sum_{p_4 \geq 0} x_1^{p_1} x_2^{p_2} x_2^{p_3} x_4^{p_4} \;\; a_1^{p_1 + p_2 - p_3 - p_4} \;\; a_2^{p_1 - p_2 \; + \; p_3 - p_4} \\ &= \frac{1}{1 - x_1 a_1 a_2} \frac{1}{1 - x_2 a_1 / a_2} \frac{1}{1 - x_3 a_2 / a_1} \frac{1}{1 - x_4 / a_1 a_2} \end{aligned}$$

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and then

$$GF_{\mathcal{S}_2}(x_1,x_2,x_3,x_4) \;=\; \Omega_{\mathcal{S}}\Big|_{a_1^0a_2^0}$$

MacMahon “Partition Analysis”

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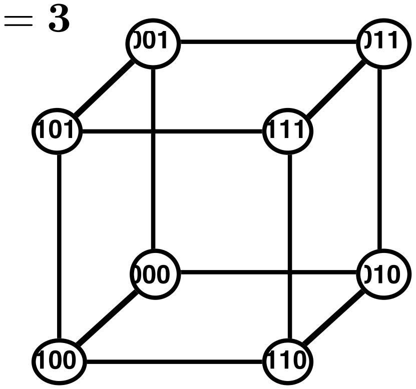
and then

$$GF_{\mathcal{S}_2}(x_1, x_2, x_3, x_4) \;=\; \Omega_{\mathcal{S}} \Big|_{a_1^0 a_2^0}$$

In particular

$$GF_{\mathcal{S}_2}(q,q,q,q) = \sum_{p \in \mathcal{S}_2} q^{p_1+p_2+p_3+p_4} = \frac{1}{\prod_{S \subseteq [1,2]} \left(1 - q \, \frac{\prod_{i \in S} a_i}{\prod_{i \notin S} a_i}\right)} \Bigg|_{a_1^0 a_2^0}$$

The Diophantine System for $k = 3$



Equating the weights of opposite faces gives

$$\begin{aligned}
p_{000} + p_{001} + p_{010} + p_{011} &= p_{100} + p_{101} + p_{110} + p_{111} \\
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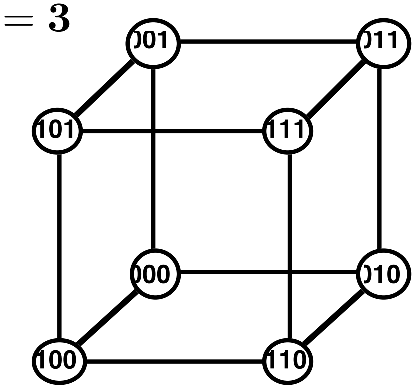
The Diophantine System for $\mathbf{k = 3}$



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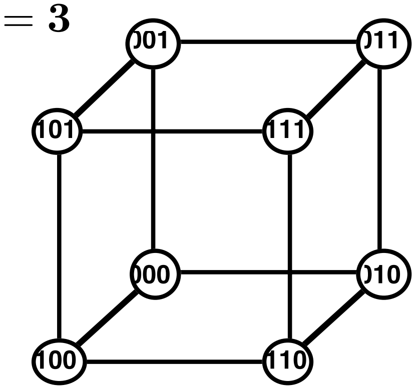
and we get the system

$$\mathcal{S}_3 = \left\| \begin{aligned} p_{000} + p_{001} + p_{010} + p_{011} - p_{100} - p_{101} - p_{110} - p_{111} &= 0 \\ p_{000} + p_{001} - p_{010} - p_{011} + p_{100} + p_{101} - p_{110} - p_{111} &= 0 \\ p_{000} - p_{001} + p_{010} - p_{011} + p_{100} - p_{101} + p_{110} - p_{111} &= 0 \end{aligned} \right.$$

The Diophantine System for $k = 3$



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or in simpler notation

$$\mathcal{S}_3 = \left\| \begin{aligned} p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_7 - p_8 &= 0 \\ p_1 + p_2 - p_3 - p_4 + p_5 + p_6 - p_7 - p_8 &= 0 \\ p_1 - p_2 + p_3 - p_4 + p_5 - p_6 + p_7 - p_8 &= 0 \end{aligned} \right.$$

q-Counting by “total weight” for k=3

Applying MacMahon Partition analysis to the system

$$\mathcal{S}_3 \quad = \quad \left\Vert \begin{array}{rcl} \mathbf{p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_7 - p_8} & = & \mathbf{0} \\ \mathbf{p_1 + p_2 - p_3 - p_4 + p_5 + p_6 - p_7 - p_8} & = & \mathbf{0} \\ \mathbf{p_1 - p_2 + p_3 - p_4 + p_5 - p_6 + p_7 - p_8} & = & \mathbf{0} \end{array} \right.$$

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$$\mathcal{S}_3 = \left\| \begin{array}{lcl} p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_7 - p_8 & = & 0 \\ p_1 + p_2 - p_3 - p_4 + p_5 + p_6 - p_7 - p_8 & = & 0 \\ p_1 - p_2 + p_3 - p_4 + p_5 - p_6 + p_7 - p_8 & = & 0 \end{array} \right.$$

we obtain that

$$\begin{aligned} GF_{\mathcal{S}_3}(q) &= \sum_{p \in \mathcal{S}_3} q^{p_1+p_2+p_3+p_4+p_5+p_6+p_7+p_8} \\ &= \sum_{p_1 \geq 0} \sum_{p_2 \geq 0} \cdots \sum_{p_8 \geq 0} q^{p_1+p_2+p_3+p_4+p_5+p_6+p_7+p_8} \; a_1^{p_1+p_2+p_3+p_4-p_5-p_6-p_7-p_8} \times \\ &\qquad \times a_2^{p_1+p_2-p_3-p_4+p_5+p_6-p_7-p_8} a_3^{p_1-p_2+p_3-p_4+p_5-p_6+p_7-p_8} \Big|_{a_1^0 a_2^0 a_3^0} \end{aligned}$$

q-Counting by “total weight” for k=3

Applying MacMahon Partition analysis to the system

$$\mathcal{S}_3 \quad = \quad \left\| \begin{array}{lcl} p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_7 - p_8 & = & 0 \\ p_1 + p_2 - p_3 - p_4 + p_5 + p_6 - p_7 - p_8 & = & 0 \\ p_1 - p_2 + p_3 - p_4 + p_5 - p_6 + p_7 - p_8 & = & 0 \end{array} \right.$$

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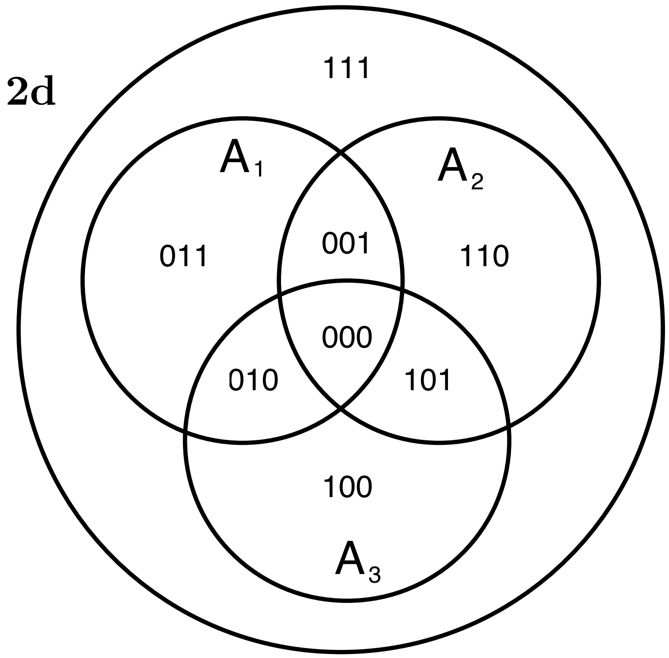
and summing the series we get

$$GF_{\mathcal{S}_3}(q) \quad = \quad \frac{1}{\prod_{S \subseteq [1,2,3]} \left(1 - q \, \frac{\prod_{i \in S} a_i}{\prod_{i \notin S} a_i}\right)} \Bigg|_{a_1^0 a_2^0 a_3^0}$$

Combinatorics of Venn Diagrams of 3 sets

Three subsets of cardinality d of a set B of cardinality 2d



Let

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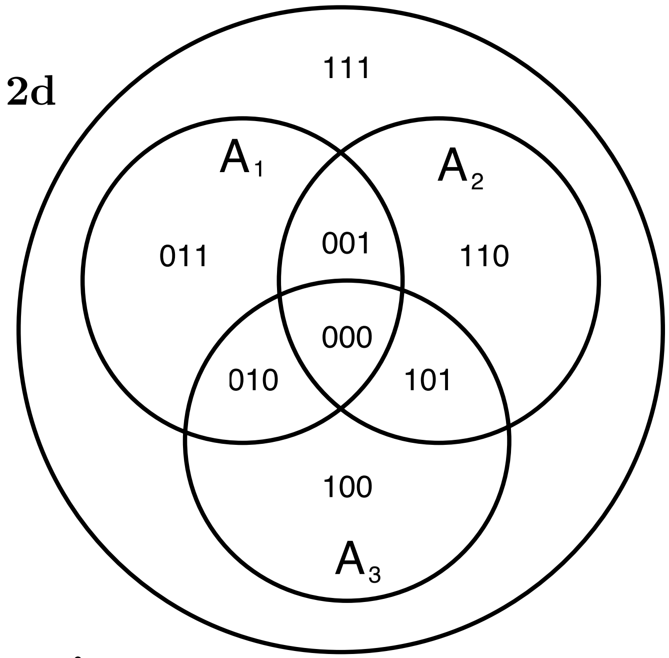

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Same Diophantine System!

$$|\mathbf{A_1}| = \mathbf{p_{000}} + \mathbf{p_{001}} + \mathbf{p_{010}} + \mathbf{p_{011}} \quad = \quad |\mathbf{A_1^c}| = \mathbf{p_{100}} + \mathbf{p_{101}} + \mathbf{p_{110}} + \mathbf{p_{111}}$$

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$$|\mathbf{A_3}| = \mathbf{p_{000}} + \mathbf{p_{010}} + \mathbf{p_{100}} + \mathbf{p_{110}} \quad = \quad |\mathbf{A_3^c}| = \mathbf{p_{001}} + \mathbf{p_{011}} + \mathbf{p_{101}} + \mathbf{p_{111}}$$

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(6) *In conclusion the Kronecker coefficient in (*) is equal to the number of solutions of the Diophantine system*

$$|\mathbf{A_1}| = \mathbf{p_{000} + p_{001} + p_{010} + p_{011} = d} \ , \qquad |\mathbf{A_1^c}| = \mathbf{p_{100} + p_{101} + p_{110} + p_{111} = d}$$

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Recall that we proved that if

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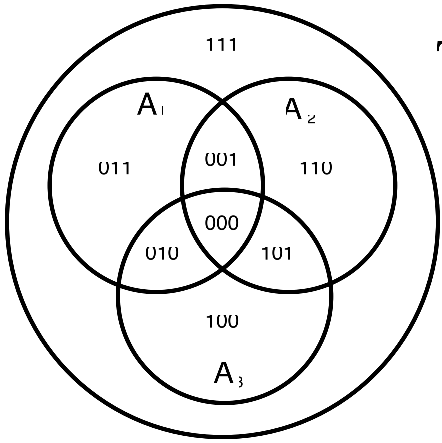
Notice the similarity?

Closely related Diophantine Systems



Three subsets A_1, A_2, A_3 of a set B of cardinality 2d with

$$|A_1| = d+1, \;\; |A_2| = d+1, \;\; |A_3| = d.$$

This gives

$$|A_1| - |A_1^c| = 2\,, \quad |A_2| - |A_2^c| = 2\,, \quad |A_3| - |A_3^c| = 0$$

With the previous notation the new Diophantine system is

$$\mathcal{S}_3^* \;=\; \left\| \begin{array}{lcl} p_1+p_2+p_3+p_4-p_5-p_6-p_7-p_8 &=& 2 \\ p_1+p_2-p_3-p_4+p_5+p_6-p_7-p_8 &=& 2 \\ p_1-p_2+p_3-p_4+p_5-p_6+p_7-p_8 &=& 0 \end{array} \right.$$

and we obtain that

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Summing

$$GF_{\mathcal{S}_3^*}(q) \;=\; \frac{\frac{1}{a_1^2}\,\frac{1}{a_2^2}}{\prod_{S \subseteq [1,2,3]} \left(1 - q\,\frac{\prod_{i \in S} a_i}{\prod_{i \notin S} a_i}\right)} \Bigg|_{a_1^0 a_2^0 a_3^0}$$

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$$\mathbf{a_d(2,2,0)} \;=\; \left\langle \mathbf{h_{d+1} h_{d-1} * h_{d+1} h_{d-1} * h_d h_d} \;,\; \mathbf{S_{2d}} \right\rangle \qquad (*)$$

gives the number of occurrences of the trivial representation under this action and thus it also gives the number of distinct orbits.

(5) *From this it follows that*

$$\sum_{\mathbf{d} \geq \mathbf{0}} \mathbf{a_d(2,2,0)\, q^{2d}} \;=\; \frac{\frac{\mathbf{1}}{\mathbf{a_1^2}} \frac{\mathbf{1}}{\mathbf{a_2^2}}}{\prod_{\mathbf{S} \subseteq [\mathbf{1},\mathbf{k}]} \left(\mathbf{1} - \mathbf{q} \frac{\prod_{\mathbf{i} \in \mathbf{S}} \mathbf{a_i}}{\prod_{\mathbf{j} \notin \mathbf{S}} \mathbf{a_j}} \right)} \Bigg|_{\mathbf{a_1^0 a_2^2 \cdots a_k^0}}$$

Diophantine systems and Schur functions

Note that the Jacobi-Trudi identity gives

$$S_{d,d} = h_d h_d - h_{d-1} h_{d+1}$$

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Thus

$$\begin{aligned} c_d(3) &= \left\langle S_{d,d} * S_{d,d} * S_{d,d} \, , \, S_{2d} \right\rangle = \\ &= \left\langle (h_d h_d - h_{d-1} h_{d+1}) * (h_d h_d - h_{d-1} h_{d+1}) * (h_d h_d - h_{d-1} h_{d+1}) \, , \, S_{2d} \right\rangle \\ &= \left\langle h_d h_d * h_d h_d * h_d h_d \, , \, S_{2d} \right\rangle \\ &\qquad - 3 \left\langle h_d h_d * h_d h_d * h_{d-1} h_{d+1} \, , \, S_{2d} \right\rangle \\ &\qquad\qquad + 3 \left\langle h_d h_d * h_{d-1} h_{d+1} * h_{d-1} h_{d+1} \, , \, S_{2d} \right\rangle \\ &\qquad\qquad\qquad \left\langle h_{d-1} h_{d+1} * h_{d-1} h_{d+1} * h_{d-1} h_{d+1} * \, , \, S_{2d} \right\rangle \end{aligned}$$

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from which we derive a combinatorial-representation theoretical proof of the identity

Diophantine systems and Schur functions

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\end{aligned}$$

from which we derive a combinatorial-representation theoretical proof of the identity

$$W_3(q) \;=\; \sum_{d \geq 0} c_d(3)\, q^{2d} \;=\; \frac{\displaystyle\prod_{i=1}^{3}\left(1 - \frac{1}{a_i^2}\right)}{\displaystyle\prod_{S \subseteq [1,3]}\left(1 - q\frac{\prod_{i \in S} a_i}{\prod_{j \notin S} a_j}\right)} \Bigg|_{a_1^0 a_2^2 a_3^0}$$

Diophantine systems and Schur functions

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How do we compute these constant terms

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Available Tools by hand:

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(a) Some of MacMahon's auxiliary identities

$$\frac{1}{1-xa}\frac{1}{1-y/a}\Big|_{a^0} = \frac{1}{1-xy}$$

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(b) The more general partial fraction algorithm of Guoce Xin

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(*Guoce Xin* >>>>>>>> *Andrews et all*)

Purely Combinatorially by hand

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Say

$$\mathcal{S}_2 = \left\| \begin{array}{rcl} \mathbf{p_1 + p_2 - p_3 - p_4} & = & \mathbf{0} \\ \mathbf{p_1 - p_2 + p_3 - p_4} & = & \mathbf{0} \end{array} \right. \qquad (\textbf{with} \quad \mathbf{p_i \geq 0})$$

Set

$$\mathbf{e} = \mathbf{min(p_1, p_4)}\ , \qquad \mathbf{f} = \mathbf{min(p_2, p_3)}$$

and note that

$$\mathbf{(e, f, f, e)} \in \mathcal{S}_\mathbf{2}$$

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$$(\mathbf{e}, \mathbf{f}, \mathbf{f}, \mathbf{e}) \in \mathcal{S}_2$$

so

$$(\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}) \;=\; (\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}) - (\mathbf{e}, \mathbf{f}, \mathbf{f}, \mathbf{e}) \in \mathcal{S}$$

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Possible patterns

$$\mathbf{(u_1, u_2, 0, 0)}\;,\; \mathbf{(u_1, 0, u_3, 0)}\;,\; \mathbf{(0, u_2, 0, u_4)}\;,\; \mathbf{(0, 0, u_3, u_4)}\;,$$

Purely Combinatorially by hand

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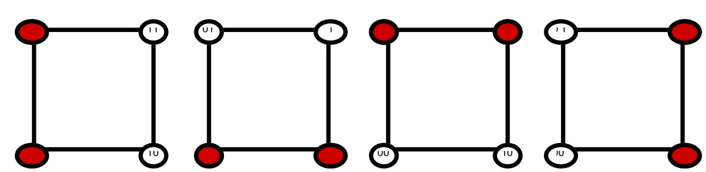
all impossible



Purely Combinatorially by hand

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$$\mathcal{S}_2 \;=\; \left\| \begin{array}{l} p_1 + p_2 - p_3 - p_4 \;=\; 0 \\ p_1 - p_2 + p_3 - p_4 \;=\; 0 \end{array} \right. \qquad (\text{with} \quad p_i \geq 0)$$

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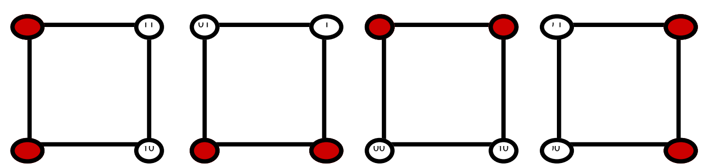
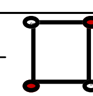
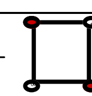


Solution

$$GF_{\mathcal{S}_2}(x_1, x_2, x_3, x_4) \;=\; \sum_{e\geq 0}\sum_{f\geq 0} x_1^e x_2^f x_3^f x_4^e \;=\; \frac{1}{(1 - x_1x_4)(1 - x_2x_3)} \;=$$

1 / ((1 − ) (1 − ))

The general solution for k=3 by Computer

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Applying MacMahon Partition Analysis

$$\mathbf{GF}_{\mathcal{S}_3}(x_1, x_2, \ldots, x_8) = \frac{1}{\left(1 - x_1 a_1 a_2 a_3\right)\left(1 - x_2 \frac{a_2 a_3}{a_1}\right)\left(1 - x_3 \frac{a_1 a_3}{a_2}\right)\left(1 - x_4 \frac{a_1 a_2}{a_3}\right)}$$

$$\left. \times \frac{1}{\left(1 - x_5 \frac{a_3}{a_1 a_2}\right)\left(1 - x_6 \frac{a_2}{a_1 a_3}\right)\left(1 - x_7 \frac{a_1}{a_2 a_3}\right)\left(1 - x_8 \frac{1}{a_1 a_2 a_3}\right)} \right|_{a_1^0 a_2^0 a_3^0}$$

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Applying MacMahon Partition Analysis

$$\mathbf{GF}_{\mathcal{S}_3}(x_1,x_2,\ldots,x_8)=\frac{1}{\left(1-x_1a_1a_2a_3\right)\left(1-x_2\frac{a_2a_3}{a_1}\right)\left(1-x_3\frac{a_1a_3}{a_2}\right)\left(1-x_4\frac{a_1a_2}{a_3}\right)}$$

$$\left. x\frac{1}{\left(1-x_5\frac{a_3}{a_1a_2}\right)\left(1-x_6\frac{a_2}{a_1a_3}\right)\left(1-x_7\frac{a_1}{a_2a_3}\right)\left(1-x_8\frac{1}{a_1a_2a_3}\right)}\right|_{a_1^0a_2^0a_3^0}$$

and the MAPLE package of Guoce Xin gives

$$\mathbf{GF}_{\mathcal{S}}(x_1,x_2,\ldots,x_8) = \frac{1}{(1-x_1x_8)(1-x_2x_7)(1-x_3x_6)(1-x_4x_5)}\times$$

$$\times\left(1+\frac{x_1x_4x_6x_7}{1-x_1x_4x_6x_7}+\frac{x_2x_3x_5x_8}{1-x_2x_3x_5x_8}\right)$$

The general solution for k=3 by Computer

Applying MacMahon Partition Analysis

$$\mathbf{GF}_{\mathcal{S}_3}(\mathbf{x_1, x_2, \ldots, x_8}) = \frac{1}{\left(1 - x_1 a_1 a_2 a_3\right)\left(1 - x_2 \frac{a_2 a_3}{a_1}\right)\left(1 - x_3 \frac{a_1 a_3}{a_2}\right)\left(1 - x_4 \frac{a_1 a_2}{a_3}\right)}$$

$$\text{x} \frac{1}{\left(1 - x_5 \frac{a_3}{a_1 a_2}\right)\left(1 - x_6 \frac{a_2}{a_1 a_3}\right)\left(1 - x_7 \frac{a_1}{a_2 a_3}\right)\left(1 - x_8 \frac{1}{a_1 a_2 a_3}\right)}\Bigg|_{a_1^0 a_2^0 a_3^0}$$

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$$\times \left(1 + \frac{x_1 x_4 x_6 x_7}{1 - x_1 x_4 x_6 x_7} + \frac{x_2 x_3 x_5 x_8}{1 - x_2 x_3 x_5 x_8}\right)$$

Visually

The general solution for k=3 by Computer

Applying MacMahon Partition Analysis

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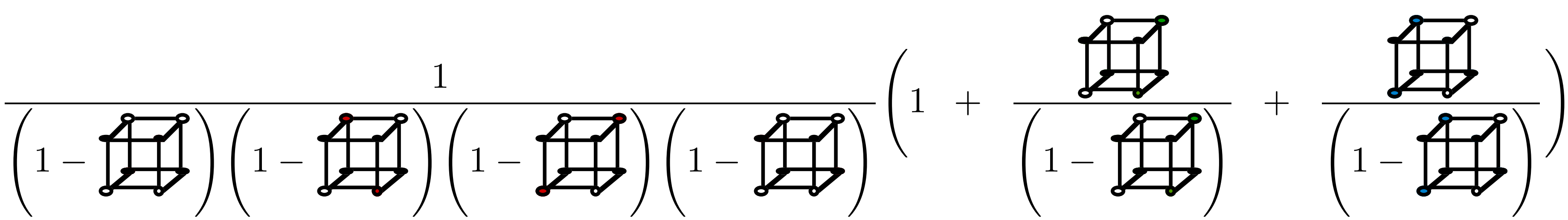
Visually



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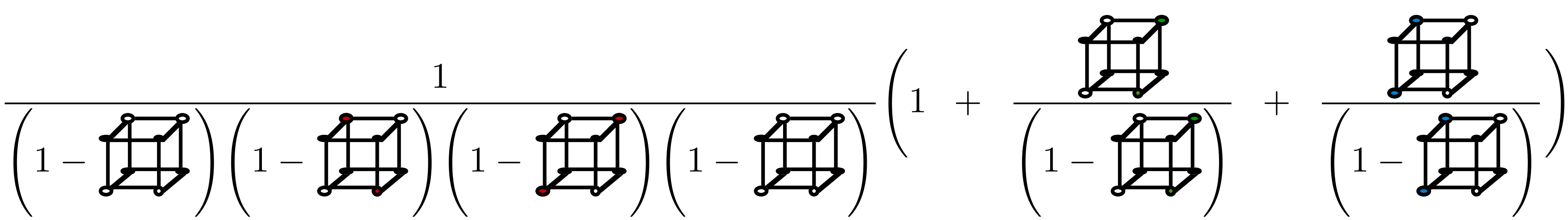


Working by hand

Working by hand

Note that the system

$$\mathcal{S}_3 = \left\| \begin{array}{rcl} \mathbf{p_1} + \mathbf{p_2} + \mathbf{p_3} + \mathbf{p_4} - \mathbf{p_5} - \mathbf{p_6} - \mathbf{p_7} - \mathbf{p_8} & = & \mathbf{0} \\ \mathbf{p_1} + \mathbf{p_2} - \mathbf{p_3} - \mathbf{p_4} + \mathbf{p_5} + \mathbf{p_6} - \mathbf{p_7} - \mathbf{p_8} & = & \mathbf{0} \\ \mathbf{p_1} - \mathbf{p_2} + \mathbf{p_3} - \mathbf{p_4} + \mathbf{p_5} - \mathbf{p_6} + \mathbf{p_7} - \mathbf{p_8} & = & \mathbf{0} \end{array} \right.$$

Working by hand

Note that the system

$$\mathcal{S}_3 \;=\; \left\| \begin{array}{lcl} \mathbf{p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_7 - p_8} & = & \mathbf{0} \\ \mathbf{p_1 + p_2 - p_3 - p_4 + p_5 + p_6 - p_7 - p_8} & = & \mathbf{0} \\ \mathbf{p_1 - p_2 + p_3 - p_4 + p_5 - p_6 + p_7 - p_8} & = & \mathbf{0} \end{array} \right.$$

has the following obvious solutions

$$(\mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}) \qquad (\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0})$$

$$(\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}) \qquad (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0})$$

Working by hand

Note that the system

$$\mathcal{S}_3 \quad = \quad \left\| \begin{array}{rcl} \mathbf{p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_7 - p_8} & = & \mathbf{0} \\ \mathbf{p_1 + p_2 - p_3 - p_4 + p_5 + p_6 - p_7 - p_8} & = & \mathbf{0} \\ \mathbf{p_1 - p_2 + p_3 - p_4 + p_5 - p_6 + p_7 - p_8} & = & \mathbf{0} \end{array} \right.$$

has the following obvious solutions

$$(\mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}) \qquad (\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0})$$

$$(\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}) \qquad (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0})$$

So we set $\quad \mathbf{e = \min(p_1, p_8)}\ , \quad \mathbf{f = \min(p_2, p_7)}\ , \quad \mathbf{g = \min(p_3, p_6)}\ , \quad \mathbf{h = \min(p_4, p_5)}\ ,$

Working by hand

Note that the system

$$\mathcal{S}_3 \quad = \quad \left\| \begin{array}{lcl} p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_7 - p_8 & = & 0 \\ p_1 + p_2 - p_3 - p_4 + p_5 + p_6 - p_7 - p_8 & = & 0 \\ p_1 - p_2 + p_3 - p_4 + p_5 - p_6 + p_7 - p_8 & = & 0 \end{array} \right.$$

has the following obvious solutions

$$(1\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,1) \qquad (0\,,\,1\,,\,0\,,\,0\,,\,0\,,\,0\,,\,1\,,\,0)$$
$$(0\,,\,0\,,\,1\,,\,0\,,\,0\,,\,1\,,\,0\,,\,0) \qquad (0\,,\,0\,,\,0\,,\,1\,,\,1\,,\,0\,,\,0\,,\,0)$$

So we set $\quad e = \min(p_1, p_8)\,, \quad f = \min(p_2, p_7)\,, \quad g = \min(p_3, p_6)\,, \quad h = \min(p_4, p_5)\,,$
and we are reduced to determine the solutions

$$(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8) \;=\; (p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) - (e\,,\,f\,,\,g\,,\,h\,,\,h\,,\,g\,,\,f\,,\,e)$$

Working by hand

Note that the system

$$\mathcal{S}_3 \;=\; \left\| \begin{array}{rcl} p_1 + p_2 + p_3 + p_4 - p_5 - p_6 - p_7 - p_8 &=& 0 \\ p_1 + p_2 - p_3 - p_4 + p_5 + p_6 - p_7 - p_8 &=& 0 \\ p_1 - p_2 + p_3 - p_4 + p_5 - p_6 + p_7 - p_8 &=& 0 \end{array} \right.$$

has the following obvious solutions

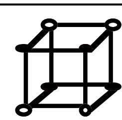
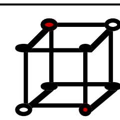
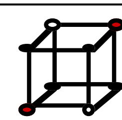
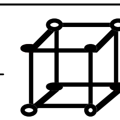
$$(1,0,0,0,0,0,0,1) \qquad (0,1,0,0,0,0,1,0)$$
$$(0,0,1,0,0,1,0,0) \qquad (0,0,0,1,1,0,0,0)$$

So we set $e = \min(p_1,p_8)$, $f = \min(p_2,p_7)$, $g = \min(p_3,p_6)$, $h = \min(p_4,p_5)$, and we are reduced to determine the solutions

$$(u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8) \;=\; (p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8) - (e,f,g,h,h,g,f,e)$$

This proves

$\mathrm{GF}_{\mathcal{S}_3}(x_1,x_2,\ldots,x_8) \;=\; \dfrac{1}{(1 - \square)(1 - \square)(1 - \square)(1 - \square)}$ x (something), where the four factors of the denominator are, in order:

$(1 -$  $)$ $(1 -$  $)$ $(1 -$  $)$ $(1 -$  $)$

Determining the asymmetric solutions

The differences

$$(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8) \;=\; (p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8) - (e\,,\ f\,,\ g\,,\ h\,,\ h\,,\ g\,,\ f\,,\ e)$$

have the property that

$$u_1 u_8 = 0; \quad u_2 u_7 = 0; \quad u_3 u_6 = 0; \quad u_4 u_5 = 0.$$

The support of one of these solutions could, in principle, be given by any one of $2^4 = 16$ patterns:

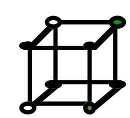
$$(x,x,x,x,0,0,0,0)\,,\ (0,x,x,x,0,0,0,x)\,,\ (x,0,x,x,0,0,x,0)\,,\ (0,0,x,x,0,0,x,x)\,,$$
$$(x,x,0,x,0,x,0,0)\,,\ (0,x,0,x,0,x,0,x)\,,\ (x,0,0,x,0,x,x,0)\,,\ (0,0,0,x,0,x,x,x)\,,$$
$$(x,x,x,0,x,0,0,0)\,,\ (0,x,x,0,x,0,0,x)\,,\ (x,0,x,0,x,0,x,0)\,,\ (0,0,x,0,x,0,x,x)\,,$$
$$(x,x,0,0,x,x,0,0)\,,\ (0,x,0,0,x,x,0,x)\,,\ (x,0,0,0,x,x,x,0)\,,\ (0,0,0,0,x,x,x,x)\,,$$

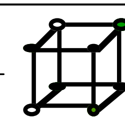
However, all but two of these patterns can be the support of a solution, namely

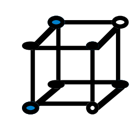
$$(0,x,x,0,x,0,0,x) \quad \text{and} \quad (x,0,0,x,0,x,x,0)$$

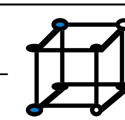
This proves the identity

$$(\text{something}) = \left(1 + \frac{x_1 x_4 x_6 x_7}{1 - x_1 x_4 x_6 x_7} + \frac{x_2 x_3 x_5 x_8}{1 - x_2 x_3 x_5 x_8}\right) = \Bigg(1 +$$



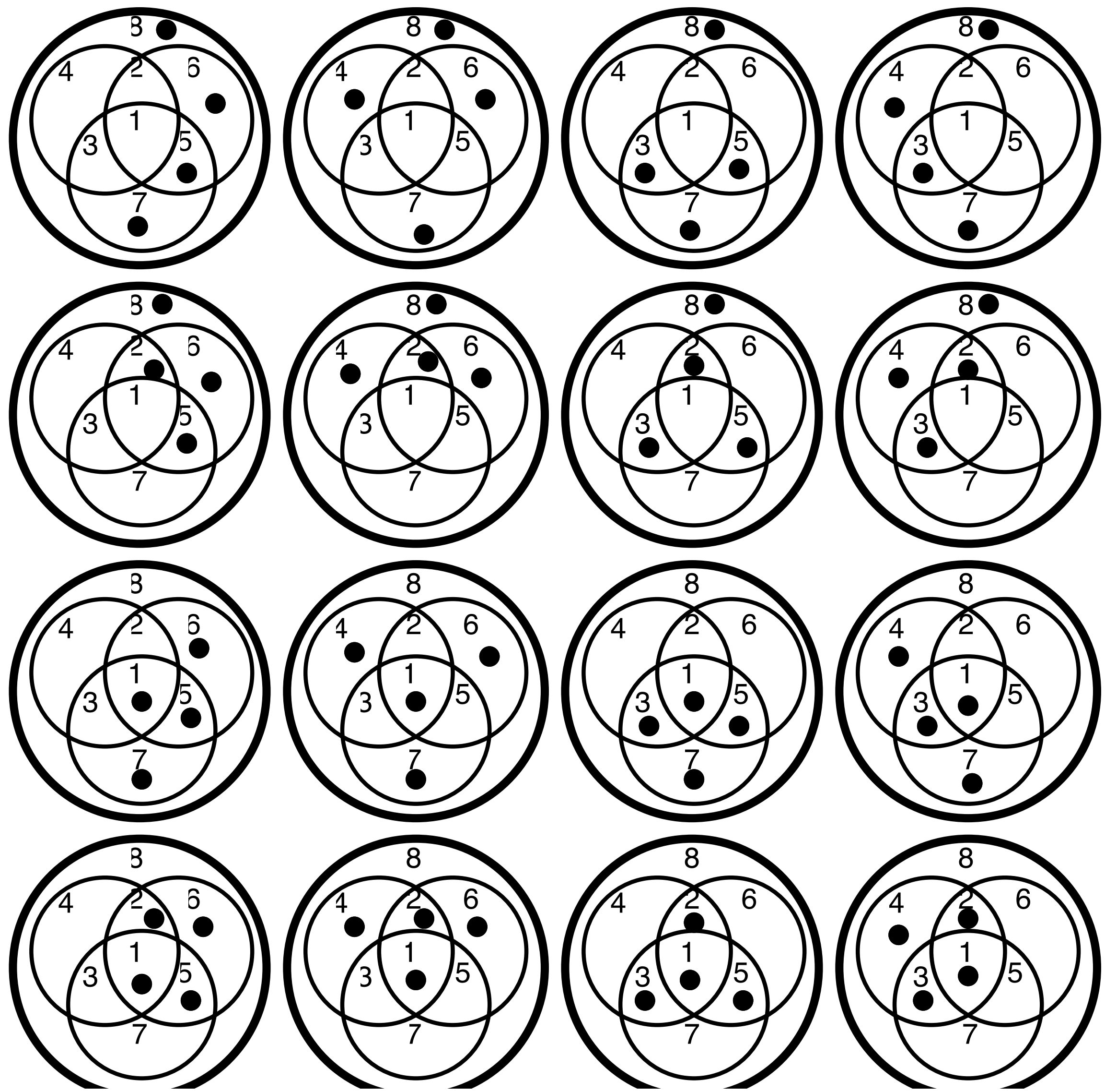
over $\left(1 - \right.$  $\left.\right)$ $+$



over $\left(1 - \right.$  $\left.\right)$ $\Bigg)$

A Venn diagram view of the 14 excluded patterns

Here a black dot in a subset represents an assignment of an integer giving the cardinality of that subset. Remarkably, all but two of these assignemts can be eliminated regardless of the actual assigned values.



Connections with Invariant Theory

If

$$\mathbf{A_1} = \begin{bmatrix} a_1 & 0 \\ 0 & 1/a_1 \end{bmatrix} \quad \text{and} \quad \mathbf{A_2} = \begin{bmatrix} a_2 & 0 \\ 0 & 1/a_2 \end{bmatrix}$$

then

$$\mathbf{A_1} \otimes \mathbf{A_2} = \begin{bmatrix} a_1 a_2 & 0 & 0 & 0 \\ 0 & a_2/a_1 & 0 & 0 \\ 0 & 0 & a_1/a_2 & 0 \\ 0 & 0 & 0 & 1/a_1 a_2 \end{bmatrix}$$

More generally given k matrices

$$\mathbf{A_1} = \begin{bmatrix} a_1 & 0 \\ 0 & 1/a_1 \end{bmatrix}, \ \mathbf{A_2} = \begin{bmatrix} a_2 & 0 \\ 0 & 1/a_2 \end{bmatrix}, \ \dots \ , \mathbf{A_2} = \begin{bmatrix} a_k & 0 \\ 0 & 1/a_k \end{bmatrix}$$

then

$$\mathbf{A_1} \otimes \mathbf{A_2} \otimes \otimes \cdots \otimes \mathbf{A_k} = \text{diag} \left\| \frac{\prod_{i \in S} a_i}{\prod_{i \notin S} a_i} \right\|_{S \subseteq [1,k]}$$

Don't you find these eigenvalues somewhat familiar?

MacMahon strikes again

It follows from Moliens Theorem that the Hilbert series of the ring of invariants of the group

$$G_k = \{A_1 \otimes A_2 \otimes \otimes \cdots \otimes A_k\}$$

with $a_1 = e^{i\theta_1}, a_2 = e^{i\theta_2}, \ldots, a_k = e^{i\theta_k}$ is the rational function

$$F_G(q) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{1}{\prod_{S \subseteq [1,k]} \left(1 - q \frac{\prod_{i \in S} a_i}{\prod_{i \notin S} a_i}\right)} d\theta_1 d\theta_2 \cdots d\theta_k$$

But we can easily see that this is same as

$$F_G(q) = \frac{1}{\prod_{S \subseteq [1,k]} \left(1 - q \frac{\prod_{i \in S} a_i}{\prod_{i \notin S} a_i}\right)} \Bigg|_{a_1^0 a_2^0 \cdots a_k^0}$$

This should not be surprising since, for instance the monomials $x_1^{p_1} x_2^{p_2} \cdots x_8^{p_8}$ that are invariant under the group G_3 are precisely those for which

$$(p_1, p_2, \ldots, p_8) \in \mathcal{S}_3$$

The final (Schur function) surprise

In the same vein, we may ask for the Hilbert series of the group

$$H_k = \{A_1 \otimes A_2 \otimes \cdots \otimes A_k : A_i \in SU[2]\}$$

In this case Moliens theorem gives that the Hilbert Series of the corresponding ring of invariants is (here again $a_1 = e^{i\theta_1}, a_2 = e^{i\theta_2}, \ldots, a_k = e^{i\theta_k},$)

$$F_{H_k}(q) = \frac{1}{(\pi)^k} \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{1}{\prod_{S \subseteq [1,k]} \left(1 - q \frac{\prod_{i \in S} a_i}{\prod_{i \notin S} a_i}\right)} \prod_{i=1}^{k} \left(\sin^2(\theta_i)\right) d\theta_1 d\theta_2 \cdots d\theta_k$$

and is easily seen that upon making the substitution

$$\sin^2(\theta_i) = \frac{1 - \cos(2\theta_i)}{2} = \frac{1 - \frac{a_i^2 + 1/a_i^2}{2}}{2}$$

that this is the same as

$$F_{H_k}(q) = \frac{\prod_{i=1}^{k} \left(1 - \frac{1}{a_i^2}\right)}{\prod_{S \subseteq [1,k]} \left(1 - q \frac{\prod_{i \in S} a_i}{\prod_{j \notin S} a_j}\right)} \Bigg|_{a_1^0 a_2^2 \cdots a_k^0}$$

THE SAGA IS TILL CONTINUING

with Diophantine systems

and constant term identities galore!