Quartic Curves and Their Bitangents

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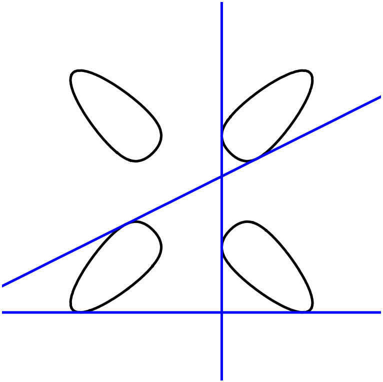
joint work with Daniel Plaumann and Cynthia Vinzant



Control, Optimization, and Functional Analysis: Synergies and Perspectives

Workshop in honor of Bill Helton, San Diego, October 2-4, 2010

Three representations of a quartic curve

We consider smooth curves in $\mathbb{P}^2$ defined by ternary quartics

$$f(x, y, z) \;=\; c_{400}x^4 + c_{310}x^3y + c_{301}x^3z + \cdots + c_{004}z^4,$$

whose 15 coefficients c_{ijk} lie in the field $\mathbb{Q}$ of rational numbers.

Our paper gives *exact algorithms* for computing, over the real numbers $\mathbb{R}$ whenever possible, the two alternate representations

$$f(x, y, z) \quad = \quad \det\bigl(xA + yB + zC\bigr),$$

where A, B, C are symmetric 4×4-matrices, and

$$f(x, y, z) \quad = \quad q_1(x, y, z)^2 + q_2(x, y, z)^2 + q_3(x, y, z)^2,$$

where the $q_i(x, y, z)$ are quadratic forms.

Example: The Edge Quartic

$$25 \cdot (x^4 + y^4 + z^4) \; - \; 34 \cdot (x^2y^2 + x^2z^2 + y^2z^2)$$

$$= \quad \det \begin{pmatrix} 0 & x+2y & 2x+z & y-2z \\ x+2y & 0 & y+2z & -2x+z \\ 2x+z & y+2z & 0 & x-2y \\ y-2z & -2x+z & x-2y & 0 \end{pmatrix}$$

[W.L. Edge: Determinantal representations of $x^4 + y^4 + z^4$, *Math. Proc. Cambridge Phil. Society* **34** (1938) 6–21]

The sum of three squares representation is derived from

$$\begin{pmatrix} x^2 \\ y^2 \\ z^2 \\ xy \\ xz \\ yz \end{pmatrix}^T \begin{pmatrix} 25 & -55/2 & -55/2 & 0 & 0 & 21 \\ -55/2 & 25 & 25 & 0 & 0 & 0 \\ -55/2 & 25 & 25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 21 & -21 & 0 \\ 0 & 0 & 0 & -21 & 21 & 0 \\ 21 & 0 & 0 & 0 & 0 & -84 \end{pmatrix} \begin{pmatrix} x^2 \\ y^2 \\ z^2 \\ xy \\ xz \\ yz \end{pmatrix}$$

Twenty-Eight Bitangents

Theorem (Plücker 1834)

Every smooth quartic curve has precisely 28 bitangent lines.

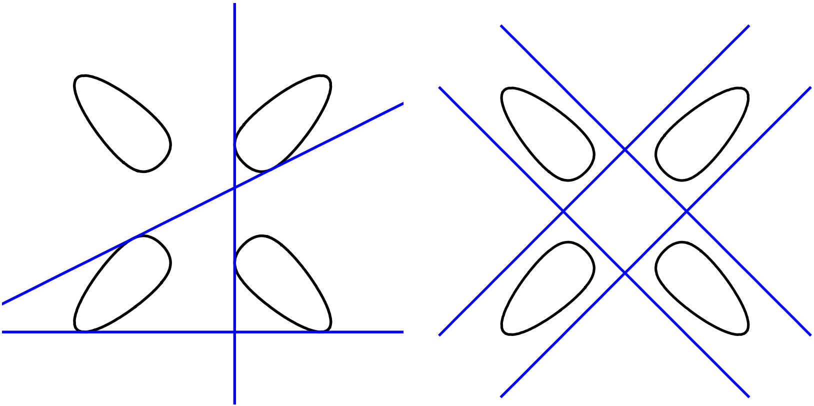


Figure: The Edge quartic and some of its 28 bitangents

Computing the Bitangents Symbolically?

Let K denote the splitting field of the 28 bitangents, that is, the smallest field extension of $\mathbb{Q}$ over which they are defined.

The Galois group $\mathrm{Gal}(K,\mathbb{Q})$ is much smaller than the symmetric group S_{28}. If the coefficients c_{ijk} of $f(x,y,z)$ are general enough, it is the Weyl group of E_7 modulo its center,

$$\mathrm{Gal}(K,\mathbb{Q}) \;\simeq\; W(E_7)/\{\pm 1\} \;\simeq\; \mathrm{Sp}_6(\mathbb{Z}/2\mathbb{Z}).$$

This group has order $\mathbf{8}! \cdot \mathbf{36} = 1\,451\,520$, and it is not solvable.

Some 19th century mathematicians who worked on quartic curves (and abelian functions of genus 3): Aronhold, Cayley, Frobenius, Hesse, Klein, Riemann, Schottky, Steiner, Sturm, Zeuthen, ...

The Real Picture

Theorem (Zeuthen 1873; Klein 1876)

There are six possible topological types for a smooth quartic curve $\mathcal{V}_{\mathbb{R}}(f)$ in the real projective plane. Each of the types corresponds to precisely one connected component in the complement of the discriminant Δ in the 14-dimensional projective space of quartics.

The real curve	Cayley octad	real bitangents	real Gram matrices
4 ovals	8 real points	28	63
3 ovals	6 real points	16	31
2 non-nested ovals	4 real points	8	15
1 oval	2 real points	4	7
2 nested ovals	0 real points	4	15
empty curve	0 real points	4	15

Table: The six types of smooth quartics in the real projective plane.

Convex Algebraic Geometry

Theorem (Hilbert 1888)

A ternary quartic is non-nonnegative if and only if it can be written as a sum of squares of quadrics. Here, three squares always suffice.

Theorem (Coble 1929; Powers-Reznick-Scheiderer-Sottile 2004)

Every smooth quartic has 63 *representations as sums of three squares over* $\mathbb{C}$. *Precisely eight of these are real sums of squares.*

Theorem (Helton-Vinnikov 2007)

Every real quartic can be written as $f(x,y,z) = \det(xA+yB+zC)$ *where* A, B *and* C *are real symmetric* 4×4*-matrices. This net of quadrics contains a matrix* $x_0A + y_0B + z_0C$ *that is positive definite if and only if the real curve* $\mathcal{V}_{\mathbb{R}}(f)$ *consists of two nested ovals.*

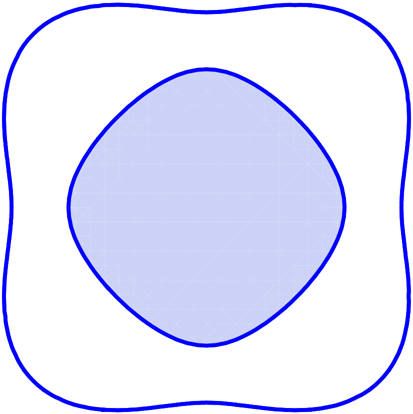
Point: Our algorithms compute these representations (in sage).

Input: A Helton-Vinnikov Curve

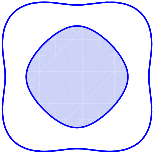
The following quartic defines two nested ovals

$$f(x, y, z) \quad = \quad 2x^4 + y^4 + z^4 - 3x^2y^2 - 3x^2z^2 + y^2z^2.$$

Helton-Vinnikov: The interior convex region is a spectrahedron.



Output: A Linear Matrix Inequality



$$\begin{pmatrix} ux+y & 0 & az & bz \\ 0 & ux-y & cz & dz \\ az & cz & x+y & 0 \\ bz & dz & 0 & x-y \end{pmatrix} \succeq 0$$

The scalars in this matrix are

$$\begin{aligned} u &= \sqrt{2} = 1.414213562373095048.... \\ a &= -0.57464203209296160548032752478263... \\ b &= 1.03492595196395554058118944258225... \\ c &= 0.69970597091301262923557093892256... \\ d &= 0.48004865038024320108560278354988... \end{aligned}$$

Their maximal ideal in $\mathbb{Q}[a, b, c, d, u]$ expresses them in radicals:

$$\begin{aligned} \langle u^2 - 2\,,\ & 256d^8 - 384d^6u + 256d^6 - 384d^4u + 672d^4 - 336d^2u + 448d^2 - 84u + 121, \\ & 23c + 7584d^7u + 10688d^7 - 5872d^5u - 8384d^5 + 1806d^3u + 2452d^3 - 181du - 307d, \\ & 23b + 5760d^7u + 8192d^7 - 4688d^5u - 6512d^5 + 1452d^3u + 2200d^3 - 212du - 232d, \\ & 23a - 1440d^7u - 2048d^7 + 1632d^5u + 2272d^5 - 570d^3u - 872d^3 + 99du + 81d \,\rangle. \end{aligned}$$

An Algorithm for Quartic Curves

Theorem: Let $f \in \mathbb{Q}[x, y, z]$ be a quartic whose curve $\mathcal{V}_{\mathbb{C}}(f)$ is smooth. Suppose $f(x, 0, 0) = x^4$ and $f(x, y, 0)$ is squarefree. Then we can compute a determinantal representation

$$f(x, y, z) \;=\; \det(xI + yD + zR) \tag{1}$$

where I is the identity matrix, D is a diagonal matrix, R is a symmetric matrix, and the entries of D and R are expressed in radicals over the splitting field K. Here, the entries of D and R can be real numbers if and only if $\mathcal{V}_{\mathbb{R}}(f)$ consists of two nested ovals.

Algorithm: We write a Helton-Vinnikov curve as a spectrahedron in radicals over the splitting field K of its 28 bitangents.

Details: The identity (1) specifies a system of 14 polynomial equations in the 14 unknown entries of D and R. This system has $6912 = \mathbf{36} \cdot 24 \cdot 8$ complex solutions. We compute these.

Sums of Squares

A *Gram matrix* for f is a symmetric 6×6 matrix G over $\mathbb{C}$ such that

$$f \;=\; v^T \cdot G \cdot v \quad \text{where} \quad v \;=\; (x^2, y^2, z^2, xy, xz, yz)^T.$$

If $G = H^T H$, where H is an $r \times 6$-matrix and $r = \mathrm{rank}(G)$, then the factorization $f \;=\; (Hv)^T(Hv)$ writes f as the sum of r squares.

No Gram matrix with $r \leq 2$ exists when f is smooth, and there are infinitely many for $r \geq 4$. We compute all Gram matrices for $r = 3$.

Theorem (and Algorithm)

Let $f \in \mathbb{Q}[x, y, z]$ be a smooth quartic and K the splitting field for its 28 bitangents. Then f has precisely 63 Gram matrices G of rank 3. We compute them all using rational arithmetic over K.

Example: An Empty Curve

Let $f = \det(M)$ where M is the matrix

$$\begin{pmatrix} 52x+12y-60z & -26x-6y+30z & 48z & 48y \\ -26x-6y+30z & 26x+6y-30z & -6x+6y-30z & -45x-27y-21z \\ 48z & -6x+6y-30z & -96x & 48x \\ 48y & -45x-27y-21z & 48x & -48x \end{pmatrix}$$

Here $\mathcal{V}_{\mathbb{C}}(f)$ is smooth and $\mathcal{V}_{\mathbb{R}}(f)$ is empty. The corresponding Cayley octad O consists of four pairs of complex conjugates:

$$\begin{pmatrix} i & -i & 0 & 0 & -6+4i & -6-4i & 3+2i & 3-2i \\ 1+i & 1-i & 0 & 0 & -4+4i & -4-4i & 7-i & 7+i \\ 0 & 0 & i & -i & -3+2i & -3-2i & -\frac{86}{39}-\frac{4}{13}i & -\frac{86}{39}+\frac{4}{13}i \\ 0 & 0 & 1+i & 1-i & 1-i & 1+i & \frac{4}{39}-\frac{20}{39}i & \frac{4}{39}+\frac{20}{39}i \end{pmatrix}$$

The bitangent matrix $O^T M O$ is defined over $K = \mathbb{Q}(i)$, and hence so are all 63 rank-3 Gram matrices. Precisely 15 of these are real:

$$f = 288 \begin{pmatrix} x^2 \\ y^2 \\ z^2 \\ xy \\ xz \\ yz \end{pmatrix}^T \begin{pmatrix} 45500 & 3102 & -9861 & 5718 & -9246 & 4956 \\ 3102 & 288 & -747 & 882 & -18 & -144 \\ -9861 & -747 & 3528 & -864 & -1170 & -504 \\ 5718 & 882 & -864 & 4440 & 1104 & -2412 \\ -9246 & -18 & -1170 & 1104 & 11814 & -5058 \\ 4956 & -144 & -504 & -2412 & -5058 & 3582 \end{pmatrix} \begin{pmatrix} x^2 \\ y^2 \\ z^2 \\ xy \\ xz \\ yz \end{pmatrix}$$

The Gram Spectrahedron

of a quartic f is the set of its positive semidefinite Gram matrices.

This spectrahedron is the intersection of the cone of positive semidefinite 6×6-matrices with a 6-dimensional affine subspace:

$$\mathrm{Gram}(f) = \left\{\lambda \in \mathbb{R}^6 : \begin{bmatrix} c_{400} & \lambda_1 & \lambda_2 & \frac{1}{2}c_{310} & \frac{1}{2}c_{301} & \lambda_4 \\ \lambda_1 & c_{040} & \lambda_3 & \frac{1}{2}c_{130} & \lambda_5 & \frac{1}{2}c_{031} \\ \lambda_2 & \lambda_3 & c_{004} & \lambda_6 & \frac{1}{2}c_{103} & \frac{1}{2}c_{013} \\ \frac{1}{2}c_{310} & \frac{1}{2}c_{130} & \lambda_6 & c_{220}-2\lambda_1 & \frac{1}{2}c_{211}-\lambda_4 & \frac{1}{2}c_{121}-\lambda_5 \\ \frac{1}{2}c_{301} & \lambda_5 & \frac{1}{2}c_{103} & \frac{1}{2}c_{211}-\lambda_4 & c_{202}-2\lambda_2 & \frac{1}{2}c_{112}-\lambda_6 \\ \lambda_4 & \frac{1}{2}c_{031} & \frac{1}{2}c_{013} & \frac{1}{2}c_{121}-\lambda_5 & \frac{1}{2}c_{112}-\lambda_6 & c_{022}-2\lambda_3 \end{bmatrix} \succeq 0 \right\}$$

Hilbert: $\mathrm{Gram}(f)$ is non-empty if and only if f is non-negative.

The *Steiner graph* of the Gram spectrahedron is the graph on the eight vertices of rank 3 whose edges represent edges of $\mathrm{Gram}(f)$.

Theorem

The Steiner graph of the Gram spectrahedron of a general positive quartic f is the disjoint union $K_4 \sqcup K_4$ of two complete graphs. The relative interiors of these edges consist of rank-5 matrices.

The Bigger Picture

- Plane quartics are canonical curves of genus 3
- The 28 bitangents are the odd theta characteristics
- The 36 Cayley octads are the even theta characteristics
- The 63 Steiner complexes and rank-3 Gram matrices correspond to the 2-torsion points on the Jacobian
- 3-phase solutions of the Kadomtsev-Petviashvili equation
- Period matrices to theta functions to plane quartics (and back)

	Classical	**Tropical**
Concrete	Today's talk on plane quartics	Tropical quartics tropical bitangents
Abstract	Abelian varieties moduli of curves	The tropical Torelli map

How to manipulate genus 3 curves over a field such as $K = \overline{\mathbb{Q}(\epsilon)}$?

Trichotomy for Nets of Quadrics in $\mathbb{P}^3$

Proposition (Calabi 1964; "S-Lemma")

Let $\mathcal{P}$ be a pencil of homogeneous quadrics in n unknowns. Then precisely one of the following two cases holds:

(a) *The quadrics in $\mathcal{P}$ have a common point in $\mathbb{P}^{n-1}(\mathbb{R})$.*

(b) *The pencil $\mathcal{P}$ contains a positive definite quadric.*

Theorem

Let $\mathcal{N}$ be a net of homogeneous quadrics in four unknowns with $\Delta(\mathcal{N}) \neq 0$. Then precisely one of the following three cases holds:

(a) *The quadrics in $\mathcal{N}$ have a common point in $\mathbb{P}^3(\mathbb{R})$.*

(b) *The net $\mathcal{N}$ contains a positive definite quadric.*

(c) *The 4×4-determinant restricted to $\mathcal{N}$ is a sum of squares.*

Proof.

The net $\mathcal{N}$ defines a Cayley octad O and ternary quartic f. Either O has a real point, or $\mathcal{V}_{\mathbb{R}}(f)$ is Helton-Vinnikov, or $\mathcal{V}_{\mathbb{R}}(f) = \emptyset$. $\square$