# Positive level, negative level and level zero 

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#### Abstract

This is a survey on the combinatorics and geometry of integrable representations of quantum affine algebras with a particular focus on level 0 . Pictures and examples are included to illustrate the affine Weyl group orbits, crystal graphs and Macdonald polynomials that provide detailed understanding of the structure of the extremal weight modules and their characters. The final section surveys the alcove walk method of working with the positive level, negative level and level zero affine flag varieties and describes the corresponding actions of the affine Hecke algebra.


Key words - affine flag varieties, integrable representations, quantum affine algebras】

## 0 Introduction

This paper is about positive level, negative level and level 0 . It was motivated by the striking result of [KNS17, which establishes a Pieri-Chevalley formula for the K-theory of the semiinfinite (level 0) affine flag variety. This made us want to learn more about the level 0 integrable modules of quantum affine algebras. Our trek brought us face to face with a huge literature, including important contributions from Drinfeld, Kashiwara, Beck, Chari, Nakajima, Lenart-Schilling-Shimozono, Cherednik-Orr, Naito-Sagaki, Feigin-Makedonskyi, Kato, their coauthors and many others. It is a beautiful theory and we count ourselves lucky to have been drawn into it.

The main point is that the integrable modules for quantum affine algebras $\mathbf{U}$ naturally partition themselves into families: positive level, negative level and level zero. Their structure is shadowed by the orbits of the affine Weyl group on the lattice of weights for the affine Lie algebra, which take the shape of a concave up paraboloid at positive level, a concave down paraboloid at negative level and a tube at level 0 . These integrable modules have crystal bases which provide detailed control of their characters. At level 0 the characters are (up to a factor similar to a Weyl denominator) Macdonald polynomials specialised at $t=0$. The next amazing feature is that there are Borel-Weil-Bott theorems for each case: positive level, negative level and level 0 , where, respectively, the appropriate geometry is a positive level (thin) affine flag variety, a negative level (thick) affine flag variety, and a level zero (semi-infinite) affine flag variety.

This paper is a survey of the general picture of positive level, negative level, and level zero, in the context of the combinatorics of affine Weyl groups and crystals, of the representation theory of integrable modules for quantum affine algebras, and of the geometry of affine flag varieties. In recent years, the picture has become more and more rich and taken clearer focus. We hope that this paper will help to bring this story to a wider audience by providing pictures and some explicit small examples for $\widehat{\mathfrak{s l}}_{2}$ and $\widehat{\mathfrak{s l}}_{3}$.

[^0]
### 0.1 Orbits of the affine Weyl group $W$ action on $\mathfrak{h}^{*}$

For $\widehat{\mathfrak{s l}}_{2}$ the vector space $\mathfrak{h}^{*}$ is three dimensional with basis $\left\{\delta, \omega_{1}, \Lambda_{0}\right\}$, and the orbits of the action of the affine Weyl group $W^{\text {ad }}$ on $\mathfrak{h}^{*}$ on different levels look like


Although informative, the picture above is misleading as it is a two dimensional projection $(\bmod \delta)$ of what is actually going on. The $W^{\text {ad }}$-action fixes the level (the coefficient of $\Lambda_{0}$ ) but it actually changes the $\delta$ coordinate significantly. Let us look at the orbits in $\delta$ and $\omega_{1}$ coordinates (i.e. $\bmod \Lambda_{0}$ ).


The positive level orbit $W^{\text {ad }}\left(\omega_{1}+2 \Lambda_{0}\right) \bmod \Lambda_{0}$
When the level (coefficient of $\Lambda_{0}$ ) is large the parabola is wide, and it gets tighter as the level decreases.


The parabolas bounding the orbits $W^{\text {ad }}\left(\omega_{1}+2 \Lambda_{0}\right)$ and $W^{\text {ad }}\left(\omega_{1}+\Lambda_{0}\right) \bmod \Lambda_{0}$ At level 0 , the parabola pops and becomes two straight lines.


The level 0 orbit $W^{\text {ad }} \omega_{1} \bmod \Lambda_{0}$
At negative level the parabola forms again, but this time facing the opposite way, and getting wider as the level gets more and more negative.


The orbit $W^{\text {ad }}\left(-\omega_{1}-2 \Lambda_{0}\right)($ negative level $) \bmod \Lambda_{0}$

The three different Bruhat orders on the affine Weyl group are visible on the $W^{\text {ad }}$-orbits:

$$
\begin{array}{ll}
v \leq w & \text { if } v\left(\omega_{1}+\Lambda_{0}\right) \text { is higher than } w\left(\omega_{1}+\Lambda_{0}\right) \text { in } W^{\text {ad }}\left(\omega_{1}+\Lambda_{0}\right) \\
v \leq 0 w & \text { if } v \omega_{1} \text { is higher than } w\left(\omega_{1}+0 \Lambda_{0}\right) \text { in } W^{\text {ad }}\left(\omega_{1}+0 \Lambda_{0}\right) . \\
v \leq w & \text { if } v\left(-\omega_{1}-\Lambda_{0}\right) \text { is higher than } w\left(-\omega_{1}-\Lambda_{0}\right) \text { in } W^{\text {ad }}\left(-\omega_{1}-\Lambda_{0}\right) .
\end{array}
$$

The definitions of the Bruhat orders on $W^{\text {ad }}$ and their relation to the closure order for Schubert cells in affine flag varieties is made precise in Section 1.3. Indicative relations illustrating the from of the Hasse diagrams of the positive level, negative level, and level zero Bruhat orders for the Weyl group of $\widehat{\mathfrak{s l}}_{3}$ are pictured in Plate A (there are additional relations which are not displayed in the pictures - in an effort to make the periodicity pattern easily visible).

For the case of $\mathfrak{g}=\widehat{\mathfrak{s l}}_{3}$ the affine Weyl group orbits take a similar form, with the points sitting on a downward paraboloid at positive level, on an upward paraboloid at negative level and with the paraboloid popping and becoming a tube at level 0 (for an example tube see the picture of $B\left(\omega_{1}+\omega_{2}\right)$ for $\widehat{\mathfrak{s l}}_{3}$ in Plate D).


Positive level orbit $W^{\text {ad }}\left(\omega_{1}+\omega_{2}+2 \Lambda_{0}\right)$ for $\widehat{\mathfrak{s l}}_{3}$ Negative level orbit $W^{\text {ad }}\left(-\omega_{1}-\omega_{2}-2 \Lambda_{0}\right)$ for $\widehat{\mathfrak{s l}}_{3}$

### 0.2 Extremal weight modules $L(\Lambda)$ and their crystals $B(\Lambda)$

For the affine Lie algebra $\mathfrak{g}=\widehat{\mathfrak{s l}}_{2}$ the weights of integrable $\mathfrak{g}$-modules always lie in the set

$$
\mathfrak{h}_{\mathbb{Z}}^{*}=\mathbb{C} \delta+\mathbb{Z} \omega_{1}+\mathbb{Z} \Lambda_{0} .
$$

A set of representatives for the orbits of the action of $W^{\text {ad }}$ on $\mathfrak{h}_{\mathbb{Z}}^{*}$ is

$$
\left(\mathfrak{h}^{*}\right)_{\text {int }}=\left(\mathfrak{h}^{*}\right)_{\text {int }}^{+} \cup\left(\mathfrak{h}^{*}\right)_{\text {int }}^{0} \cup\left(\mathfrak{h}^{*}\right)_{\text {int }}^{-}, \quad \text { where }, \begin{aligned}
& \left(\mathfrak{h}^{*}\right)_{\text {int }}^{0}=\left\{a \delta+n \omega_{1} \in \mathfrak{h}_{\mathbb{Z}}^{*} \mid 0 \leq n\right\}, \\
& \left(\mathfrak{h}^{*}\right)_{\text {int }}^{+}=\left\{a \delta+m \omega_{1}+n \Lambda_{0} \in \mathfrak{h}_{\mathbb{Z}}^{*} \mid 0 \leq m \leq n\right\}, \\
& \left(\mathfrak{h}^{*}\right)_{\text {int }}^{-}=\left\{a \delta-m \omega_{1}-n \Lambda_{0} \in \mathfrak{h}_{\mathbb{Z}}^{*} \mid 0 \leq m \leq n\right\} .
\end{aligned}
$$

These sets are illustrated $(\bmod \delta)$ below.
For each of the $\Lambda \in\left(\mathfrak{h}^{*}\right)_{\text {int }}$, there is a (universal) integrable extremal weight module $L(\Lambda)$, which is highest weight if $\Lambda \in\left(\mathfrak{h}^{*}\right)_{\text {int }}^{+}$, is lowest weight (and not highest weight) if $\Lambda \in\left(\mathfrak{h}^{*}\right)_{\text {int }}^{-}$, and which is neither highest or lowest weight when $\Lambda \in\left(\mathfrak{h}^{*}\right)_{\text {int }}^{0}$. The module $L(\Lambda)$ has a crystal $B(\Lambda)$.

At positive level and negative levels the crystals $B(\Lambda)$ are connected, but the crystal $B(\Lambda)$ is usually not connected in level 0 . The connected components and their structure are known
explicitly from a combination of results of Kashiwara, Beck-Chari-Pressley, Nakajima, BeckNakajima, Fourier-Littelmann, Ion and others. These results are collected in Theorem 2.4 and equation (2.12) expresses the characters of the $L(\Lambda)$ in terms of Macdonald polynomials specialised at $t=0$.


### 0.3 Affine flag varieties $G / I^{+}, G / I^{0}$ and $G / I^{-}$

There are three kinds of affine flag varieties for the loop group $G=G(\mathbb{C}((\epsilon)))$ : the positive level (thin) affine flag variety $G / I^{+}$, the negative level (thick) affine flag variety $G / I^{-}$and the level 0 (semi-infinite) affine flag variety $G / I^{0}$. A combination of results of Kumar, Mathieu, Kashiwara, Kashiwara-Tanisaki, Kashiwara-Shimozono, Varagnolo-Vasserot, Lusztig and Braverman-Finkelberg have made it clear that there is a Borel-Weil-Bott theorem for each of these:

$$
\begin{gathered}
H^{0}\left(G / I^{+}, \mathcal{L}_{\Lambda}\right) \cong L(\Lambda), \quad \text { for positive level } \Lambda \in\left(\mathfrak{h}^{*}\right)_{\text {int }}^{+}, \\
H^{0}\left(G / I^{0}, \mathcal{L}_{\lambda}\right) \cong L(\lambda), \quad \text { for level zero } \lambda \in\left(\mathfrak{h}^{*}\right)_{\text {int }}^{0}, \\
H^{0}\left(G / I^{-}, \mathcal{L}_{-\Lambda}\right) \cong L(-\Lambda), \quad \text { for negative }-\Lambda \in\left(\mathfrak{h}^{*}\right)_{\text {int }}^{-} .
\end{gathered}
$$

These Borel-Weil-Bott theorems tightly connect the representation theory with the geometry. In all essential aspects the combinatorics of the positive level affine flag variety and the loop Grassmannian generalizes to the negative level and the level 0 affine flag varieties.

Section 5 extends the results of PRS and displays the alcove walk combinatorics for each of the three cases (positive level, negative level and level 0 ) in parallel. In addition it describes the method for deriving the natural affine Hecke algebra actions on the function spaces $C\left(G / I^{+}\right)$, $C\left(I^{+} \backslash G / I^{+}\right), C\left(I^{+} \backslash G / I^{+}\right)$and $C\left(I^{-} \backslash G / I^{+}\right)$.

### 0.4 References, technicalities and acknowledgements

Section 1.1 introduces the affine Lie algebra and the homogeneous Heisenberg subalgebra following [Kac and Section 1.2 gives explicit matrices describing the actions of the affine Weyl
group $W^{\text {ad }}$ on the affine Cartan $\mathfrak{h}$ and its dual $\mathfrak{h}^{*}$. Section 1.3 defines the Bruhat orders on the affine Weyl group and explains their relation to the corresponding affine flag varieties following Kum and LuICM, $\S 7$ and 11]. Sections 1.4 and 1.5 introduce the affine braid groups and Macdonald polynomials following [RY11. Section 1.6 treats the specializations of (nonsymmetric) Macdonald polynomials at $t=0, t=\infty, q=0$ and $q=\infty$ and reviews the result of Ion Ion01 that relates Macdonald polynomials at $t=0$ to Demazure operators.

Section 2 follows BN02, B94, and BCP98, introducing the quantum affine algebra U, the conversion to its loop presentation, the PBW-type elements and the quantum homogeneous Heisenberg subalgebra. Section 2.2 defines integrable U-modules and Section 2.3 introduces the extremal weight modules $L(\Lambda)$ following [Kas94, (8.2.2)] and [Kas02, §3.1]. Section [2.4 reviews the Demazure character formulas for extremal weight modules. Section 2.5 discusses the loop presentation of the level 0 extremal weight modules and the fact that these coincide with the universal standard modules of [Nak99] and the global Weyl modules of [P01]. Letting $\mathbf{U}^{\prime}$ be $\mathbf{U}$ without the element $D$, Section 2.6 explains how to shrink the extremal weight module to a local Weyl module and how this provides a classification of finite dimensional simple $\mathbf{U}^{\prime}$-modules by Drinfeld polynomials.

We have made a concerted effort to make a useful survey. In order to simplify the exposition we have brushed under the rug a number of technicalities which are wisely ignored when one learns the subject (particularly (a) the difference between simply laced cases and the general case requires proper attention to the diagonal matrix which symmetrizes the affine Cartan matrix [Kac, (2.1.1)] causing the constants $d_{i}$ which pepper the quantum group literature and (b) the machinations necessary for allowing multiple parameters $t_{i}^{\frac{1}{2}}$ in Macdonald polynomials). The reader who needs to sort out these features is advised to drink a strong double espresso to optimise clear thinking, consult the references (particularly [BN02] and [RY11]) and not trust our exposition. Perhaps in the future a more complete (probably book length) version of this paper will be completed which allows us to attend more carefully to these nuances and include more detailed proofs. Having made this point, we can say that a careful effort has been made to provide specific references to the literature at every step and we hope that this will be useful for the reader that wishes to go further.

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Finally, it is a pleasure to dedicate this paper to Ian Macdonald and Alun Morris who forcefully led the way to the kinds explorations of affine combinatorial representation theory that are happening these days.

## PLATE A: Bruhat orders on the affine Weyl group (partial, indicative, relations)


postive level Bruhat order for $\widehat{\mathfrak{s l}}_{2}$
1 is minimal

level zero Bruhat order for $\widehat{\mathfrak{s l}}_{2}$ translation invariant

negative level Bruhat order for $\widehat{\mathfrak{s}}_{2}$ 1 is maximal

postive level Bruhat order for $\widehat{\mathfrak{s l}}_{3}$
1 is minimal

level zero Bruhat order for $\widehat{\mathfrak{s l}}_{3}$ translation invariant

negative level Bruhat order for $\widehat{\mathfrak{s}}_{3}$
1 is maximal

PLATE B: Pictures of $B\left(\omega_{1}+\Lambda_{0}\right), B\left(\omega_{1}+0 \Lambda_{0}\right)$ and $B\left(-\omega_{1}-\Lambda_{0}\right)$ for $\widehat{\mathfrak{s}}_{2}$


Initial portion of the crystal graph of $B\left(\omega_{1}+\Lambda_{0}\right)$ for $\widehat{\mathfrak{s}}_{2}$


Final portion of the crystal graph of $B\left(-\omega_{1}-\Lambda_{0}\right)$ for $\widehat{\mathfrak{s}}_{2}$


Middle portion of the crystal graph of $B\left(\omega_{1}+0 \Lambda_{0}\right)$ for $\mathfrak{g}=\widehat{\mathfrak{s l}}{ }_{2}$

PLATE C: Pictures for $B\left(2 \omega_{1}\right)$. Representative paths from the (first five) connected components of $B\left(2 \omega_{1}\right)$ are

and the paths in $B\left(2 \omega_{1}\right)_{0} \subseteq B\left(\omega_{1}\right) \otimes B\left(\omega_{1}\right)$ are


Paths in $B\left(2 \omega_{1}\right)_{0}$


The crystal graph of $B\left(2 \omega_{1}\right)_{0}$

PLATE D: Pictures and characters of $B\left(\omega_{1}+\omega_{2}\right)$ for $\widehat{\mathfrak{s}}_{3}$. The colour red indicates change in the $\delta$-coordinate.


$$
\begin{aligned}
\operatorname{gchar}\left(B^{\operatorname{fin}}\left(\omega_{1}+\omega_{2}\right)\right)= & X^{\rho}+X^{\alpha_{1}}+X^{\alpha_{2}}+X^{-\alpha_{1}} \\
& +X^{-\alpha_{2}}+X^{-\rho}+2+q^{-1}
\end{aligned}
$$



At $t=0$ and $t=\infty$ the normalized nonsymmetric Macdonald polynomials $\tilde{E}_{s_{1} s_{2} s_{1} \rho}(q, t)$ are

$$
\begin{aligned}
\tilde{E}_{s_{1} s_{2} s_{1} \rho}(q, 0) & =X^{s_{1} s_{2} s_{1} \rho}+X^{s_{1} s_{2} \rho}+X^{s_{2} s_{1} \rho}+X^{s_{2} \rho}+X^{s_{1} \rho}+X^{\rho}+2+q, \\
\tilde{E}_{s_{1} s_{2} s_{1} \rho}(q, \infty) & =X^{s_{1} s_{2} s_{1} \rho}+q^{-1}\left(X^{s_{1} s_{2} \rho}+X^{s_{2} s_{1} \rho}\right)+q^{-2}\left(X^{s_{1} \rho}+X^{s_{2} \rho}+X^{\rho}\right)+2 q^{-2}+q^{-1} .
\end{aligned}
$$

Letting $q=e^{-\delta}\left(\right.$ as in [Kac, (12.1.9)]), the Demazure module $L\left(\omega_{1}+\omega_{2}\right)_{\leq s_{1} s_{2} s_{1}}$ has character

$$
\begin{aligned}
\operatorname{char}\left(L\left(\omega_{1}+\omega_{2}\right)_{\leq s_{1} s_{2} s_{1}}\right) & =\frac{1}{\left(1-q^{-1}\right)^{2}} \tilde{E}_{s_{1} s_{2} s_{1} \rho}\left(q^{-1}, 0\right) \quad \text { and } \\
\operatorname{char}\left(L\left(\omega_{1}+\omega_{2}\right)\right) & =0_{q} 0_{q} \tilde{E}_{s_{1} s_{2} s_{1} \rho}\left(q^{-1}, 0\right)=0_{q} 0_{q} \tilde{E}_{s_{1} s_{2} s_{1} \rho}\left(q^{-1}, \infty\right)
\end{aligned}
$$

where $0_{q}=\cdots+q^{-3}+q^{-2}+q^{-1}+1+q+q^{2}+\cdots$ as in Remark 2.2,
Remark 0.1. The expansion
$\frac{1}{\left(1-q^{-1}\right)^{2}}=\left(1+q^{-1}+q^{-2}+\cdots\right)\left(1+q^{-1}+q^{-2}+\cdots\right)=1+2 q^{-1}+3 q^{-2}+4 q^{-3}+5 q^{-4}+\cdots$
show that the sizes of the weight spaces of $L\left(\omega_{1}+\omega_{2}\right)_{\leq s_{1} s_{2} s_{1}}$ are growing as $\delta$ increases. Similarly, in the character formula of $L\left(\omega_{1}+\omega_{2}\right)$, the factor $0_{q} 0_{q}$ has coefficient of $q^{n}$ equal to $\operatorname{Card}\left(\left\{\left(k_{1}, k_{2}\right) \in \mathbb{Z}^{2} \mid k_{1}+k_{2}=n\right\}\right)=\infty$. This shows that every weight space of the extremal weight module $L\left(\omega_{1}+\omega_{2}\right)$ is infinite dimensional.

## 1 Affine Weyl groups, braid groups and Macdonald polynomials

### 1.1 The affine Lie algebra $\mathfrak{g}$

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and fix a Cartan subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$ and a symmetric, ad-invariant, nondegenerate, bilinear form $\langle\rangle:, \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$. The affine KacMoody algebra is

$$
\begin{gather*}
\mathfrak{g}=\left(\bigoplus_{k \in \mathbb{Z}} \mathfrak{g} \epsilon^{k}\right) \oplus \mathbb{C} K \oplus \mathbb{C} d, \quad \text { with bracket given by } \quad\left[K, x \epsilon^{k}\right]=0, \quad[K, d]=0, \\
{\left[d, x \epsilon^{k}\right]=k x \epsilon^{k}, \quad \text { and } \quad\left[x \epsilon^{k}, y \epsilon^{\ell}\right]=[x, y] \epsilon^{k+\ell}+k \delta_{k,-\ell}\langle x, y\rangle K,} \tag{1.1}
\end{gather*}
$$

for $x, y \in \mathfrak{g}$ and $k, \ell \in \mathbb{Z}$ (see [Kac, (7.2.2)]). Let $\theta$ be the highest root of $\mathfrak{g}$ (the highest weight of the adjoint representation) and define

$$
e_{0}=f_{\theta} \epsilon, \quad f_{0}=e_{\theta} \epsilon^{-1}, \quad \text { and } \quad h_{0}=\left[e_{0}, f_{0}\right]=-h_{\theta}+K .
$$

The miracle is that $\mathfrak{g}$ is a Kac-Moody Lie algebra with Chevalley generators

$$
\begin{equation*}
e_{0}, \ldots, e_{n}, h_{0}, \ldots h_{n}, d, f_{0}, \ldots f_{n}, \quad \text { which satisfy Serre relations. } \tag{1.2}
\end{equation*}
$$

Because of (1.2), $\mathfrak{g}$ has a corresponding quantum enveloping algebra $\mathbf{U}=U_{q} \mathfrak{g}$.
The Cartan subalgebra of $\mathfrak{g}$ is

$$
\mathfrak{h}=\mathfrak{a} \oplus \mathbb{C} K \oplus \mathbb{C} d, \quad \text { where } \mathfrak{a} \subseteq \mathfrak{g} \text { is the Cartan subalgebra of } \mathfrak{g} .
$$

Let $\stackrel{\circ}{R}^{+}$be the set of positive roots for $\mathfrak{g}$. For $\alpha \in \grave{R}^{+}, k \in \mathbb{Z}, \ell \in \mathbb{Z}_{\neq 0}$ and $i \in\{1, \ldots, n\}$, let

$$
x_{\alpha+k \delta}=e_{\alpha} \epsilon^{k}, \quad x_{-\alpha+k \delta}=f_{\alpha} \epsilon^{k}, \quad h_{i, \ell}=h_{i} \epsilon^{\ell} .
$$

The homogeneous Heisenberg subalgebra (see [Kac, $\S 8.4$ and $\S 14.8]$ ) is

$$
\begin{equation*}
\mathbb{C} K \oplus \mathfrak{a}\left[\epsilon, \epsilon^{-1}\right] \quad \text { with } \quad\left[h_{i} \epsilon^{k}, h_{j} \epsilon^{\ell}\right]=k \delta_{k,-\ell} \frac{2}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle} \alpha_{i}\left(h_{j}\right) K, \tag{1.3}
\end{equation*}
$$

and $\mathfrak{a}[\epsilon]$ is a commutative Lie algebra with basis $\left\{h_{i} \epsilon^{k} \mid i \in\{1, \ldots, n\}, k \in \mathbb{Z}_{\geq 0}\right\}$.

### 1.2 The affine Weyl group $W^{\text {ad }}$ and its action on $\mathfrak{h}^{*}$ and $\mathfrak{h}$

Let $\delta, \omega_{1}, \ldots, \omega_{n}, \Lambda_{0}$ be the basis in $\mathfrak{h}^{*}$ which is the dual basis to the basis $d, h_{1}, \ldots, h_{n}, K$ of $\mathfrak{h}$. The affine Weyl group $W^{\text {ad }}$ is the subgroup of $G L\left(\mathfrak{h}^{*}\right)$ generated by the linear transformations $s_{0}, s_{1}, \ldots, s_{n}$ which, in the basis $\delta, \omega_{1}, \ldots, \omega_{n}, \Lambda_{0}$, are

$$
s_{i}=\left(\begin{array}{cccccccc}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0  \tag{1.4}\\
0 & 1 & \cdots & 0 & -\alpha_{i}\left(h_{1}\right) & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & -\alpha_{i}\left(h_{2}\right) & 0 & \cdots & 0 \\
\vdots & & & & \vdots & & & \vdots \\
0 & 0 & \cdots & 1 & -\alpha_{i}\left(h_{i-1}\right) & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & -\alpha_{i}\left(h_{i+1}\right) & 1 & \cdots & 0 \\
\vdots & & & & \vdots & & & \vdots \\
0 & 0 & \cdots & 0 & -\alpha_{i}\left(h_{n}\right) & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1
\end{array}\right), \quad \text { for } i \in\{1, \ldots, n\}
$$

and, writing $\theta=a_{1} \alpha_{1}+\cdots+a_{n} \alpha_{n}$ and $h_{\theta}=\left[e_{\theta}, f_{\theta}\right]=a_{1}^{\vee} h_{1}+\cdots+a_{n}^{\vee} h_{n}$,

$$
s_{0}=\left(\begin{array}{cccccc}
1 & a_{1}^{\vee} & a_{2}^{\vee} & \cdots & a_{n}^{\vee} & -1  \tag{1.5}\\
0 & 1-a_{1} a_{1}^{\vee} & -a_{1} a_{2}^{\vee} & \cdots & -a_{1} a_{n}^{\vee} & a_{1} \\
0 & -a_{2} a_{1}^{\vee} & 1-a_{2} a_{2}^{\vee} & \cdots & -a_{2} a_{n}^{\vee} & a_{2} \\
\vdots & & & \vdots & & \vdots \\
0 & -a_{n} a_{1}^{\vee} & -a_{n} a_{2}^{\vee} & \cdots & 1-a_{n} a_{n}^{\vee} & a_{n} \\
0 & 0 & 0 & \cdots & 0 & 1
\end{array}\right)
$$

Let $\mathfrak{a}_{\mathbb{R}}^{*}=\mathbb{R}$-span $\left\{\alpha_{1}, \ldots, \alpha_{n}\right\}$. An alcove is a fundamental region for the action of $W^{\text {ad }}$ on $\left(\mathbb{R} \delta+\mathfrak{a}_{\mathbb{R}}^{*}+\Lambda_{0}\right) / \mathbb{R} \delta$. As explained (for example) in [Ra06] and [RY11], there is a bijection

$$
\begin{array}{ccc}
W^{\text {ad }} & \longleftrightarrow \text { alcoves }\} \\
1 & \longmapsto\left\{x+\Lambda_{0} \in \mathfrak{a}_{\mathbb{R}}^{*}+\Lambda_{0} \mid x\left(h_{i}\right)>0 \text { for } i \in\{0, \ldots, n\}\right\} \tag{1.6}
\end{array}
$$

Let $\mathfrak{a}_{\mathbb{Z}}^{\text {ad }}=\mathbb{Z}$-span $\left\{h_{1}, \ldots, h_{n}\right\}$. The finite Weyl group $W_{\text {fin }}$ is generated by $s_{1}, \ldots, s_{n}$. The translation presentation of the affine Weyl group is

$$
W^{\text {ad }}=\mathfrak{a}_{\mathbb{Z}}^{\text {ad }} \rtimes W_{\text {fin }}=\left\{t_{\mu^{\vee}} u \mid \mu^{\vee} \in \mathfrak{a}_{\mathbb{Z}}^{\text {ad }}, u \in W_{\text {fin }}\right\} \quad \text { with } \quad \begin{align*}
& t_{\mu \vee} t_{\nu^{\vee}}=t_{\mu \vee+\nu^{\vee}} \text { and }  \tag{1.7}\\
& u t_{\mu \vee}=t_{u \mu \vee},
\end{align*}
$$

for $\mu^{\vee}, \nu^{\vee} \in \mathfrak{a}_{\mathbb{Z}}^{\text {ad }}$ and $u \in W_{\text {fin }}$.
Let $\alpha_{i}^{\vee}$ be the image of $h_{i}$ under the isomorphism $\mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^{*}$ coming from the nondegenerate bilinear form on $\mathfrak{a}$ which is the restriction of the nondegerate bilinear form on $\mathfrak{g}$. In matrix form with respect to the basis $\delta, \omega_{1}, \ldots, \omega_{n}, \Lambda_{0}$ of $\mathfrak{h}^{*}$ the action of $W^{\text {ad }}$ on $\mathfrak{h}^{*}$ is given by
so that $-\frac{1}{2}\left\langle\mu^{\vee}, \mu^{\vee}\right\rangle=-\frac{1}{2}\left(\mu_{1}^{\vee} k_{1}+\cdots \mu_{n}^{\vee} k_{n}\right)$. (In (1.8) $d_{1}, \ldots, d_{n}$ are the minimal positive integers such that the product of the diagonal matrix $\operatorname{diag}\left(d_{1}, \ldots, d_{n}\right)$ with the Cartan matrix is symmetric, see [Kac, (2.1.1)].)

The basis $\left\{d, h_{1}, \ldots, h_{n}, K\right\}$ of $\mathfrak{h}$ is the dual basis to the basis $\left\{\delta, \omega_{1}, \ldots, \omega_{n}, \Lambda_{0}\right\}$ of $\mathfrak{h} *$. Using the $W^{\text {ad }}$-action on $\mathfrak{h}$ given by

$$
s_{i} \mu^{\vee}=\mu^{\vee}-\alpha_{i}\left(\mu^{\vee}\right) h_{i}, \quad \text { for } i \in\{0, \ldots, n\} \text { and } \mu^{\vee} \in \mathfrak{h},
$$

the matrices for the action of $s_{0}, s_{1}, \ldots, s_{n}$ on $\mathfrak{h}$, in the basis $\left\{d, h_{1}, \ldots, h_{n}, K\right\}$, are the transposes of the matrices in (1.4) and (1.5).

### 1.3 The positive level, negative level and level 0 Bruhat orders on $W^{\text {ad }}$

In the framework of Section 5, where $G=\dot{G}(\mathbb{C}((\epsilon)))$ is the loop group, the closure orders for the Schubert cells in the positive level (thin) affine flag variety $G / I^{+}$, the negative level (thick)
affine flag variety $G / I^{-}$, and the level 0 (semi-infinite) affine flag variety $G / I^{0}$ give partial orders on the affine Weyl group $W^{\text {ad }}$ :

$$
\overline{I^{+} w I^{+}}=\bigsqcup_{x \leq+w} I^{+} x I^{+}, \quad \overline{I^{+} w I^{0}}=\bigsqcup_{x \leq 0} I^{+} x I^{0}, \quad \overline{I^{+} w I^{-}}=\bigsqcup_{x \leq w} I^{+} x I^{-}
$$

These orders can be described combinatorially as follows.
An element $w \in W^{\text {ad }}$ is dominant if

$$
w\left(\rho+\Lambda_{0}\right) \in \mathbb{R}_{\geq 0}-\operatorname{span}\left\{\omega_{1}, \ldots, \omega_{n}\right\}+\Lambda_{0}, \quad \text { where } \quad \rho=\omega_{1}+\cdots+\omega_{n}
$$

In the identification (1.6) of elements of $W^{\text {ad }}$ with alcoves, the dominant elements of $W^{\text {ad }}$ are the alcoves in the dominant Weyl chamber.

Let $x, w \in W^{\text {ad }}$ and let $w=s_{i_{1}} \cdots s_{i_{\ell}}$ be a reduced word for $w$ in the generators $s_{0}, \ldots, s_{n}$. The positive level Bruhat order on $W^{\text {ad }}$ is defined by

$$
x \leq \pm \quad \text { if } x \text { has a reduced word which is a subword of } w=s_{i_{1}} \cdots s_{i_{\ell}}
$$

The negative level Bruhat order on $W^{\text {ad }}$ is defined by $\quad x \leq w$ if $x \pm w$. The level 0 Bruhat order on $W^{\text {ad }}$ is determined by
(a) $\leq 0$ for dominant elements: If $x, w$ are dominant then $x \leq 0 w$ if and only if $x \leq w$,
(b) $\leq 0$ translation invariance: If $\mu^{\vee} \in \mathfrak{a}_{\mathbb{Z}}^{\text {ad }}$ and $x, w \in W$ then $x \leq 0 w$ if and only if $x t_{\mu \vee} \leq 0 w t_{\mu^{\vee}}$.

The positive level length is $\ell^{+}: W^{\text {ad }} \rightarrow \mathbb{Z}_{\geq 0}$ given by $\ell^{+}(w)=($ length of a reduced word for $w)$. The negative level length is $\ell^{-}: W^{\text {ad }} \rightarrow \mathbb{Z}_{\leq 0}$ given by $\ell^{-}(w)=-\ell^{+}(w)$.
The level 0 length is $\ell^{0}: W^{\text {ad }} \rightarrow \mathbb{Z}$ given by

$$
\ell^{0}(w)=\ell^{+}(w) \text { if } w \text { is dominant } \quad \text { and } \quad \ell^{0}\left(x t_{\mu^{\vee}}\right)-\ell^{0}\left(y t_{\mu^{\vee}}\right)=\ell^{0}(x)-\ell^{0}(y)
$$

for $x, y \in W^{\text {ad }}$ and $\mu^{\vee} \in \mathfrak{a}_{\mathbb{Z}}^{\text {ad }}$. Using the formula for $\ell^{+}$given in Mac96, (2.8)], gives a formula for $\ell^{0}$,

$$
\begin{equation*}
\ell^{0}\left(u t_{\mu^{\vee}}\right)=\ell^{+}(u)+2\left\langle\rho, \mu^{\vee}\right\rangle, \quad \text { for } u \in W_{\text {fin }}, \mu^{\vee} \in \mathfrak{a}_{\mathbb{Z}}^{\text {ad }} \tag{1.9}
\end{equation*}
$$

The length functions $\ell^{+}, \ell^{-}$and $\ell^{0}$ return, respectively, the dimension, the codimension and the relative dimension of Schubert cells in the positive level, negative level and level 0 affine flag varieties.

### 1.4 The affine braid groups $\mathcal{B}^{\text {sc }}$ and $\mathcal{B}^{\text {ad }}$

Let $\omega_{1}^{\vee}, \ldots, \omega_{n}^{\vee}$ be the basis of $\mathfrak{a}$ which is dual to the basis $\alpha_{1}, \ldots, \alpha_{n}$ of $\mathfrak{a}^{*}$. Let

$$
\mathfrak{a}_{\mathbb{Z}}^{\text {ad }}=\mathbb{Z}-\operatorname{span}\left\{h_{1}, \ldots, h_{n}\right\} \quad \subseteq \quad \mathfrak{a}_{\mathbb{Z}}^{\text {sc }}=\mathbb{Z}-\operatorname{span}\left\{\omega_{1}^{\vee}, \ldots, \omega_{n}^{\vee}\right\}
$$

The affine braid group $\mathcal{B}^{\text {ad }}$ (resp. $\mathcal{B}^{\text {sc }}$ ) is generated by $T_{1}, \ldots, T_{n}$ and $Y^{\lambda^{\vee}}, \lambda^{\vee} \in \mathfrak{a}_{\mathbb{Z}}^{\text {ad }}$ (resp. $\left.\lambda^{\vee} \in \mathfrak{a}_{\mathbb{Z}}^{\text {sc }}\right)$, with relations

$$
Y^{\lambda^{\vee}} Y^{\sigma^{\vee}}=Y^{\lambda^{\vee}+\sigma^{\vee}}, \quad \underbrace{T_{i} T_{j} \cdots}_{m_{i j} \text { factors }}=\underbrace{T_{j} T_{i} \cdots}_{m_{i j} \text { factors }}, \begin{aligned}
& T_{i}^{-1} Y^{\lambda^{\vee}}=Y^{s_{i} \lambda^{\vee}} T_{i}^{-1}, \quad \text { if }\left\langle\lambda^{\vee}, \alpha_{i}\right\rangle=0, \\
& T_{i}^{-1} Y^{\lambda^{\vee}} T_{i}^{-1}=Y^{s_{i} \lambda^{\vee}}, \quad \text { if }\left\langle\lambda^{\vee}, \alpha_{i}\right\rangle=1,
\end{aligned}
$$

for $i \in\{1, \ldots, n\}$ and $\lambda^{\vee} \in \mathfrak{a}_{\mathbb{Z}}^{\text {ad }}\left(\right.$ resp. $\left.\mathfrak{a}_{\mathbb{Z}}^{\text {sc }}\right)$ and $m_{i j}=\alpha_{i}\left(h_{j}\right) \alpha_{j}\left(h_{i}\right)$ for $i, j \in\{1, \ldots, n\}$ with $i \neq j$.

### 1.5 Macdonald polynomials

Let

$$
\mathfrak{h}_{\mathbb{Z}}^{*}=\mathbb{Z} \text {-span }\left\{\delta, \omega_{1}, \ldots, \omega_{n}, \Lambda_{0}\right\} \quad \text { and } \quad \mathfrak{a}_{\mathbb{Z}}^{*}=\mathbb{Z} \text {-span }\left\{\omega_{1}, \ldots, \omega_{n}\right\} .
$$

The double affine Hecke algebra $\tilde{H}$ is presented by generators $T_{0}, \ldots, T_{n}$ and $X^{\mu}, \mu \in \mathfrak{h}_{\mathbb{Z}}^{*}$, with relations

$$
\begin{gather*}
X^{\lambda} X^{\mu}=X^{\lambda+\mu}, \quad \underbrace{T_{i} T_{j} \cdots}_{m_{i j} \text { factors }}=\underbrace{T_{j} T_{i} \cdots}_{m_{i j} \text { factors }}, \quad T_{i}^{2}=\left(t^{\frac{1}{2}}-t^{-\frac{1}{2}}\right) T_{i}+1,  \tag{1.10}\\
T_{i} X^{\mu}=X^{s_{i} \mu} T_{i}+\left(t^{\frac{1}{2}}-t^{-\frac{1}{2}}\right) \frac{X^{\mu}-X^{s_{i} \mu}}{1-X^{\alpha_{i}}}, \quad T_{i}^{-1} X^{\mu}=X^{s_{i} \mu} T_{i}^{-1}-\left(t^{\frac{1}{2}}-t^{-\frac{1}{2}}\right) \frac{X^{\mu}-X^{s_{i} \mu}}{1-X^{-\alpha_{i}}} .
\end{gather*}
$$

for $i \in\{0, \ldots, n\}$ and $\mu \in \mathfrak{h}_{\mathbb{Z}}^{*}$. For $w \in W^{\text {ad }}$ put

$$
Y^{w}=\left\{\begin{array}{ll}
Y^{w s_{i}} T_{i}^{-1}, & \text { if } w<o w s_{i},  \tag{1.11}\\
Y^{w s_{i}} T_{i}, & \text { if } w \triangleright w s_{i},
\end{array} \quad \text { and let } \quad Y^{\lambda^{\vee}}=Y^{t_{\lambda} \vee} \text { for } \lambda^{\vee} \in \mathfrak{a}_{\mathbb{Z}}^{\text {ad }}\right.
$$

Putting $q=X^{\delta}=Y^{-K}$ then, as an algebra over $\mathbb{C}\left[q^{ \pm 1}, t^{ \pm \frac{1}{2}}\right]$,
$\tilde{H}$ has basis $\quad\left\{X^{\mu} T_{u} Y^{\lambda^{\vee}} \mid \mu \in \mathfrak{a}_{\mathbb{Z}}^{*}+\mathbb{Z} \Lambda_{0}, u \in W_{\text {fin }}, \lambda^{\vee} \in \mathfrak{a}_{\mathbb{Z}}^{\text {ad }}\right\}, \quad$ where $\quad T_{u}=T_{i_{1}} \cdots T_{i_{\ell}}$,
for a reduced word $u=s_{i_{1}} \cdots s_{i_{\ell}}$. The affine Hecke algebra is the subalgebra $H$ of $\tilde{H}$ with basis $\left\{T_{u} Y^{\lambda^{\vee}} \mid u \in W_{\mathrm{fin}}, \lambda^{\vee} \in \mathfrak{a}_{\mathbb{Z}}^{\text {ad }}\right\}$. The polynomial representation of $\tilde{H}$ is

$$
\begin{equation*}
\mathbb{C}[X]=\operatorname{Ind}_{H}^{\tilde{H}}(\mathbf{1}) \quad \text { with basis } \quad\left\{X^{\mu} \mathbf{1} \mid \mu \in \mathfrak{a}_{\mathbb{Z}}^{*}\right\} \tag{1.12}
\end{equation*}
$$

and $\quad Y^{K} \mathbf{1}=q^{-1} \mathbf{1}, \quad Y^{-\alpha_{i}^{\vee}} \mathbf{1}=t \mathbf{1}, \quad$ and $\quad T_{i} \mathbf{1}=t^{\frac{1}{2}} \mathbf{1}$ for $i \in\{1, \ldots, n\}$.
Let

$$
\begin{equation*}
T_{0}^{\vee}=Y^{\alpha_{0}^{\vee}} X^{-\Lambda_{0}} T_{0}^{-1} X^{\Lambda_{0}} \quad \text { and } \quad T_{i}^{\vee}=T_{i} \text { for } i \in\{1, \ldots, n\} . \tag{1.13}
\end{equation*}
$$

The automorphism of $\tilde{H}$ given by conjugation by $X^{-\Lambda_{0}}$ is the automorphism $\tau: \tilde{H} \rightarrow \tilde{H}$ of Ch96, (2.8)]. Extend $\tilde{H}$ to allow rational functions in the $Y^{\lambda^{\vee}}$. For each $i \in\{0,1, \ldots, n\}$, the intertwiner $\tau_{i}^{\vee} \in \widetilde{H}$ is

$$
\begin{equation*}
\tau_{i}^{\vee}=T_{i}^{\vee}+\frac{t^{-\frac{1}{2}}(1-t)}{1-Y^{-\alpha_{i}^{\vee}}}=\left(T_{i}^{\vee}\right)^{-1}+\frac{t^{-\frac{1}{2}}(1-t) Y^{-\alpha_{i}^{\vee}}}{1-Y^{-\alpha_{i}^{\vee}}} \quad \text { so that } \quad Y^{\lambda^{\vee}} \tau_{i}^{\vee}=\tau_{i}^{\vee} Y^{s_{i} \lambda^{\vee}} \tag{1.14}
\end{equation*}
$$

Let, for simplicity, $\mu \in \mathbb{Z}$-span $\left\{\alpha_{1}, \ldots, \alpha_{n}\right\}$ (the general case $\mu \in \mathfrak{a}_{\mathbb{Z}}^{*}$ requires consideraton of the group $\Omega^{\vee}$, the quotient of the weight lattice by the root lattice, and is treated in detail in [RY11]. The nonsymmetric Macdonald polynomial $E_{\mu}=E_{\mu}(q, t)$ is

$$
\begin{equation*}
E_{\mu}=E_{\mu}(q, t)=\tau_{i_{1}}^{\vee} \ldots \tau_{i_{\ell}}^{\vee} \mathbf{1}, \quad \text { where } m_{\mu}=s_{i_{1}} \ldots s_{i_{\ell}} \text { is a reduced word } \tag{1.15}
\end{equation*}
$$

for the minimal length element in the coset $t_{\mu} W_{\text {fin }}$. The $E_{\mu}$ form a basis of $\mathbb{C}[X]$ consisting of eigenvectors for the $Y^{\lambda^{\vee}}$ (the Cherednik-Dunkl operators).

Fix a reduced word $m_{\mu}=s_{i_{1}} \ldots s_{i_{\ell}}$ as in (1.15). Identifying the elements of $W^{\text {ad }}$ with alcoves as in (1.6), an alcove walk of type $\vec{m}_{\mu}=\left(i_{1}, \ldots, i_{\ell}\right)$ beginning at 1 (the fundamental alcove) is
a sequence of steps, of types $i_{1}, \ldots, i_{\ell}$, where a step of type $j$ is (the signs - and + indicate that $z s_{j} \triangleright z$ )

positive $j$-crossing negative $j$-crossing positive $j$-fold negative $j$-fold
Let $\mathcal{B}\left(1, \vec{m}_{\mu}\right)$ be the set of alcove walks of type $\vec{m}_{\mu}=\left(i_{1}, \ldots, i_{\ell}\right)$ beginning at 1 . For a walk $p \in \mathcal{B}\left(1, \vec{m}_{\mu}\right)$ let

$$
\begin{aligned}
& f^{+}(p)=\{k \in\{1, \ldots, \ell\} \mid \text { the } k \text { th step of } p \text { is a positive fold }\}, \\
& f^{-}(p)=\{k \in\{1, \ldots, \ell\} \mid \text { the } k \text { th step of } p \text { is a negative fold }\}, \\
& f(p)=f^{+}(p) \cup f^{-}(p)=\{k \in\{1, \ldots, \ell\} \mid \text { the } k \text { th step of } p \text { is a fold }\} .
\end{aligned}
$$

For $p \in \mathcal{B}\left(1, \vec{m}_{\mu}\right)$ let $\operatorname{end}(p)$ be the endpoint of $p$ (an element of $W^{\text {ad }}$ ) and define the weight $\mathrm{wt}(p)$ and the final direction $\varphi(p)$ of $p$ by

$$
X^{\operatorname{end}(p)}=X^{\mathrm{wt}(p)} T_{\varphi(p)}^{\vee}, \quad \text { with } \mathrm{wt}(p) \in \mathfrak{a}_{\mathbb{Z}}^{*} \text { and } \varphi(p) \in W_{\text {fin }}
$$

Using (1.14) and doing a left to right expansion of the terms of $\tau_{i_{1}}^{\vee} \cdots \tau_{i_{\ell}}^{\vee} \mathbf{1}$ produces the monomial expansion of $E_{\mu}$ as sum over alcove walks as given in the following theorem. For simplicity we state the following theorem for $\mu \in \mathbb{Z}$-span $\left\{\alpha_{1}, \ldots, \alpha_{n}\right\}$. It holds, after a small technical adjustment to the statement, for all $\mu \in \mathfrak{a}_{\mathbb{Z}}^{*}$, see RY11] for details.

Theorem 1.1. RY11, Theorem 3.1 and Remark 3.3] Let $\mu \in \mathbb{Z}-\operatorname{span}\left\{\alpha_{1}, \ldots, \alpha_{n}\right\}$ and let $m_{\mu}$ be the minimal length element in the coset $t_{\mu} W_{\text {fin }}$. Fix a reduced word $\vec{m}_{\mu}=s_{i_{1}} \cdots s_{i_{\ell}}$, let

$$
\beta_{1}^{\vee}=s_{i_{\ell}} \cdots s_{i_{2}} \alpha_{i_{1}}^{\vee}, \quad \beta_{2}^{\vee}=s_{i_{\ell}} \cdots s_{i_{3}} \alpha_{i_{2}}^{\vee}, \quad \cdots, \quad \beta_{\ell}^{\vee}=\alpha_{i_{\ell}}^{\vee}
$$

and let $\operatorname{sh}\left(\beta_{k}^{\vee}\right)$ and $\operatorname{ht}\left(\beta_{k}^{\vee}\right)$ be defined by $Y^{\beta_{k}^{\vee}} \mathbf{1}=q^{\operatorname{sh}\left(\beta_{k}^{\vee}\right)} t^{\text {ht }\left(\beta_{k}^{\vee}\right)} \mathbf{1}$ for $k \in\{1, \ldots, \ell\}$. Then

$$
E_{\mu}(q, t)=\sum_{p \in \mathcal{B}\left(1, \vec{m}_{\mu}\right)} X^{w t(p)} t^{\frac{1}{2}(\ell(\varphi(p))} \prod_{k \in f^{+}(p)} \frac{t^{-\frac{1}{2}}(1-t)}{1-q^{\operatorname{sh}\left(\beta_{k}^{\vee}\right)} t^{\mathrm{ht}\left(\beta_{k}^{\vee}\right)}} \prod_{k \in f^{-}(p)} \frac{t^{-\frac{1}{2}}(1-t) q^{\operatorname{sh}\left(\beta_{k}^{\vee}\right)} t^{\mathrm{ht}\left(\beta_{k}^{\vee}\right)}}{1-q^{\operatorname{sh}\left(\beta_{k}^{\vee}\right)} t^{\mathrm{ht}\left(\beta_{k}^{\vee}\right)}}
$$

### 1.6 Specializations of the normalized Macdonald polynomials $\tilde{E}_{\mu}(q, t)$

If $m_{\mu}=t_{\mu} m$ with $m \in W_{\text {fin }}$ then $E_{\mu}(q, t)$ has top term $t^{\frac{1}{2} \ell(m)} X^{\mu}$ (this term is the term corresponding to the unique alcove walk in $\mathcal{B}\left(\vec{m}_{\mu}\right)$ with no folds). The normalized nonsymmetric Macdonald polynomial is

$$
\tilde{E}_{\mu}(q, t)=t^{-\frac{1}{2} \ell(m)} E_{\mu}(q, t) \quad \text { so that } \tilde{E}_{\mu}(q, t) \text { has top term } X^{\mu}
$$

A path $p \in \mathcal{B}\left(1, \vec{m}_{\mu}\right)$ is positively folded if there are no negative folds, i.e. $\# f^{-}(p)=0$.
A path $p \in \mathcal{B}\left(1, \vec{m}_{\mu}\right)$ is negatively folded if there are no positive folds, i.e. $\# f^{+}(p)=0$.
A path $p \in \mathcal{B}\left(1, \vec{m}_{\mu}\right)$ is positive semi-infinite if $\ell(\varphi(p))-\ell(m)-\# f(p)+2 \sum_{k \in f^{-}(p)} h t\left(\beta_{k}^{\vee}\right)=0$.
A path $p \in \mathcal{B}\left(1, \vec{m}_{\mu}\right)$ is negative semi-infinite if $\ell(m)-\ell(\varphi(p))+\# f(p)+2 \sum_{k \in f^{+}(p)} h t\left(\beta_{k}^{\vee}\right)=0$.

Proposition 1.2. Let $\mu \in \mathfrak{a}_{\mathbb{Z}}^{*}$. The specializations $q=0, t=0, q^{-1}=0$ and $t^{-1}=0$ are well defined and given, respectively, by

$$
\begin{aligned}
\tilde{E}_{\mu}(0, t) & =\sum_{\substack{p \in \mathcal{B}\left(1, \vec{m}^{\prime} \mu\right) \\
p \text { pos folded }}} t^{\frac{1}{2}(\ell(\varphi(p))-\ell(m)-\# f(p))}(1-t)^{\# f(p)} X^{\mathrm{wt}(p)} \\
\tilde{E}_{\mu}(q, 0) & =\sum_{\substack{p \in \mathcal{B}\left(1, \vec{m}_{\mu}\right) \\
p \text { neg semi-inf }}} q^{\sum_{k \in f^{-}(p)} \operatorname{sh}\left(\beta_{k}^{\vee}\right)} X^{\mathrm{wt}(p)} \\
\tilde{E}_{\mu}(\infty, t) & =\sum_{\substack{p \in \mathcal{B}\left(1, \vec{m}_{\mu}\right) \\
p \text { neg folded }}} t^{-\frac{1}{2}(\ell(\varphi(p))-\# f(p))}\left(1-t^{-1}\right)^{\# f(p)} X^{\mathrm{wt}(p)} \\
\tilde{E}_{\mu}(q, \infty) & =\sum_{\substack{p \in \mathcal{B}\left(1, \vec{m}^{\prime} \mu\right) \\
p \text { pos semi-inf }}} q^{-\sum_{k \in f^{+}(p)} \operatorname{sh}\left(\beta_{k}^{\vee}\right)} X^{\mathrm{wt}(p)}
\end{aligned}
$$

For $i \in\{0,1, \ldots, n\}$ and $f \in \mathbb{C}\left[\mathfrak{h}_{\mathbb{Z}}^{*}\right]$ define

$$
\begin{equation*}
\Delta_{i} f=\frac{f-s_{i} f}{1-X^{-\alpha_{i}}} \quad \text { and } \quad D_{i} f=\left(1+s_{i}\right) \frac{1}{1-X^{-\alpha_{i}}} f=\frac{f-X^{-\alpha_{i}} s_{i} f}{1-X^{-\alpha_{i}}} \tag{1.16}
\end{equation*}
$$

Equations (1.14), (1.13) and the last relation in (1.10) give that, as operators on the polynomial representation,

$$
\begin{aligned}
t^{\frac{1}{2}} \tau_{i}^{\vee} & =X^{-\Lambda_{0}}\left(D_{i}-t \Delta_{i}\right) X^{\Lambda_{0}}+\frac{(1-t) Y^{-\alpha_{i}^{\vee}}}{1-Y^{-\alpha_{i}^{\vee}}} \quad \text { for } i \in\{1, \ldots, n\}, \quad \text { and } \\
t^{\frac{1}{2}} \tau_{0}^{\vee} Y^{\alpha_{0}^{\vee}} & =Y^{-\alpha_{0}^{\vee}} t^{\frac{1}{2}} \tau_{0}^{\vee}=X^{-\Lambda_{0}}\left(D_{0}-t \Delta_{0}\right) X^{\Lambda_{0}}+\frac{(1-t) Y^{-\alpha_{0}^{\vee}}}{1-Y^{-\alpha_{0}^{\vee}}}
\end{aligned}
$$

When applied in the formula of (1.15), these formulas for (normalized) intertwiners are specailizable at $t=0$, giving the following result (see the examples computed in Section 3.2).

Theorem 1.3. ( $\overline{\text { Ion01, }} \S 4.1])$ Let $\mu \in \mathfrak{a}_{\mathbb{Z}}^{*}$. There are unique $\nu \in \mathfrak{a}_{\mathbb{Z}}^{*}$ and $j \in \mathbb{Z}$ such that

$$
\Lambda_{0}+\nu \in\left(\mathfrak{h}^{*}\right)_{\mathrm{int}}^{+} \quad \text { and } \quad-j \delta+\mu+\Lambda_{0} \in W^{\mathrm{ad}}\left(\Lambda_{0}+\nu\right)
$$

Let $w \in W^{\text {ad }}$ be minimal length such that $-j+\mu+\Lambda_{0}=w\left(\Lambda_{0}+\nu\right)$ and let $w=s_{i_{1}} \cdots s_{i_{\ell}}$ be a reduced word. Letting $D_{0}, \ldots, D_{n}$ be the Demazure operators given in (1.16),

$$
\tilde{E}_{\mu}(q, 0)=q^{-j} X^{-\Lambda_{0}} D_{i_{1}} \cdots D_{i_{\ell}} X^{\nu} X^{\Lambda_{0}} \mathbf{1}
$$

## 2 Quantum affine algebras U and integrable modules

### 2.1 The quantum affine algebra U

The quantum affine algebra $\mathbf{U}$ is the $\mathbb{C}(q)$-algebra generated by

$$
E_{0}, \ldots, E_{n}, F_{0}, \ldots, F_{n},, K_{0}^{ \pm 1}, \ldots, K_{n}^{ \pm 1}, C^{ \pm \frac{1}{2}}, D^{ \pm 1}
$$

with Chevalley-Serre type relations corresponding to the affine Dynkin diagram. Following [Dr88, (11)], Lu93, §39], [B94, end of $\S 1]$, there is an action of the affine braid group $\mathcal{B}^{\text {sc }}$ on $\mathbf{U}$ by automorphisms.

Let $\mathbf{U}^{+}$be the subalgebra of $\mathbf{U}$ generated by $E_{0}, E_{1}, \ldots, E_{n}$. As explained in BCP98, Lemma 1.1 (iv)] and [BN02, (3.1)], there is a doubly infinite "longest element" for the affine Weyl group with a favourite reduced expression $w_{\infty}=\cdots s_{i_{-1}} s_{i_{0}} s_{i_{1}} \cdots$. This reduced word is used, with the braid group action, to define root vectors in $\mathbf{U}^{+}$by

$$
\begin{equation*}
E_{\beta_{0}}=E_{i_{0}}, \quad \text { and } \quad E_{\beta_{-k}}=T_{i_{0}}^{-1} T_{i_{-1}}^{-1} \cdots T_{i_{-(k-1)}}^{-1} E_{i_{-k}} \quad \text { and } \quad E_{\beta_{k}}=T_{i_{1}} T_{i_{2}} \ldots T_{i_{k-1}} E_{i_{k}}, \tag{2.1}
\end{equation*}
$$

for $k \in \mathbb{Z}_{>0}$. For $i \in\{1, \ldots, n\}$ and $r, s \in \mathbb{Z}$ define the loop generators in $\mathbf{U}^{+}$

$$
\begin{equation*}
\mathbf{x}_{i, r}^{+}=E_{\alpha_{i}+r \delta}=Y^{-r \omega_{i}^{\vee}} E_{i} \quad \text { and } \quad \mathbf{x}_{i, s}^{-}=E_{-\alpha_{i}+s \delta}=Y^{s \omega_{i}^{\vee}} F_{i}, \tag{2.2}
\end{equation*}
$$

where $Y^{r \omega_{i}^{\vee}}$ and $Y^{s \omega_{i}^{\vee}}$ are elements of the braid group $\mathcal{B}^{\text {sc }}$ as defined in Section 1.4. For $r \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{Z}_{>0}$ these are special cases of the root vectors in (2.1). For $i \in\{1, \ldots, n\}$ and $r \in \mathbb{Z}_{>0}$ define $\mathbf{q}_{r}^{(i)}$ by

$$
\begin{equation*}
\mathbf{q}_{r}^{(i)}=\mathbf{x}_{i, r}^{-} \mathbf{x}_{i, 0}^{+}-q^{-2} \mathbf{x}_{i, 0}^{+} \mathbf{x}_{i, r}^{-}, \quad \text { let } \quad \mathbf{q}_{+}^{(i)}(z)=1+\left(q-q^{-1}\right) \sum_{s \in \mathbb{Z}_{>0}} \mathbf{q}_{s}^{(i)} z^{s} \tag{2.3}
\end{equation*}
$$

and define $\mathbf{p}_{r}^{(i)}$ and $\mathbf{e}_{r}^{(i)}$ by

$$
\begin{equation*}
\mathbf{q}_{+}^{(i)}(z)=\exp \left(\sum_{r \in \mathbb{Z}_{>0}}\left(q-q^{-1}\right) \mathbf{p}_{r}^{(i)} z^{r}\right) \quad \text { and } \quad \exp \left(\sum_{r \in \mathbb{Z}_{>0}} \frac{\mathbf{p}_{r}^{(i)}}{[r]} z^{r}\right)=1+\sum_{k \in \mathbb{Z}_{>0}} \mathbf{e}_{k}^{(i)} z^{k} \tag{2.4}
\end{equation*}
$$

For a sequence of partitions $\vec{\kappa}=\left(\kappa^{(1)}, \ldots, \kappa^{(n)}\right)$ define

$$
\begin{equation*}
\mathbf{s}_{\vec{\kappa}}=\mathbf{s}_{\kappa^{(1)}} \cdots \mathbf{s}_{\kappa^{(n)}}, \quad \text { where } \quad \mathbf{s}_{\kappa^{(i)}}=\operatorname{det}\left(\mathbf{e}_{\left(\kappa^{(i)}\right)_{r}^{\prime}-r+s}^{(i)}\right)_{1 \leq r, s \leq m_{i}}, \tag{2.5}
\end{equation*}
$$

where $\left(\kappa^{(i)}\right)_{r}^{\prime}$ is the length of the $r$ th column of $\kappa^{(i)}$ and $m_{i}=\ell\left(\kappa^{(i)}\right)$ (see [Mac, Ch. I (3.5)]). For a sequence $\mathbf{c}=\left(\cdots, c_{-3}, c_{-2}, c_{-1}, \vec{\kappa}, c_{1}, c_{2}, c_{3}, \ldots\right)$ with $c_{i} \in \mathbb{Z}_{\geq 0}$ and all but a finite number of $c_{i}$ equal to 0 . The corresponding $P B W$-type element of $\mathbf{U}^{+}$is

$$
\begin{equation*}
E_{\mathbf{c}}=\left(E_{\beta_{0}}^{\left(c_{0}\right)} E_{\beta_{-1}}^{\left(c_{-1}\right)} E_{\beta_{-2}}^{\left(c_{-2}\right)} \cdots\right)\left(\mathbf{s}_{\kappa^{(1)}} \cdots \mathbf{s}_{\kappa^{(n)}}\right)\left(\cdots E_{\beta_{3}}^{\left(c_{3}\right)} E_{\beta_{2}}^{\left(c_{2}\right)} E_{\beta_{1}}^{\left(c_{1}\right)}\right) \tag{2.6}
\end{equation*}
$$

The Cartan involution is the $\mathbb{C}$-linear anti-automorphism $\Omega: \mathbf{U} \rightarrow \mathbf{U}$ given by

$$
\Omega\left(E_{i}\right)=F_{i}, \quad \Omega\left(F_{i}\right)=E_{i}, \quad \Omega\left(K_{i}\right)=K_{i}^{-1}, \quad \Omega(D)=D^{-1}, \quad \Omega(q)=q^{-1}
$$

and $\mathbf{U}^{-}=\Omega\left(\mathbf{U}^{+}\right)$. Putting $\mathbf{p}_{-r}^{(i)}=\Omega\left(\mathbf{p}_{r}^{(i)}\right)$, then (see Dr88, Th. 2 (6)] and [B94, Th. 4.7 (2)])

$$
\begin{equation*}
\left[\mathbf{p}_{k}^{(i)}, \mathbf{p}_{l}^{(i)}\right]=\delta_{k,-l} \frac{1}{k}\left(\frac{q^{k \alpha_{i}\left(h_{j}\right)}-q^{-k \alpha_{i}\left(h_{j}\right)}}{q-q^{-1}}\right) \frac{C^{k}-C^{-k}}{q-q^{-1}} . \tag{2.7}
\end{equation*}
$$

Define $q_{-s}^{(i)}=\Omega\left(q_{s}^{(i)}\right)$ and

$$
\begin{equation*}
\mathbf{q}_{-}^{(i)}\left(z^{-1}\right)=1+\left(q-q^{-1}\right) \sum_{s \in \mathbb{Z}_{>0}} \mathbf{q}_{-s}^{(i)} z^{-s} \tag{2.8}
\end{equation*}
$$

The Heisenberg subalgebra $\mathbf{H}$ is the subalgebra of $\mathbf{U}$ generated by $\left\{\mathbf{p}_{k}^{(i)} \mid i \in\{1, \ldots, n\}, k \in\right.$ $\left.\mathbb{Z}_{\neq 0}\right\}$. In $\mathbf{H} \cap \mathbf{U}^{+}$the $\mathbf{p}_{r}^{(i)}\left(r \in \mathbb{Z}_{>0}\right)$ are the power sums, the $\mathbf{q}_{r}^{(i)}\left(r \in \mathbb{Z}_{>0}\right)$ are the HallLittlewoods and the $\mathbf{e}_{r}^{(i)}\left(r \in \mathbb{Z}_{>0}\right)$ are the elementary symmetric functions and the $\mathbf{s}_{\vec{\kappa}}$ are the Schur functions.

### 2.2 Integrable U-modules

As in (1.5), let $h_{\theta}=a_{1}^{\vee} h_{1}+\cdots+a_{n}^{\vee} h_{n}$ be the highest root of $\mathfrak{g}$ and let

$$
\Lambda_{i}=\omega_{i}+a_{i}^{\vee} \Lambda_{0}, \quad \text { for } i \in\{1, \ldots, n\},
$$

so that $\left\{\delta, \Lambda_{1}, \ldots, \Lambda_{n}, \Lambda_{0}\right\}$ is the dual basis in $\mathfrak{h}^{*}$ to the basis $\left\{d, h_{1}, \ldots, h_{n}, h_{0}\right\}$ of $\mathfrak{h}$. Let

$$
\mathfrak{h}_{\mathbb{Z}}^{*}=\left\{\Lambda \in \mathfrak{h}^{*} \mid\left\langle\Lambda, \alpha_{i}^{\vee}\right\rangle \in \mathbb{Z} \text { for } i \in\{0,1, \ldots, n\}\right\}=\mathbb{C} \delta+\mathbb{Z}-\operatorname{span}\left\{\Lambda_{0}, \ldots, \Lambda_{n}\right\} .
$$

A set of representatives for the $W^{\text {ad }}$-orbits on $\mathfrak{h}_{\mathbb{Z}}^{*}$ is

$$
\left(\mathfrak{h}^{*}\right)_{\text {int }}=\left(\mathfrak{h}^{*}\right)_{\text {int }}^{+} \cup\left(\mathfrak{h}^{*}\right)_{\text {int }}^{0} \cup\left(\mathfrak{h}^{*}\right)_{\text {int }}^{-}, \quad\left(\mathfrak{h}^{*}\right)_{\text {int }}^{+}=\mathbb{C} \delta+\mathbb{Z}_{\geq 0}-\operatorname{span}\left\{\Lambda_{0}, \ldots, \Lambda_{n}\right\},, ~\left(\mathfrak{h}^{*}\right)_{\text {int }}^{0}=\mathbb{C} \delta+0 \Lambda_{0}+\mathbb{Z}_{\geq 0}-\operatorname{span}\left\{\omega_{1}, \ldots, \omega_{n}\right\},, ~\left(\mathfrak{h}^{*}\right)_{\text {int }}^{-}=\mathbb{C} \delta+\mathbb{Z}_{\leq 0-\operatorname{span}\left\{\Lambda_{0}, \ldots, \Lambda_{n}\right\} .} .
$$

For $\widehat{\mathfrak{s l}}_{2}$ these sets are pictured $(\bmod \delta)$ in (0.2).
For $i \in\{0,1, \ldots, n\}$ let $\mathbf{U}_{(i)}$ be the subalgebra of $\mathbf{U}$ generated by $\left\{E_{i}, F_{i}, K_{i}^{ \pm 1}\right\}$. An integrable $\mathbf{U}$-module is a $\mathbf{U}$-module $M$ such that if $i \in\{0, \ldots, n\}$ then

$$
\operatorname{Res}_{\mathbf{U}_{(i)}}^{\mathbf{U}}(M) \quad \text { is a direct sum of finite dimensional } \mathbf{U}_{(i)} \text {-modules, }
$$

where $\operatorname{Res}_{\mathbf{U}_{(i)}}^{\mathbf{U}}(M)$ denotes the restriction of the $\mathbf{U}$-module $M$ to a $\mathbf{U}_{(i)}$-module.
Let $M$ be an integrable U-module. Following [Lu93, §5], for each $w \in W^{\text {ad }}$ there is a linear map

$$
T_{w}: M \rightarrow M \quad \text { such that } \quad T_{w}(u m)=T_{w}(u) T_{w}(m),
$$

for $u \in \mathbf{U}$ and $m \in M$ (here $T_{w}(u)$ refers to the braid group action on $\mathbf{U}$ ). Thus, every integrable module $M$ is a module for the semidirect product $\mathcal{B}^{\text {ad }} \ltimes \mathbf{U}$ where $\mathcal{B}^{\text {ad }}$ is the braid group of $W^{\text {ad }}$.

### 2.3 Extremal weight modules $L(\Lambda)$

Let $\Lambda \in\left(\mathfrak{h}^{*}\right)_{\text {int }}$. Following [Kas94, (8.2.2)] and [Kas02, §3.1], the extremal weight module $L(\Lambda)$ is the $\mathbf{U}$-module
generated by $\left\{u_{w \Lambda} \mid w \in W\right\} \quad$ with relations $\quad K_{i}\left(u_{w \Lambda}\right)=q^{\left\langle w \Lambda, \alpha_{i}^{\vee}\right\rangle} u_{w \Lambda}$,

$$
\begin{array}{cll}
E_{i} u_{w \Lambda}=0, \quad \text { and } \quad F_{i}^{\left\langle w \Lambda, \alpha_{i}^{\vee}\right\rangle} u_{w \Lambda}=u_{s_{i} w \Lambda}, & \text { if }\left\langle w \Lambda, \alpha_{i}^{\vee}\right\rangle \in \mathbb{Z}_{\geq 0},  \tag{2.10}\\
F_{i} u_{w \Lambda}=0, \quad \text { and } \quad E_{i}^{-\left\langle w \Lambda, \alpha_{i}^{\vee}\right\rangle} u_{w \Lambda}=u_{s_{i} w \Lambda}, & \text { if }\left\langle w \Lambda, \alpha_{i}^{\vee}\right\rangle \in \mathbb{Z}_{\leq 0},
\end{array}
$$

for $i \in\{0, \cdots, n\}$. The module $L(\Lambda)$ has a crystal $B(\Lambda)$ (Kas94, Prop. 8.2.2(ii)], [Kas02, §3.1]).

- If $\Lambda \in\left(\mathfrak{h}^{*}\right)_{\text {int }}^{+}$then $L(\Lambda)$ is the simple $\mathbf{U}$-module of highest weight $\Lambda$ (see [Kac, (10.4.6)]).
- If $\Lambda \notin\left(\mathfrak{h}^{*}\right)_{\text {int }}^{+}$then $L(\Lambda)$ is not a highest weight module.
- If $\Lambda \in\left(\mathfrak{h}^{*}\right)_{\text {int }}^{-}$then $L(\Lambda)$ is the simple $\mathbf{U}$-module of lowest weight $\Lambda$.

The finite dimensional simple modules, denoted $\left.L^{\text {fin }}(a(u))\right)$, are integrable weight modules which are not extremal weight modules. The connection between the $L^{\text {fin }}(a(u))$ and the $L(\lambda)$ for $\lambda \in\left(\mathfrak{h}^{*}\right)_{\text {int }}^{0}$ is given by Theorem 2.3 below.

The module $L(\Lambda)$ is universal (see [Kas05, §2.6], [B02, §2.1], [Nak02, §2.5]). One way to formulate this universality is to let $\mathbf{U}_{0}$ be the subalgebra generated by $K_{1}^{ \pm 1}, \ldots, K_{n}^{ \pm 1}, C^{ \pm \frac{1}{2}}, D^{ \pm 1}$, let intInd be an induction functor in the category of integrable U-modules and write

$$
L(\Lambda)=\operatorname{int} \operatorname{Ind}_{\mathbf{U}_{0} \rtimes \mathcal{B}^{\text {ad }}}^{\mathbf{U}}(S(\Lambda)), \quad \text { where } \quad S(\Lambda)=\operatorname{span}\left\{u_{w \Lambda} \mid w \in W^{\operatorname{ad}}\right\}
$$

is the $\mathbf{U}_{0} \rtimes \mathcal{B}^{\text {ad }}$-module with action given by $T_{i} u_{w \Lambda}=(-q)^{\left\langle w \Lambda_{i}, \alpha_{i}^{\vee}\right\rangle} u_{s_{i} w \Lambda}$ and $K_{i} u_{w \Lambda}=q^{\left\langle w \Lambda, \alpha_{i}^{\vee}\right\rangle} u_{w \Lambda}$, for $i \in\{0,1, \ldots, n\}$ and $w \in W^{\text {ad }}$.

### 2.4 Demazure submodules $L(\Lambda)_{\leq w}$

Let $w \in W^{\text {ad }}$. The Demazure module $L(\Lambda)_{\leq w}$ is the $\mathbf{U}^{+}$-submodule of $L(\Lambda)$ given by

$$
L(\Lambda)_{\leq w}=\mathbf{U}^{+} u_{w \Lambda} \quad \text { and } \quad \operatorname{char}\left(L(\Lambda)_{\leq w}\right)=\sum_{p \in B(\Lambda)_{\leq w}} e^{\mathrm{wt}(p)}
$$

since $L(\Lambda)_{\leq w}$ has a crystal $B(\Lambda)_{\leq w}$. The $B G G$-Demazure operator on $\mathbb{C}\left[\mathfrak{h}_{\mathbb{Z}}^{*}\right]=\mathbb{C}$-span $\left\{X^{\lambda} \mid \lambda \in\right.$ $\left.\mathfrak{h}_{\mathbb{Z}}^{*}\right\}$ is given by

$$
D_{i}=\left(1+s_{i}\right) \frac{1}{1-X^{-\alpha_{i}}}, \quad \text { for } i \in\{0,1, \ldots, n\}
$$

Let $\Lambda \in\left(\mathfrak{h}^{*}\right)_{\text {int }}, w \in W$ and $i \in\{0,1, \ldots, n\}$.

$$
\begin{aligned}
& \text { If } \left.\Lambda \in\left(\mathfrak{h}^{*}\right)_{\text {int }}^{+} \quad \text { then } \quad D_{i} \operatorname{char}\left(L(\Lambda)_{\leq w}\right)\right)= \begin{cases}\operatorname{char}\left(L(\Lambda)_{\leq s_{i} w}\right), & \text { if } s_{i} w \pm w, \\
\operatorname{char}\left(L(\Lambda)_{\leq w}\right), & \text { if } s_{i} w \leq w ;\end{cases} \\
& \text { if } \lambda \in\left(\mathfrak{h}^{*}\right)_{\text {int }}^{0} \quad \text { then } \quad D_{i} \operatorname{char}\left(L(\lambda)_{\leq w}\right)= \begin{cases}\operatorname{char}\left(L(\lambda) \leq s_{i} w\right), & \text { if } s_{i} w \stackrel{ }{ } \geq w, \\
\operatorname{char}\left(L(\lambda)_{\leq w}\right), & \text { if } s_{i} w \leq 0 w ;\end{cases} \\
& \text { if } \Lambda \in\left(\mathfrak{h}^{*}\right)_{\text {int }}^{-} \text {then } \quad D_{i} \operatorname{char}\left(L(\Lambda)_{\leq w}\right)= \begin{cases}\operatorname{char}\left(L(\Lambda)_{\leq s_{i} w}\right), & \text { if } s_{i} w \geq w, \\
\operatorname{char}\left(L(\Lambda)_{\leq w}\right), & \text { if } s_{i} w \leq w ;\end{cases}
\end{aligned}
$$

(see [Kum, Theorem 8.2.9], Kas93], Kas05, §2.8] and [Kt16, Theorems 4.7 and 4.11]).

### 2.5 An alternate presentation for level 0 extremal weight modules

For $\lambda=m_{1} \omega_{1}+\cdots+m_{n} \omega_{n} \in\left(\mathfrak{h}^{*}\right)_{\text {int }}$ and let $x_{1,1}, \ldots, x_{m_{1}, 1}, x_{1,2}, \ldots, x_{m_{2}, 2}, \ldots x_{1, n}, \ldots, x_{m_{n}, n}$ be $n$ sets of formal variables. Letting $e_{i}^{(j)}=e_{i}\left(x_{1, j}, \ldots, x_{m_{j}, j}\right)$ denote the elementary symmetric function in the variables $x_{1, j}, \ldots, x_{m_{j}, j}$ define

$$
\begin{aligned}
R G_{\lambda} & =\mathbb{C}\left[x_{1,1}^{ \pm 1}, \ldots, x_{m_{1}, 1}^{ \pm 1}\right]^{S_{m_{1}}} \otimes \cdots \otimes \mathbb{C}\left[x_{1, n}^{ \pm 1}, \ldots, x_{m_{n}, n}^{ \pm 1}\right]^{S_{m_{n}}} \\
& =\mathbb{C}\left[e_{1}^{(1)}, \ldots, e_{m_{1}-1}^{(1)},\left(e_{m_{1}}^{(1)}\right)^{ \pm 1}\right] \otimes \cdots \otimes \mathbb{C}\left[e_{1}^{(n)}, \ldots, e_{m_{n}-1}^{(n)},\left(e_{m_{n}}^{(n)}\right)^{ \pm 1}\right] \\
R G_{\lambda}^{+} & =\mathbb{C}\left[x_{1,1}, \ldots, x_{m_{1}, 1}\right]^{S_{m_{1}}} \otimes \cdots \otimes \mathbb{C}\left[x_{1, n}, \ldots, x_{m_{n}, n}\right]^{S_{m_{n}}}, \quad \text { and } \\
R G_{\lambda}^{-} & =\mathbb{C}\left[x_{1,1}^{-1}, \ldots, x_{m_{1}, 1}^{-1}\right]^{S_{m_{1}}} \otimes \cdots \otimes \mathbb{C}\left[x_{1, n}^{-1}, \ldots, x_{m_{n}, n}^{-1}\right]^{S_{m_{n}}}
\end{aligned}
$$

Let

$$
\begin{aligned}
e_{+}^{(i)}(u) & =\left(1-x_{1, i} u\right)\left(1-x_{2, i} u\right) \cdots\left(1-x_{m_{i}, i} u\right) \quad \text { and } \\
e_{-}^{(i)}\left(u^{-1}\right) & =\left(1-x_{1, i}^{-1} u^{-1}\right)\left(1-x_{2, i}^{-1} u^{-1}\right) \cdots\left(1-x_{m_{i}, i}^{-1} u^{-1}\right)
\end{aligned}
$$

Let $\mathbf{U}^{\prime}$ be the subalgebra of $\mathbf{U}$ without the generator $D$.
Theorem 2.1. (see $\left.\mathrm{Nak02}^{2}, \S 3.4\right]$ ) The extremal weight module $L(\lambda)$ is the $\left(\mathbf{U}^{\prime} \otimes_{\mathbb{Z}} R G_{\lambda}\right)$-module generated by a single vector $m_{\lambda}$ with relations

$$
\begin{gathered}
\mathbf{x}_{i, r}^{+} m_{\lambda}=0, \quad K_{i} m_{\lambda}=q^{m_{i}} m_{\lambda}, \quad C m_{\lambda}=m_{\lambda} \\
\mathbf{q}_{+}^{(i)}(u) m_{\lambda}=K_{i} \frac{e_{+}^{(i)}\left(q^{-1} u\right)}{e_{+}^{(i)}(q u)} m_{\lambda} \quad \text { and } \quad \mathbf{q}_{-}^{(i)}\left(u^{-1}\right) m_{\lambda}=K_{i} \frac{e_{-}^{(i)}\left(q u^{-1}\right)}{e_{-}^{(i)}\left(q^{-1} u^{-1}\right)} m_{\lambda}
\end{gathered}
$$

where $\mathbf{q}_{+}^{(i)}(u)$ and $\mathbf{q}_{-}^{(i)}\left(u^{-1}\right)$ are as defined in (2.3) and (2.8).
In this form $L(\lambda)$ has been termed the universal standard module [Nak02, §3.4]) or the global Weyl module [CP01, §2]. See [Nak02, Theorem 2] and [Nak02, Remark 2.15] for discussion of how to see that the extremal weight module, the universal standard module and the global Weyl module coincide.

Remark 2.2. Let

$$
0_{q}=\frac{1}{1-q}+\frac{q^{-1}}{1-q^{-1}}=\cdots+q^{-3}+q^{-2}+q^{-1}+1+q+q^{2}+\cdots
$$

(although $\frac{q^{-1}}{1-q^{-1}}=\frac{1}{q-1}=\frac{-1}{1-q}$, it is important to note that $0_{q}$ is not equal to 0 , it is a doubly infinite formal series in $q$ and $\left.q^{-1}\right)$. Since $\operatorname{deg}\left(e_{j}^{(i)}\right)=j$,

$$
\begin{aligned}
& \operatorname{gchar}\left(R G_{\lambda}^{+}\right)=\left(\prod_{i=1}^{n} \prod_{k=1}^{m_{i}} \frac{1}{1-q^{k}}\right) \quad \operatorname{gchar}\left(R G_{\lambda}^{-}\right)=\left(\prod_{i=1}^{n} \prod_{k=1}^{m_{i}} \frac{1}{1-q^{-k}}\right) \text { and } \\
& \operatorname{gchar}\left(R G_{\lambda}\right)=\left(0_{q^{m_{1}}} \prod_{k=1}^{m_{1}-1} \frac{1}{1-q^{k}}\right)\left(0_{q^{m_{2}}} \prod_{k=1}^{m_{2}-1} \frac{1}{1-q^{k}}\right) \cdots\left(0_{q^{m_{n}}} \prod_{k=1}^{m_{n}-1} \frac{1}{1-q^{k}}\right)
\end{aligned}
$$

### 2.6 Level $0 L(\lambda)$ and finite dimensional simple U-modules $L^{\text {fin }}(a(u))$

The loop presentation provides a triangular decomposition of $\mathbf{U}$ (different form the usual triangular decomposition coming from the Kac-Moody presentation). The extremal weight module $L(\lambda)$ is the standard (Verma type) module for the loop triangular decomposition (see CP95, Theorem 2.3(b)], [Nak02, Lemma 2.14] and [CP94, outline of proof of Theorem 12.2.6]).

A Drinfeld polynomial is an $n$-tuple of polynomials $a(u)=\left(a^{(1)}(u), \ldots, a^{(n)}(u)\right)$ with $a^{(i)}(u) \in$ $\mathbb{C}[u]$, represented as

$$
a(u)=a^{(1)}(u) \omega_{1}+\cdots+a^{(n)}(u) \omega_{n}, \quad \text { with } \quad a^{(i)}(u)=\left(u-a_{1, i}\right) \cdots\left(u-a_{m_{i}, i}\right)
$$

so that the coefficient of $u^{j}$ in $a^{(i)}(u)$ is $e_{m_{i}-j}^{(i)}\left(a_{1, i}, \ldots, a_{m_{i}, i}\right)$, the $\left(m_{i}-j\right)$ th elementary symmetric function evaluated at the values $a_{1, i}, \ldots, a_{m_{i}, i}$. The local Weyl module (a finite dimensional standard module) is defined by

$$
M^{\mathrm{fin}}(a(u))=L(\lambda) \otimes_{R G_{\lambda}} m_{a(u)}, \quad \text { where } \quad e_{k}^{(i)}\left(x_{1, i}, x_{2, i}, \ldots\right) m_{a(u)}=e_{k}^{(i)}\left(a_{1, i}, \ldots, a_{m_{i}, i}\right) m_{a(u)}
$$

specifies the $R G_{\lambda}$-action on $m_{a(u)}$. In other words, the module $M^{\mathrm{fin}}(a(u))$ is $L(\lambda)$ except that variables $x_{j, i}$ from 2.5 have been specialised to the values $a_{j, i}$. As in Theorem 2.1, let $\mathbf{U}^{\prime}$ be the subalgebra of $\mathbf{U}$ without the generator $D$.

Theorem 2.3. (see Dr88, Theorem 2] and CP95, Theorem 3.3]) The standard module $M^{\mathrm{fin}}(a(u))$ has a unique simple quotient $L^{\text {fin }}(a(u))$ and

$$
\begin{array}{ccc}
\begin{array}{c}
\text { \{Drinfeld polynomials }\} \\
a(u)=a^{(1)}(u) \omega_{1}+\cdots+a^{(n)}(u) \omega_{n}
\end{array} & \longrightarrow & \text { \{finite dimensional simple } \left.\mathbf{U}^{\prime} \text {-modules }\right\} \\
& L^{\text {fin }}(a(u))
\end{array}
$$

is a bijection.

### 2.7 Path models for the crystals $B(\Lambda)$

The work of Littelmann Li94, Li95 provided a particularly convenient model for the crystals $B(\Lambda)$ when $\Lambda$ is positive or negative level. This model realizes the crystal as a set of paths $p: \mathbb{R}_{[0,1]} \rightarrow \mathfrak{h}^{*}$ with combinatorially defined Kashiwara operators $\tilde{e}_{0}, \ldots, \tilde{e}_{n}, \tilde{f}_{0}, \ldots, \tilde{f}_{n}$. In the LS (Lakshmibai-Seshadri) model the generator of the crystal $B(\Lambda)$ is the straight line path to $\Lambda$. When $\lambda \in\left(\mathfrak{h}^{*}\right)_{\text {int }}^{0}$ using

$$
\begin{aligned}
p_{\lambda}: \mathbb{R}_{[0,1]} & \rightarrow \mathfrak{h}^{*} \\
t & \rightarrow t \lambda, \quad \text { the straight line path from } 0 \text { to } \lambda,
\end{aligned}
$$

as a generator for $B(\lambda)$ may not be the optimal choice. Remarkably, Naito and Sagaki (see [INS16, Definition 3.1.4 and Theorem 3.2.1] and [NNS15, Theorem 4.6.1(b)]), have shown that $B(\lambda)$ can be constructed with sequences of Weyl group elements and rational numbers as in LLi94, §1.2, 1.3 and 2.2] but with the positive level length $\ell^{+}$and Bruhat order $\leq+$replaced by the level zero length $\ell^{0}$ and Bruhat order $\leq 0$. However, when working with the Naito-Sagaki construction one must be very careful not to identify the Naito-Sagaki sequences with actual paths (piecewise linear maps from $\mathbb{R}_{[0,1]}$ to $\mathfrak{h}^{*}$ ) because the natural map from Naito-Sagaki sequences to paths is not always injective (an example is provided by Kas02, Remark 5.10]).

If $\Lambda \in\left(\mathfrak{h}^{*}\right)_{\text {int }}^{+}$then $-\Lambda \in\left(\mathfrak{h}^{*}\right)_{\text {int }}^{-}$and

$$
\begin{aligned}
B(\Lambda) & =\left\{\tilde{f}_{i_{1}} \cdots \tilde{f}_{i_{k}} p_{\Lambda} \mid k \in \mathbb{Z}_{\geq 0} \text { and } i_{1}, \ldots, i_{k} \in\{0,1, \ldots, n\}\right\} \\
B(-\Lambda) & =\left\{\tilde{e}_{i_{1}} \cdots \tilde{e}_{i_{k}} p_{-\Lambda} \mid k \in \mathbb{Z}_{\geq 0} \text { and } i_{1}, \ldots, i_{k} \in\{0,1, \ldots, n\}\right\} .
\end{aligned}
$$

are each a single connected component and their characters are determined by the Weyl-Kac character formula [Kac, Theorem 11.13.3].

### 2.8 Crystals for level 0 extremal weight modules $L(\lambda)$

For general $\lambda \in\left(\mathfrak{h}^{*}\right)_{\text {int }}^{0}$ the crystal $B(\lambda)$ is not connected (as a graph with edges determined by the Kashiwara operators $\left.\tilde{e}_{0}, \ldots, \tilde{e}_{n}, \tilde{f}_{0}, \ldots \tilde{f}_{n}\right)$. Let $\lambda=m_{1} \omega_{1}+\cdots+m_{n} \omega_{n}$, with $m_{1}, \ldots, m_{n} \in \mathbb{Z} \geq 0$. By BN02, Corollary 4.15], the map

$$
\begin{array}{rlll}
\Phi_{\lambda}: \quad L(\lambda) & \longrightarrow L\left(\omega_{1}\right)^{\otimes m_{1}} \otimes \cdots \otimes L\left(\omega_{n}\right)^{\otimes m_{n}} \quad \text { is injective }  \tag{2.11}\\
u_{\lambda} & \longmapsto & u_{\omega_{1}}^{\otimes m_{1}} \otimes \cdots \otimes u_{\omega_{n}}^{\otimes m_{n}} & \quad \text { in }
\end{array}
$$

and gives rise to an injection of crystals

$$
B(\lambda) \hookrightarrow B\left(\omega_{1}\right)^{\otimes m_{1}} \otimes \cdots \otimes B\left(\omega_{n}\right)^{\otimes m_{n}}
$$

which takes the connected component of $B(\lambda)$ containing $b_{\lambda}$ to the connected component of $B\left(\omega_{1}\right)^{\otimes m_{1}} \otimes \cdots \otimes B\left(\omega_{n}\right)^{\otimes m_{n}}$ containing $b_{\omega_{1}}^{\otimes m_{1}} \otimes \cdots \otimes b_{\omega_{n}}^{\otimes m_{n}}$. Kashiwara Kas02, Theorem 5.15] fully described the structure of $L\left(\omega_{i}\right)$ (see [BN02, Theorem 2.16]). Beck-Nakajima analyzed the PBW basis of (2.6) and use (2.11) to show that the connected components of $B(\lambda)$ are labeled by $n$-tuples of partitions $\kappa=\left(\kappa^{(1)}, \ldots, \kappa^{(n)}\right)$ such that $\ell\left(\kappa^{(i)}\right)<m_{i}$. Together with a result of Fourier-Littelmann which shows that the crystal of the level 0 module $M^{\text {fin }}(a(u))$ is isomorphic to a level one Demazure crystal $B\left(\nu+\Lambda_{0}\right)_{\leq w}$, the full result is as detailed in Theorem 2.4] below.

A labeling set for a basis of $R G_{\lambda} /\left\langle e_{m_{i}}^{(i)}=1\right\rangle$ is

$$
S^{\lambda}=\left\{\vec{\kappa}=\left(\kappa^{(1)}, \ldots, \kappa^{(n)}\right) \mid \kappa^{(i)} \text { is a partition with } \ell\left(\kappa^{(i)}\right)<m_{i} \text { for } i \in\{1, \ldots, n\}\right\}
$$

The connected component of $b_{\lambda}$ in $B(\lambda)$ is

$$
B(\lambda)_{0}=\left\{\tilde{r}_{i_{1}} \cdots \tilde{r}_{i_{k}} b_{\lambda} \mid k \in \mathbb{Z}_{\geq 0} \text { and } \tilde{r}_{i_{1}}, \ldots, \tilde{r}_{i_{k}} \in\left\{\tilde{e}_{0}, \ldots, \tilde{e}_{n}, \tilde{f}_{0}, \ldots, \tilde{f}_{n}\right\}\right\} .
$$

Define $B^{\text {fin }}(\lambda)$ to be the "crystal" which has a crystal graph which is the "quotient" of the crystal graph of $L(\lambda)$ obtained by identifying the vertices $b$ and $b^{\prime}$ if there is an element $\mathbf{s} \in R G_{\lambda}$ such that $\mathbf{s} G(b)=G\left(b^{\prime}\right)$, where $G(b)$ denotes the canonical basis element of $L(\lambda)$ corresponding to $b$.

$$
B^{\mathrm{fin}}(\lambda) \text { is the crystal of the finite dimensional standard module } M^{\mathrm{fin}}(a(u)) \text {. }
$$

Theorem 2.4. (see BN02, Theorem 4.16], [Nak02, §3.4]) and [FL05, Proposition 3]) Let $\lambda=$ $m_{1} \omega_{1}+\cdots+m_{n} \omega_{n} \in\left(\mathfrak{h}^{*}\right)_{\text {int }}^{0}$. As in Theorem 1.3, let $\nu \in \mathfrak{a}_{\mathbb{Z}}^{*}, j \in \mathbb{Z}_{\geq 0}$ and $w \in W^{\text {ad }}$ such that $w\left(\nu+\Lambda_{0}\right)=-j \delta+\lambda+\Lambda_{0}$ and $w$ is minimal length. Then

$$
B(\lambda) \simeq B(\lambda)_{0} \times S^{\lambda} \quad \text { and } \quad B(\lambda)_{0} \simeq \mathbb{Z}^{k} \times B^{\mathrm{fin}}(\lambda) \quad \text { and } \quad B^{\mathrm{fin}}(\lambda) \simeq B\left(\nu+\Lambda_{0}\right)_{\leq w}
$$

where $k$ is the number of elements of $m_{1}, \ldots, m_{n}$ which are nonzero.
Additional useful references for Theorem [2.4 are [Nak02, Theorem 1] and [B02, Theorem 1]. The first two statements in Theorem [2.4] are reflections of the very important fact that $L(\lambda)$ is free as an $R G_{\lambda}$-module. This fact that $L(\lambda)$ is free as an $R G_{\lambda}$-module was deduced geometrically, via the quiver variety, in [Nak99, Theorem 7.3.5] (note that property $\left(T_{G_{\mathbf{w}} \times \mathbb{C}^{\times}}\right)$there includes the freeness, see the definition of property $\left(T_{G}\right)$ after Nak99, (7.1.1)]). This freeness was understood more algebraically in the work of Fourier-Littelmann [FL05] and Chari-Ion [CI13, Cor. 2.10]. Further understanding of the $R G_{\lambda}$-action in terms of the geometry of the semi-infinite flag variety is in [BF13, §5.1].

The last statement of Theorem [2.4 is proved by considering the map

$$
\begin{array}{cccc}
B(\lambda) & \longrightarrow & B\left(\Lambda_{0}\right) \otimes B(\lambda) \\
\cup \mid & \cup \mid & & \\
B^{\mathrm{fin}}(\lambda) & \xrightarrow{\sim} & B\left(\nu+\Lambda_{0}\right)_{\leq w}
\end{array} \quad \text { given by } \quad b \mapsto b_{\Lambda_{0}} \otimes b .
$$

where $b_{\Lambda_{0}}$ is the highest weight of the crystal $B\left(\Lambda_{0}\right)$. Combining the isomorphism $B^{\mathrm{fin}}(\lambda) \simeq$ $B\left(\nu+\Lambda_{0}\right)_{\leq w}$ with Theorem 1.3 and the positive level formula in $\$ 2.4$ gives

$$
\begin{gather*}
\operatorname{char}(M(a(u)))=\operatorname{char}\left(L\left(\nu+\Lambda_{0}\right)_{\leq w}\right)=q^{-j} X^{-\Lambda_{0}} \tilde{E}_{\lambda}(q, 0) \quad \text { and } \\
\operatorname{char}\left(L(\lambda)_{\leq w_{0}}\right)=\operatorname{gchar}\left(R G_{\lambda}^{+}\right) \tilde{E}_{w_{0} \lambda}(q, 0), \quad \operatorname{char}(L(\lambda))=\operatorname{gchar}\left(R G_{\lambda}\right) \tilde{E}_{w_{0} \lambda}(q, 0) . \tag{2.12}
\end{gather*}
$$

## 3 Examples for $\mathfrak{g}=\widehat{\mathfrak{s l}}_{2}$

Let $\mathfrak{g}=\widehat{\mathfrak{s l}}_{2}$ with $\mathfrak{h}^{*}=\mathbb{C} \omega_{1} \oplus \mathbb{C} \Lambda_{0} \oplus \mathbb{C} \delta$ with affine Cartan matrix

$$
\left(\begin{array}{ll}
\alpha_{0}\left(h_{0}\right) & \alpha_{0}\left(h_{1}\right) \\
\alpha_{1}\left(h_{0}\right) & \alpha_{1}\left(h_{1}\right)
\end{array}\right)=\left(\begin{array}{cc}
2 & -2 \\
-2 & 2
\end{array}\right) \quad \text { and } \quad \begin{array}{ll} 
& \theta=\alpha_{1}=2 \omega_{1}, \\
\Lambda_{1}=\omega_{1}+\Lambda_{0}, & \theta^{\vee}=\alpha_{1}^{\vee}=h_{1}, \\
\alpha_{0}=-\alpha_{1}+\delta,
\end{array}
$$

Using that $\left\langle\alpha_{1}, \alpha_{1}\right\rangle=2$, if $k \in \mathbb{Z}$ and $\mu^{\vee}=k \alpha_{1}^{\vee}=k \alpha_{1}=k 2 \omega_{1}=2 k \omega_{1}$ so that $\mu_{1}^{\vee}=2 k$ and $-\frac{1}{2}\left\langle\mu^{\vee}, \mu^{\vee}\right\rangle=-\frac{1}{2}(k 2 k)=-k^{2}$. Thus, following (1.8), (1.4) and (1.5), in the basis $\left\{\delta, \omega_{1}, \Lambda_{0}\right\}$ of $\mathfrak{h}^{*}$,

$$
t_{k \alpha^{\vee}}=\left(\begin{array}{ccc}
1 & -k & -k^{2} \\
0 & 1 & 2 k \\
0 & 0 & 1
\end{array}\right), \quad s_{1}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{array}\right), \quad s_{0}=\left(\begin{array}{ccc}
1 & 1 & -1 \\
0 & -1 & 2 \\
0 & 0 & 1
\end{array}\right) .
$$

These matrices are used to compute the $W$-orbits pictured in Section 0.1,

### 3.1 Macdonald polynomials

The Demazure operators are given by

$$
D_{1} f=\frac{f-X^{-2 \omega_{1}}\left(s_{1} f\right)}{1-X^{-2 \omega_{1}}} \quad \text { and } \quad D_{0} f=\frac{f-X^{-\alpha_{0}}\left(s_{0} f\right)}{1-X^{-\alpha_{0}}}=\frac{f-q^{-1} X^{2 \omega_{1}}\left(s_{0} f\right)}{1-q^{-1} X^{2 \omega_{1}}} .
$$

The normalized Macdonald polynomials $\tilde{E}_{\omega_{1}}(q, t)$ and $\tilde{E}_{-\omega_{1}}(q, t)$ are

$$
\left.\begin{aligned}
\nmid \mid<\downarrow & \mid
\end{aligned} \right\rvert\,
$$

giving $\tilde{E}_{\omega_{1}}(0, t)=\tilde{E}_{\omega_{1}}(\infty, t)=\tilde{E}_{\omega_{1}}(q, 0)=\tilde{E}_{\omega_{1}}(q, \infty)=X^{\omega_{1}}$ and

$$
\begin{array}{ll}
\tilde{E}_{-\omega_{1}}(0, t)=X^{-\omega_{1}}+(1-t) X^{\omega_{1}}, & \tilde{E}_{-\omega_{1}}(\infty, t)=X^{-\omega_{1}} \\
\tilde{E}_{-\omega_{1}}(q, 0)=X^{-\omega_{1}}+X^{\omega_{1}}, & \tilde{E}_{-\omega_{1}}(q, \infty)=X^{-\omega_{1}}+q^{-1} X^{\omega_{1}} .
\end{array}
$$

The normalized Macdonald polynomials $\tilde{E}_{2 \omega_{1}}(q, t)$ and $\tilde{E}_{-2 \omega_{1}}(q, t)$ are

$$
\begin{aligned}
& \perp \mid \rightarrow+\quad+\quad \downarrow 4 \\
& \tilde{E}_{2 \omega_{1}}(q, t)=X^{2 \omega_{1}} \quad+\frac{(1-t) q}{1-q t} \quad \text { and } \\
& =X^{2 \omega_{1}} \quad+\frac{1-t^{-1}}{1-q^{-1} t^{-1}} \\
& \tilde{E}_{-2 \omega_{1}}(q, t)=X^{-2 \omega_{1}} \\
& =X^{-2 \omega_{1}} \\
& +\frac{1-t}{1-q t} \\
& +\frac{\left(1-t^{-1}\right) q^{-1}}{1-q^{-1} t^{-1}} \\
& +\frac{1-t}{1-q^{2} t} X^{2 \omega_{1}} \\
& +\frac{(1-t)}{\left(1-q^{2} t\right)} \frac{(1-t) q}{(1-q t)} \\
& +\frac{\left(1-t^{-1}\right) q^{-2}}{1-q^{-2} t^{-1}} X^{2 \omega_{1}} \quad+\frac{\left(1-t^{-1}\right) q^{-2}}{\left(1-q^{-2} t^{-1}\right)} \frac{\left(1-t^{-1}\right)}{\left(1-q^{-1} t^{-1}\right)}
\end{aligned}
$$

giving

$$
\begin{array}{ll}
\tilde{E}_{2 \omega_{1}}(0, t)=X^{2 \omega_{1}}, & \tilde{E}_{2 \omega_{1}}(\infty, t)=X^{2 \omega_{1}}+\left(1-t^{-1}\right) \\
\tilde{E}_{2 \omega_{1}}(q, 0)=X^{2 \omega_{1}}+q, & \tilde{E}_{2 \omega_{1}}(q, \infty)=X^{2 \omega_{1}}+1
\end{array}
$$

and

$$
\begin{array}{ll}
\tilde{E}_{-2 \omega_{1}}(0, t)=X^{-2 \omega_{1}}+(1-t) X^{2 \omega_{1}}+(1-t), & \tilde{E}_{-2 \omega_{1}}(\infty, t)=X^{-2 \omega_{1}}, \\
\tilde{E}_{-2 \omega_{1}}(q, 0)=X^{-2 \omega_{1}}+X^{2 \omega_{1}}+1+q, & \tilde{E}_{-2 \omega_{1}}(q, \infty)=X^{-2 \omega_{1}}+q^{-2} X^{2 \omega_{1}}+q^{-1}+q^{-2} .
\end{array}
$$

3.2 The crystal $B\left(\Lambda_{0}\right)$ and $B\left(\omega_{1}+\Lambda_{0}\right)$


Initial portion of the crystal graph of $B\left(\Lambda_{0}\right)$ for $\widehat{\mathfrak{s}}_{2}$
The characters of the first few Demazure modules in $L\left(\Lambda_{0}\right)$ are

$$
\begin{aligned}
\operatorname{char}\left(L\left(\Lambda_{0}\right)_{\leq 1}\right) & =X^{\Lambda_{0}}=X^{\Lambda_{0}} \tilde{E}_{0}\left(q^{-1}, 0\right) \\
\operatorname{char}\left(L\left(\Lambda_{0}\right)_{\leq s_{0}}\right) & =D_{0} X^{\Lambda_{0}}=X^{\Lambda_{0}}\left(1+q X^{2 \omega_{1}}\right)=q X^{\Lambda_{0}}\left(X^{2 \omega_{1}}+q^{-1}\right)=q X^{\Lambda_{0}} \tilde{E}_{2 \omega_{1}}\left(q^{-1}, 0\right), \\
\operatorname{char}\left(L\left(\Lambda_{0}\right)_{\leq s_{1} s_{0}}\right) & =D_{1} D_{0} X^{\Lambda_{0}}=X^{\Lambda_{0}}\left(1+q X^{2 \omega_{1}}+q+q X^{-2 \omega_{1}}\right) \\
& =q X^{\Lambda_{0}}\left(X^{-2 \omega_{1}}+X^{2 \omega_{1}}+1+q^{-1}\right)=q X^{\Lambda_{0}} \tilde{E}_{-2 \omega_{1}}\left(q^{-1}, 0\right) .
\end{aligned}
$$

The crystal graph of $B\left(\omega_{1}+\Lambda_{0}\right)$ is pictured in Plate B and

$$
\begin{aligned}
\operatorname{char}\left(L\left(\omega_{1}+\Lambda_{0}\right)_{\leq 1}\right) & =X^{\Lambda_{0}} X^{\omega_{1}}=X^{\Lambda_{0}} \tilde{E}_{\omega_{1}}\left(q^{-1}, 0\right), \\
\operatorname{char}\left(L\left(\omega_{1}+\Lambda_{0}\right)_{\leq s_{1}}\right) & =D_{1} X^{\omega_{1}+\Lambda_{0}}=X^{\Lambda_{0}}\left(X^{\omega_{1}}+X^{-\omega_{1}}\right)=X^{\Lambda_{0}} \tilde{E}_{-\omega_{1}}\left(q^{-1}, 0\right), \\
\operatorname{char}\left(L\left(\omega_{1}+\Lambda_{0}\right)_{\leq s_{0} s_{1}}\right) & =D_{0} D_{1} X^{\omega_{1}+\Lambda_{0}}=X^{\Lambda_{0}}\left(X^{\omega_{1}}+X^{-\omega_{1}}+q X^{\omega_{1}}+q^{2} X^{3 \omega_{1}}\right) \\
& =q^{2} X^{\Lambda_{0}} \tilde{E}_{3 \omega_{1}}\left(q^{-1}, 0\right),
\end{aligned}
$$

### 3.3 The crystal $B\left(\omega_{1}\right)$

The crystal $B\left(\omega_{1}\right)=\left\{p_{-\omega_{1}+k \delta}, p_{\omega_{1}+k \delta} \mid k \in \mathbb{Z}\right\}$ is a single connected component. The crystal graph of $B\left(\omega_{1}\right)$ is pictured in Plate B. Following [Kac, (12.1.9)] and putting $q=X^{-\delta}$ noting
that $\tilde{E}_{-\omega_{1}}\left(q^{-1}, 0\right)=X^{\omega_{1}}+X^{-\omega_{1}}$ and $\tilde{E}_{-\omega_{1}}\left(q^{-1}, \infty\right)=X^{-\omega_{1}}+q X^{\omega_{1}}$, then

$$
\begin{aligned}
\operatorname{char}\left(L\left(\omega_{1}\right)_{\leq s_{1}}\right) & =\frac{1}{1-q^{-1}} \tilde{E}_{-\omega_{1}}\left(q^{-1}, 0\right) \quad \text { and } \\
\operatorname{char}\left(L\left(\omega_{1}\right)\right) & =0_{q} \tilde{E}_{-\omega_{1}}\left(q^{-1}, 0\right)=0_{q} \tilde{E}_{-\omega_{1}}\left(q^{-1}, \infty\right)
\end{aligned}
$$

where $0_{q}=\cdots+q^{-3}+q^{-2}+q^{-1}+1+q+q^{2}+\cdots$ as in Remark 2.2,

### 3.4 The crystal $B\left(2 \omega_{1}\right)$

On crystals, the injective $\mathbf{U}$-module homomorphism $L\left(2 \omega_{1}\right) \hookrightarrow L\left(\omega_{1}\right) \otimes L\left(\omega_{1}\right)$ given in (2.11) is the inclusion

$$
\begin{aligned}
B\left(\omega_{1}\right) \otimes B\left(\omega_{1}\right) & =\left\{p_{w_{1} \omega_{1}} \otimes p_{w_{2} \omega_{1}} \mid w_{1}, w_{2} \in W^{\text {ad }}\right\} \\
\cup \mid & =\left\{p_{w_{1} \omega_{1}} \otimes p_{w_{2} \omega_{1}} \mid w_{1}, w_{2} \in W^{\text {ad }} \text { with } w_{1} \geq w_{2}\right\}
\end{aligned}
$$

The connected components of $B\left(2 \omega_{1}\right)$ are determined by

$$
B\left(2 \omega_{1}\right)=B\left(2 \omega_{1}\right)_{0} \times S^{2 \omega_{1}}, \quad \text { where } \quad S^{2 \omega_{1}}=\{\text { partitions } \kappa \text { with } \ell(\kappa)<2\}=\mathbb{Z}_{\geq 0}
$$

For $\kappa \in \mathbb{Z}_{\geq 0}$, the connected component corresponding to $\kappa$, as a subset of $B\left(\omega_{1}\right) \otimes B\left(\omega_{1}\right)$, is

$$
B\left(2 \omega_{1}\right)_{\kappa}=\left\{p_{w_{1} \omega_{1}} \otimes p_{w_{2} \omega_{1}} \mid w_{1} \unrhd w_{2} \text { and } \ell^{0}\left(w_{1}\right)-\ell^{0}\left(w_{2}\right) \in\{\kappa, \kappa+1\}\right\}
$$

Representatives of the components and the crystal graph of the connected component $B\left(2 \omega_{1}\right)_{0}$ are pictured in Plate C. Inspection of the crystal graphs gives

$$
\begin{aligned}
\operatorname{char}\left(L\left(2 \omega_{1}\right)_{\leq s_{1}}\right) & =\frac{1}{1-q^{-2}} \frac{1}{1-q^{-1}} \tilde{E}_{-2 \omega_{1}}\left(q^{-1}, 0\right), \quad \text { and } \\
\operatorname{char}\left(L\left(2 \omega_{1}\right)\right) & =0_{q} \frac{1}{1-q} \tilde{E}_{-2 \omega_{1}}\left(q^{-1}, 0\right)=0_{q} \frac{1}{1-q} \tilde{E}_{-2 \omega_{1}}\left(q^{-1}, \infty\right)
\end{aligned}
$$

## 4 Examples for $\mathfrak{g}=\widehat{\mathfrak{s l}}_{3}$

Let $\mathfrak{g}=\widehat{\mathfrak{s}}_{3}$. The affine Dynkin diagram is

$$
\left(\begin{array}{lll}
\alpha_{0}\left(h_{0}\right) & \alpha_{0}\left(h_{1}\right) & \alpha_{0}\left(h_{2}\right) \\
\alpha_{1}\left(h_{0}\right) & \alpha_{1}\left(h_{1}\right) & \alpha_{1}\left(h_{2}\right) \\
\alpha_{2}\left(h_{0}\right) & \alpha_{2}\left(h_{1}\right) & \alpha_{2}\left(h_{2}\right)
\end{array}\right)=\left(\begin{array}{ccc}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{array}\right)
$$

and $h_{0}=-\left(h_{1}+h_{2}\right)+K, \quad \alpha_{0}=-\left(\alpha_{1}+\alpha_{2}\right)+\delta, \quad \Lambda_{1}=\omega_{1}+\Lambda_{0}, \quad \Lambda_{2}=\omega_{2}+\Lambda_{0}$.
In the basis $\left\{\delta, \omega_{1}, \omega_{2}, \Lambda_{0}\right\}$ of $\mathfrak{h}^{*}$ the action of the affine Weyl group $W^{\text {ad }}$ is given by

$$
\begin{gathered}
s_{1}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) \quad s_{2}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) \quad s_{0}=\left(\begin{array}{cccc}
1 & 1 & 1 & -1 \\
0 & 0 & -1 & 1 \\
0 & -1 & 0 & 1 \\
0 & 0 & 0 & 1
\end{array}\right), \quad \text { and } \\
t_{k_{1} h_{1}+k_{2} h_{2}}=\left(\begin{array}{cccc}
1 & -k_{1} & -k_{2} & -k_{1}^{2}-k_{2}^{2}+k_{1} k_{2} \\
0 & 1 & 0 & 2 k_{1}-k_{2} \\
0 & 0 & 1 & 2 k_{2}-k_{1} \\
0 & 0 & 0 & 1
\end{array}\right) \quad \text { for } k_{1}, k_{2} \in \mathbb{Z} .
\end{gathered}
$$

These matrices are used to compute the orbits pictured at the end of Section 0.1.

### 4.1 The extremal weight modules $L\left(\omega_{1}\right)$ and $L\left(\omega_{2}\right)$

Letting $\mathbb{C}^{3}=\mathbb{C}$-span $\left\{v_{1}, v_{2}, v_{3}\right\}$ be the standard representation of $\mathfrak{g}=\mathfrak{s l}_{3}$, the extremal weight modules $L\left(\omega_{1}\right)$ and $L\left(\omega_{2}\right)$ for $\mathfrak{g}=\widehat{\mathfrak{s l}}{ }_{3}$ are

$$
L\left(\omega_{1}\right)=\mathbb{C}^{3} \otimes_{\mathbb{C}} \mathbb{C}\left[\epsilon, \epsilon^{-1}\right] \quad \text { and } \quad L\left(\omega_{2}\right)=\left(\Lambda^{2} \mathbb{C}^{3}\right) \otimes_{\mathbb{C}} \mathbb{C}\left[\epsilon, \epsilon^{-1}\right]
$$

with $u_{\omega_{1}}=v_{1}$ and $u_{\omega_{2}}=v_{1} \wedge v_{2}$, respectively. The crystals $B\left(\omega_{1}\right)$ and $B\left(\omega_{2}\right)$ have realizations as sets of straight line paths:
$B\left(\omega_{1}\right)=\left\{p_{\omega_{1}+k \delta}, p_{s_{1} \omega_{1}+k \delta}, p_{-\omega_{2}+k \delta} \mid k \in \mathbb{Z}\right\}, \quad B\left(\omega_{2}\right)=\left\{p_{\omega_{2}+k \delta}, p_{s_{2} \omega_{2}+k \delta}, p_{-\omega_{1}+k \delta} \mid k \in \mathbb{Z}\right\}$.

$B\left(\omega_{1}\right)$ crystal graph

$B\left(\omega_{2}\right)$ crystal graph

Each of the crystals $B\left(\omega_{1}\right)$ and $B\left(\omega_{2}\right)$ has a single connected component, all weight spaces are one dimensional and

$$
\begin{aligned}
\operatorname{char}\left(L\left(\omega_{1}\right)_{\leq s_{2} s_{1}}\right) & =\frac{1}{1-q^{-1}}\left(X^{\omega_{1}}+X^{s_{1} \omega_{1}}+X^{-\omega_{2}}\right)=\frac{1}{1-q^{-1}} \tilde{E}_{-\omega_{2}}\left(q^{-1}, 0\right), \quad \text { and } \\
\operatorname{char}\left(L\left(\omega_{1}\right)\right) & =0_{q}\left(X^{\omega_{1}}+X^{-\omega_{2}}+X^{s_{1} \omega_{1}}\right)=0_{q} \tilde{E}_{-\omega_{2}}\left(q^{-1}, 0\right) \\
& =0_{q}\left(q^{-1} X^{\omega_{1}}+q^{-1} X^{\omega_{2}-\omega_{1}}+X^{-\omega_{2}}\right)=0_{q} \tilde{E}_{-\omega_{2}}\left(q^{-1}, \infty\right) .
\end{aligned}
$$

where $0_{q}=\cdots+q^{-3}+q^{-2}+q^{-1}+1+q+q^{2}+\cdots$ as in Remark 2.2,
For $a \in \mathbb{C}$, the crystals of $M^{\text {fin }}\left((u-a) \omega_{1}\right) \cong \mathbb{C}^{3}$ and $M^{\text {fin }}\left((u-a) \omega_{2}\right) \cong \Lambda^{2}\left(\mathbb{C}^{3}\right)$ have crystal graphs $B^{\text {fin }}\left(\omega_{1}\right)$ and $B^{\text {fin }}\left(\omega_{2}\right)$.


### 4.2 The extremal weight module $L\left(\omega_{1}+\omega_{2}\right)$

To construct the crystal $B\left(\omega_{1}+\omega_{2}\right)$ use

$$
B\left(\omega_{1}\right) \otimes B\left(\omega_{2}\right)=\left\{p_{v \omega_{1}+k \delta} \otimes p_{w \omega_{2}+\ell \delta} \mid v \in\left\{1, s_{1}, s_{2} s_{1}\right\}, w \in\left\{1, s_{2}, s_{1} s_{2}\right\}, k \in \mathbb{Z}, \ell \in \mathbb{Z}\right\}
$$

with the tensor product action for crystals given by (see, for example, Ra06, Prop. 5.7])
$\tilde{f}_{i}\left(p_{1} \otimes p_{2}\right)=\left\{\begin{array}{ll}\tilde{f}_{i} p_{1} \otimes p_{2}, & \text { if } d_{i}^{+}\left(p_{1}\right)>d_{i}^{-}\left(p_{2}\right), \\ p_{1} \otimes \tilde{f}_{i} p_{2}, & \text { if } d_{i}^{+}\left(p_{1}\right) \leq d_{i}^{-}\left(p_{2}\right),\end{array} \quad \tilde{e}_{i}\left(p_{1} \otimes p_{2}\right)= \begin{cases}\tilde{e}_{i} p_{1} \otimes p_{2}, & \text { if } d_{i}^{+}\left(p_{1}\right) \geq d_{i}^{-}\left(p_{2}\right), \\ p_{1} \otimes \tilde{e}_{i} p_{2}, & \text { if } d_{i}^{+}\left(p_{1}\right)<d_{i}^{-}\left(p_{2}\right) .\end{cases}\right.$
where $d_{i}^{ \pm}(p)$ are determined by

$$
\tilde{f}_{i}^{d_{i}^{+}(p)} p \neq 0 \text { and } \tilde{f}_{i}^{d_{i}^{+}(p)+1} p=0, \quad \text { and } \quad \tilde{e}_{i}^{d_{i}^{-}(p)} p \neq 0 \text { and } \tilde{e}_{i}^{d_{i}^{-}(p)+1} p=0 .
$$

The crystal $B\left(\omega_{1}+\omega_{2}\right)$ is realized as a subset of $B\left(\omega_{1}\right) \otimes B\left(\omega_{2}\right)$ via the crystal embedding

$$
\begin{aligned}
B\left(\omega_{1}+\omega_{2}\right) & \hookrightarrow \\
b_{\omega_{1}+\omega_{2}} & \longmapsto \\
\longmapsto & \left.p_{\omega_{1}} \otimes \omega_{\omega_{2}}\right) \otimes B\left(\omega_{2}\right)
\end{aligned}
$$

By Theorem 2.4, $B\left(\omega_{1}+\omega_{2}\right)$ is connected and is generated by $b_{\omega_{1}+\omega_{2}}$. The crystal $B\left(\omega_{1}+\omega_{2}\right)$ is pictured and its character is computed in Plate D.

## 5 Alcove walks for affine flag varieties

### 5.1 The affine Kac-Moody group $G$

The most visible form of the loop group is $G=\dot{G}(\mathbb{C}((\epsilon)))$, where $\mathbb{C}((\epsilon))$ is the field of formal power series in a variable $\epsilon$ and $\dot{G}$ is a reductive algebraic group. The favourite example is when

$$
\text { when } \quad \dot{G}=G L_{n} \quad \text { and the loop group is } \quad G=G L_{n}(\mathbb{C}((\epsilon))) \text {, }
$$

the group of $n \times n$ invertible matrices with entries in $\mathbb{C}((\epsilon))$. A slightly more extended, and extremely powerful, point of view is to let $G$ be the Kac-Moody group whose Lie algebra is the affine Lie algebra $\mathfrak{g}$ of Section 1.1, so that $G$ is a central extension of a semidirect product (where the semidirect product comes from the action of $\mathbb{C}^{\times}$on $\dot{G}(\mathbb{C}((\epsilon))$ by "loop rotations")

$$
\begin{equation*}
\{1\} \rightarrow \mathbb{C}^{\times} \rightarrow G \rightarrow \dot{G}(\mathbb{C}((\epsilon))) \rtimes \mathbb{C}^{\times} \rightarrow\{1\} \quad \text { so that } \quad G=\exp (\mathfrak{g}) \tag{5.1}
\end{equation*}
$$

In this section we wish to work with $G$ via generators and relations. Fortunately, presentations of $G$ are well established in the work of Steinberg [St67] and Tits Ti87] and others (see [PRS, §3] for a survey). More specifically, up to an extra (commutative) torus $T \cong\left(\mathbb{C}^{\times}\right)^{k}$ for some $k$, the group $G$ is generated by subgroups isomorphic to $S L_{2}(\mathbb{C})$, the images of homomorphisms

$$
\begin{array}{rlll}
\varphi_{i}: & & \\
S L_{2}(\mathbb{C}) & \longrightarrow & G \\
\left(\begin{array}{cc}
1 & c \\
0 & 1
\end{array}\right) & \longmapsto & x_{\alpha_{i}}(c) &  \tag{5.2}\\
\left(\begin{array}{cc}
1 & 0 \\
c & 1
\end{array}\right) & \longmapsto & x_{-\alpha_{i}}(c) & \\
\text { for each vertex } \quad i \\
\left(\begin{array}{cc}
d & 0 \\
0 & d^{-1}
\end{array}\right) & \longmapsto & h_{\alpha_{i}^{\vee}}(d) & \\
\left(\begin{array}{cc}
c & 1 \\
-1 & 0
\end{array}\right) & \longmapsto & y_{i}(c)
\end{array}
$$

For each $\alpha \in \stackrel{\circ}{R}^{+}$there is a homomorphism $\varphi_{\alpha}: S L_{2}(\mathbb{C}((\epsilon))) \rightarrow G$ and we let

$$
x_{\alpha}(f)=\varphi_{\alpha}\left(\begin{array}{cc}
1 & f  \tag{5.3}\\
0 & 1
\end{array}\right) \quad \text { and } \quad x_{-\alpha}(f)=\varphi_{\alpha}\left(\begin{array}{ll}
1 & 0 \\
f & 1
\end{array}\right) \quad \text { for } f \in \mathbb{C}((\epsilon))
$$

For each $z \in W^{\text {ad }}$ fix a reduced word $z=s_{j_{1}} \cdots s_{j_{r}}$, define

$$
\begin{equation*}
n_{z}=y_{j_{1}}(0) \cdots y_{j_{r}}(0) \quad \text { and define } \quad x_{ \pm \alpha+r \delta}(c)=x_{ \pm \alpha}\left(c \epsilon^{r}\right) \quad \text { for } \alpha \in \stackrel{\circ}{R}^{+} \text {and } r \in \mathbb{Z} \tag{5.4}
\end{equation*}
$$

In the following sections, for simplicity, we will work only with the loop group $G=\dot{G}(\mathbb{C}((\epsilon)))$ rather than the full affine Kac-Moody group as in (5.1). We shall also use a slightly more general setting where $\mathbb{C}$ is replaced by an arbitrary field $\mathbb{k}$ so that, in Proposition5.4, we may let $\mathbb{k}=\mathbb{F}_{q}$ be the finite field with $q$ elements.

### 5.2 The affine flag varieties $G / I^{+}, G / I^{0}$ and $G / I^{-}$

Let $\mathbb{k}$ be a field. Define subgroups of $\dot{G}(\mathbb{k})$ by

$$
\begin{array}{cl}
\text { "unipotent upper triangular matrices" } & U^{+}(\mathbb{k})=\left\langle x_{\alpha}(c) \mid \alpha \in \stackrel{\circ}{R}^{+}, c \in \mathbb{k}\right\rangle, \\
\text { "diagonal matrices" } & H\left(\mathbb{k}^{\times}\right)=\left\langle h_{\alpha_{i}^{\vee}}(c) \mid i \in\{1, \ldots, n\}, c \in \mathbb{k}^{\times}\right\rangle, \\
\text {"unipotent lower triangular matrices" } & \left.\left.U^{-}(\mathbb{k})=\left\langle x_{-\alpha}(c)\right| \alpha \in \stackrel{\circ}{R}^{+}, c \in \mathbb{k}\right\}\right\rangle .
\end{array}
$$

Let

$$
\begin{array}{rlll} 
& \mathbb{F}=\mathbb{k}((\epsilon)), \quad \text { which has } & \mathfrak{o}=\mathbb{k}[[\epsilon]] \text { and } & \mathfrak{o}^{\times}=\{p \in \mathbb{k}[[\epsilon]] \mid p(0) \neq 0\}, \\
\text { or let } & \mathbb{F}=\mathbb{k}\left[\epsilon, \epsilon^{-1}\right], \quad \text { which has } & \mathfrak{o}=\mathbb{k}[\epsilon] \text { and } & \mathfrak{o}^{\times}=\mathbb{k}^{\times} .
\end{array}
$$

Define subgroups of $G=\dot{G}(\mathbb{F})$ by

$$
U^{-}(\mathbb{F})=\left\langle x_{-\alpha+k \delta}(c) \mid c \in \mathbb{k}, \alpha \in \stackrel{\circ}{R}^{+}, k \in \mathbb{Z}\right\rangle \quad \text { and } \quad H\left(\mathfrak{o}^{\times}\right)=\left\langle h_{\alpha_{i}^{\vee}}(c) \mid c \in \mathfrak{o}^{\times}\right\rangle
$$

Let $g(0)$ denote $g$ evaluated at $\epsilon=0$ and let $g(\infty)$ denote $g$ evaluated at $\epsilon^{-1}=0$. Following, for example, Pet97, Lect. 11 Theorem (B)] and [uICM, §11], define subgroups of $G(\mathbb{F})$ by

$$
\begin{aligned}
\text { (positive Iwahori) } & I^{+}=\left\{g \in G \mid g(0) \text { exists and } g(0) \in U^{+}(\mathbb{k}) H\left(\mathbb{k}^{\times}\right)\right\} \\
\text {(level 0 Iwahori) } & I^{0}=U^{-}(\mathbb{F}) H\left(\mathfrak{o}^{\times}\right) \\
\text {(negative Iwahori) } & I^{-}=\left\{g \in G \mid g(\infty) \text { exists and } g(\infty) \in U^{-}(\mathbb{k}) H\left(\mathbb{k}^{\times}\right)\right\}
\end{aligned}
$$

Then

$$
\begin{aligned}
& G / I^{+} \text {is the positive level (thin) affine flag variety, } \\
& G / I^{0} \text { is the level } 0 \text { (semi-infinite) affine flag variety, } \\
& G / I^{-} \text {is the negative level (thick) affine flag variety. }
\end{aligned}
$$

These are studied with the aid of the decompositions (relax notation and write $z I^{+}$for $n_{z} I^{+}$)

$$
G=\bigsqcup_{x \in W^{\mathrm{ad}}} I^{+} x I^{+}, \quad G=\bigsqcup_{y \in W^{\mathrm{ad}}} I^{0} y I^{+}, \quad G=\bigsqcup_{z \in W^{\mathrm{ad}}} I^{-} z I^{+}
$$

### 5.3 Labeling points of $I^{+} w I^{+}, I^{+} w I^{+} \cap I^{0} v I^{+}$and $I^{+} w I^{+} \cap I^{-} z I^{+}$

Recall the bijection between elements of $W^{\text {ad }}$ and alcoves given in (1.6) and note that if $v \in W^{\text {ad }}$ and $i \in\{0, \ldots, n\}$ then the hyperplane separating the alcoves corresponding to $v$ and $s_{i} v$ is

$$
\mathfrak{h}^{v \alpha_{i}^{\vee}}=\left\{x+\Lambda_{0} \mid x \in \mathfrak{a}_{\mathbb{R}}^{*} \text { and }\left\langle x+\Lambda_{0}, v \alpha_{i}^{\vee}\right\rangle=0\right\} .
$$

A blue labeled step of type $i$ is
where $y_{i}(c)$ is as in (5.2). A blue labeled walk of type $\vec{w}=\left(i_{1}, \ldots, i_{\ell}\right)$ is a sequence $\left(p_{1}, \ldots, p_{\ell}\right)$ where $p_{k}$ is a blue labeled step of type $i_{k}$ and which begins at the alcove 1 .
A red labeled step of type $i$ is

$c \in \mathbb{k}$
or

or $\left.\quad v\right|_{\underset{c^{-1}}{\stackrel{v}{\hookrightarrow}}} ^{\mathfrak{h}_{i}^{-v \alpha_{i}^{\vee}}} \quad$ where $\quad v s_{i} \xrightarrow{\circ} v$.
$c \in \mathbb{K}^{\times}$

With notations as in (5.4), let

$$
\Phi^{0}\left(\right)=x_{v \alpha_{i}}(c), \quad \Phi^{0}\left(\right)=x_{-v \alpha_{i}}(0), \quad \Phi^{0}\left(\left.\begin{array}{c}
\mathfrak{h}^{-v \alpha_{i}^{\vee}} \\
v
\end{array} \right\rvert\, \begin{array}{c}
v s_{i} \\
c^{-1}
\end{array}\right)=x_{-v \alpha_{i}}\left(c^{-1}\right) .
$$

A red labeled walk of type $\vec{w}=\left(i_{1}, \ldots, i_{\ell}\right)$ is a sequence $\left(p_{1}, \ldots, p_{\ell}\right)$ where $p_{k}$ is a red labeled step of type $i_{k}$ and which begins at the alcove 1 .
A green labeled step of type $i$ is

| $\mathfrak{h}^{v \alpha_{i}^{v}}$ |  |  | $\mathfrak{h}^{-v \alpha_{i}^{V}}$ |  |  | $\mathfrak{h}^{-v \alpha_{i}^{\vee}}$ |  | where | $v s_{i} \geq v$. |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v$ | $v s_{i}$ | or | $v$ | $v s_{i}$ | or | $v$ |  |  |  |
|  | $\vec{c}$ |  |  | 0 |  | $\overline{c^{-1}}$ |  |  |  |
|  | $\mathfrak{k}$ |  |  |  |  |  | $\in \mathbb{k}^{\times}$ |  |  |

Let

$$
\Phi^{-}\left(\right)=x_{v \alpha_{i}}(c), \quad \Phi^{-}\left(\begin{array}{c|c}
\mathfrak{h}^{-v \alpha_{i}^{\vee}} \\
v & v s_{i} \\
\hdashline 0 &
\end{array}\right)=x_{-v \alpha_{i}}(0), \quad \Phi^{-}\left(\begin{array}{c|c}
\mathfrak{h}^{-v \alpha_{i}^{\vee}} \\
v & v s_{i} \\
\hline & \stackrel{y y}{c \mid}
\end{array}\right)=x_{-v \alpha_{i}}\left(c^{-1}\right) .
$$

A green labeled walk of type $\vec{w}=\left(i_{1}, \ldots, i_{\ell}\right)$ is a sequence $\left(p_{1}, \ldots, p_{\ell}\right)$ where $p_{k}$ is a green labeled step of type $i_{k}$ and which begins at the alcove 1 .

Theorem 5.1. Let $v, w \in W^{\text {ad }}$ and fix a reduced expression $w=s_{i_{1}} \ldots s_{i_{\ell}}$ for $w$. The maps $\Psi_{\vec{w}}^{+}, \Psi_{\vec{w}, v}^{0}, \Psi_{\vec{w}, v}^{-}$are bijections.

$$
\begin{aligned}
& \Phi_{\vec{w}}^{+}: \quad\left\{\begin{array}{l}
\text { blue labeled paths of type } \\
\vec{w}=\left(i_{1}, \ldots, i_{\ell}\right)
\end{array}\right\} \quad \stackrel{\sim}{\longrightarrow} \quad\left(I^{+} w I^{+}\right) / I^{+} \\
& \left(p_{1}, \ldots, p_{\ell}\right) \quad \longmapsto \quad \Phi^{+}\left(p_{1}\right) \ldots \Phi^{+}\left(p_{\ell}\right) I^{+} \\
& \Phi_{\vec{w}, v}^{0}:\left\{\begin{array}{l}
\text { red labeled paths of type } \\
\vec{w}=\left(i_{1}, \ldots, i_{\ell}\right) \text { ending in } v
\end{array}\right\} \quad \sim \quad\left(I^{0} v I^{+} \cap I^{+} w I^{+}\right) / I^{+} \\
& \left(p_{1}, \ldots, p_{\ell}\right) \quad \longmapsto \quad \Phi^{0}\left(p_{1}\right) \ldots \Phi^{0}\left(p_{\ell}\right) n_{v} I^{+} \\
& \Phi_{\vec{w}, v}^{-}: \quad\left\{\begin{array}{l}
\text { green labeled paths of type } \\
\vec{w}=\left(i_{1}, \ldots, i_{\ell}\right) \text { ending in } v
\end{array}\right\} \quad \xrightarrow{\sim}\left(I^{-} v I^{+} \cap I^{+} w I^{+}\right) / I^{+} \\
& \left(p_{1}, \ldots, p_{\ell}\right) \quad \longmapsto \Phi^{-}\left(p_{1}\right) \ldots \Phi^{-}\left(p_{\ell}\right) n_{v} I^{+}
\end{aligned}
$$

The proof of Theorem 5.1 is by induction on the length of $w$ following PRS, Theorem 4.1 and $\S 7$ ] where (a) and (b) are proved. The induction step can be formulated as the following proposition.

Proposition 5.2. Let $v, w \in W^{\text {ad }}$ and fix a reduced expression $\vec{w}=s_{i_{1}} \ldots s_{i_{\ell}}$ for $w$. Let $j \in\{0, \ldots, n\}$ and $c \in \mathbb{C}$. If $w s_{j}<+w$ then assume that the reduced word for $w$ is chosen with $i_{\ell}=j$. Let

$$
\tilde{c} \in \mathbb{C} \text { and } \tilde{b}_{1} \in I^{+} \quad \text { be the unique elements such that } \quad b_{1} y_{j}(c)=y_{j}(\tilde{c}) \tilde{b}_{1}
$$

(a) Let $p=\left(p_{1}, \ldots, p_{\ell}\right)$ be a blue labeled path of type $\left(i_{1}, \ldots, i_{\ell}\right)$ and let $\Phi_{\vec{w}}^{+}(p)=y_{i_{1}}\left(c_{1}\right) \ldots y_{i_{\ell}}\left(c_{\ell}\right)$. Then

$$
\begin{aligned}
& \left(y_{i_{1}}\left(c_{1}\right) \cdots y_{i_{\ell}}\left(c_{\ell}\right) b_{1}\right)\left(y_{j}(c) b_{2}\right) \\
& = \begin{cases}y_{i_{1}}\left(c_{1}\right) \cdots y_{i_{\ell}}\left(c_{\ell}\right) y_{j}(\tilde{c}) \tilde{b}_{1} b_{2}, & \text { if } w<+w s_{j}, \\
y_{i_{1}}\left(c_{1}\right) \cdots y_{i_{\ell-1}}\left(c_{\ell-1}\right) y_{i_{\ell}}\left(c_{\ell}-\tilde{c}^{-1}\right) h_{\alpha_{1}^{\vee}}(\tilde{c}) x_{\alpha_{i_{\ell}}}\left(-\tilde{c}^{-1}\right) \tilde{b}_{1} b_{2}, & \text { if } w s_{j}<w \text { and } \tilde{c} \neq 0, \\
y_{i_{1}}\left(c_{1}\right) \cdots y_{i_{\ell-1}}\left(c_{\ell-1}\right) h_{\alpha_{i_{\ell}}^{\vee}}(-1) x_{\alpha_{i_{\ell}}}(c) \tilde{b}_{1} b_{2}, & \text { if } w s_{j}<w \text { and } \tilde{c}=0,\end{cases}
\end{aligned}
$$

(b) Let $p=\left(p_{1}, \ldots, p_{\ell}\right)$ be a red labeled path of type $\left(i_{1}, \ldots, i_{\ell}\right)$ ending in $v$ and let $\Phi_{\vec{w}, v}^{0}(p)=x_{\gamma_{1}}\left(c_{1}\right) \cdots x_{\gamma_{\ell}}\left(c_{\ell}\right) n_{v} I^{+}$where the notation is as in (5.4). Then

$$
\begin{aligned}
& \left(x_{\gamma_{1}}\left(c_{1}\right) \cdots x_{\gamma_{\ell}}\left(c_{\ell}\right) n_{v} b_{1}\right)\left(y_{j}(c) b_{2}\right) \\
& = \begin{cases}x_{\gamma_{1}}\left(c_{1}\right) \cdots x_{\gamma_{\ell}}\left(c_{\ell}\right) x_{v \alpha_{j}}( \pm \tilde{c}) n_{v s_{j}} \tilde{b}_{1} b_{2} . & \text { if } w<w s_{j}, \\
x_{\gamma_{1}}\left(c_{1}\right) \cdots x_{\gamma_{\ell}}\left(c_{\ell}\right) x_{-v \alpha_{j}}\left(\tilde{c}^{-1}\right) n_{v} x_{\alpha_{j}}(-\tilde{c}) h_{\alpha_{j}^{\vee}}(\tilde{c}) \tilde{b}_{1} b_{2}, & \text { if } w s_{j}<w \text { and } \tilde{c} \neq 0, \\
x_{\gamma_{1}}\left(c_{1}\right) \cdots x_{\gamma_{\ell}}\left(c_{\ell}\right) x_{-v \alpha_{j}}(0) n_{v s_{j}} \tilde{b}_{1} b_{2}, & \text { if } w s_{j}<0 w \text { and } \tilde{c}=0\end{cases}
\end{aligned}
$$

(c) Let $p=\left(p_{1}, \ldots, p_{\ell}\right)$ be a green labeled path of type $\left(i_{1}, \ldots, i_{\ell}\right)$ ending in $v$ and let $\Phi_{\vec{w}, v}^{-}(p)=x_{\gamma_{1}}\left(c_{1}\right) \cdots x_{\gamma_{\ell}}\left(c_{\ell}\right) n_{v} I^{+}$where the notation is as in (5.4). Then

$$
\begin{aligned}
& \left(x_{\gamma_{1}}\left(c_{1}\right) \cdots x_{\gamma_{\ell}}\left(c_{\ell}\right) n_{v} b_{1}\right)\left(y_{j}(c) b_{2}\right) \\
& = \begin{cases}x_{\gamma_{1}}\left(c_{1}\right) \cdots x_{\gamma_{\ell}}\left(c_{\ell}\right) x_{v \alpha_{j}}( \pm \tilde{c}) n_{v s_{j}} \tilde{b}_{1} b_{2} . & \text { if } w<w s_{j}, \\
x_{\gamma_{1}}\left(c_{1}\right) \cdots x_{\gamma_{\ell}}\left(c_{\ell}\right) x_{-v \alpha_{j}}\left(\tilde{c}^{-1}\right) n_{v} x_{\alpha_{j}}(-\tilde{c}) h_{\alpha_{j}^{\vee}}(\tilde{c}) \tilde{b}_{1} b_{2}, & \text { if } w s_{j}<w \text { and } \tilde{c} \neq 0, \\
x_{\gamma_{1}}\left(c_{1}\right) \cdots x_{\gamma_{\ell}}\left(c_{\ell}\right) x_{-v \alpha_{j}}(0) n_{v s_{j}} \tilde{b}_{1} b_{2}, & \text { if } w s_{j}<w \text { and } \tilde{c}=0,\end{cases}
\end{aligned}
$$

### 5.4 Actions of the affine Hecke algebra

The affine flag representation is

$$
C\left(G / I^{+}\right)=\mathbb{C} \text {-span }\left\{y_{\vec{w}}(\vec{c}) I^{+} \mid w \in W^{\text {ad }}, \vec{c}=\left(c_{1}, \ldots, c_{\ell}\right) \in \mathbb{C}^{\ell(w)}\right\}
$$

where, for a fixed reduced word $w=s_{i_{1}} \cdots s_{i_{\ell}}$

$$
y_{\vec{w}}(\vec{c})=y_{i_{1}}\left(c_{1}\right) \cdots y_{i_{\ell}}\left(c_{\ell}\right) I^{+}=\sum_{g \in y_{i_{1}}\left(c_{1}\right) \cdots y_{i_{\ell}}\left(c_{\ell}\right) I^{+}} g \quad \text { (a formal sum) }
$$

Let

$$
\begin{array}{lll}
C\left(I^{+} \backslash G / I^{+}\right)=\mathbb{C}-\operatorname{span}\left\{T_{w} \mid w \in W^{\text {ad }}\right\}, & \text { where } & T_{w}=I^{+} w I^{+}, \\
C\left(I^{0} \backslash G / I^{+}\right)=\mathbb{C} \text {-span }\left\{X^{w} \mid w \in W^{\text {ad }}\right\}, & \text { where } & X^{w}=I^{0} w I^{+}, \\
C\left(I^{-} \backslash G / I^{+}\right)=\mathbb{C}-\operatorname{span}\left\{L^{w} \mid w \in W^{\text {ad }}\right\}, & \text { where } & L^{w}=I^{-} w I^{+},
\end{array}
$$

The affine Hecke algebra is $\quad h_{I^{+}}\left(G / I^{+}\right)=\mathbb{C}$-span $\left\{T_{w} \mid w \in W\right\}$,

$$
\text { where } \quad T_{w}=I^{+} n_{w} I^{+}=\sum_{x \in I^{+} n_{w} I^{+}} x \quad \text { (a formal sum). }
$$

The formal sums allow us to view all of these elements as elements of the group algebra of $G$, where infinite formal sums are allowed (to do this precisely one should use Haar measure and a convolution product). Proposition 5.3 computes the (right) action of $h_{I^{+}}\left(G / I^{+}\right)$on $C\left(G / I^{+}\right)$, and Proposition 5.4 computes the (right) action of $h_{I^{+}}\left(G / I^{+}\right)$on $C\left(I^{+} \backslash G / I^{+}\right)$on $C\left(I^{0} \backslash G / I^{+}\right)$, and on $C\left(I^{-} \backslash G / I^{+}\right)$. Proposition 5.3 follows from Proposition 5.2(a) by summing over $c$ and Proposition 5.4 follows from Proposition 5.2 by summing over the appropriate double cosets. We use the convention that the normalization (Haar measure) is such that $I^{+} \cdot I^{+}=I^{+}$.

Proposition 5.3. Let $w \in W^{\text {ad }}$, let $\vec{w}=s_{i_{1}} \cdots s_{i_{\ell}}$ be a reduced word for $w$ and let $j \in\{0, \ldots, n\}$. Assume that if $w s_{j} \ll w$ then $i_{\ell}=j$ and let

$$
y_{\overrightarrow{w s_{j}}}(\vec{c})=y_{i_{1}}\left(c_{1}\right) \cdots y_{i_{\ell-1}}\left(c_{\ell-1}\right) \quad \text { and } \quad y_{\vec{w}}(\vec{c})=y_{i_{1}}\left(c_{1}\right) \cdots y_{i_{\ell}}\left(c_{\ell}\right)=y_{\overrightarrow{w s_{j}}}(\vec{c}) y_{j}\left(c_{\ell}\right) .
$$

Then

$$
y_{\vec{w}}(\vec{c}) I^{+} \cdot T_{s_{j}}= \begin{cases}y_{i_{1}}\left(c_{1}\right) \cdots y_{i_{\ell-1}}\left(c_{\ell-1}\right) I^{+}+\sum_{\tilde{c} \in \mathbb{k}^{\times}} y_{i_{1}}\left(c_{1}\right) \cdots y_{i_{\ell}}\left(c_{\ell}-\tilde{c}^{-1}\right) I^{+}, & \text {if } w s_{j}<w, \\ \sum_{\tilde{c} \in \mathbb{k}} y_{i_{1}}\left(c_{1}\right) \cdots y_{i_{\ell}}\left(c_{\ell}\right) y_{j}(\tilde{c}) I^{+}, & \text {if } w<+w s_{j} .\end{cases}
$$

Proposition 5.4. Let $\mathbb{k}=\mathbb{F}_{q}$ the finite field with $q$ elements. Let $w \in W^{\text {ad }}$ and $j \in\{0, \ldots, n\}$. Then

$$
\begin{gathered}
T_{w} T_{s_{j}}=\left\{\begin{array}{ll}
T_{w s_{j}}, & \text { if } w<w s_{j}, \\
(q-1) T_{w}+q T_{w s_{j}} . & \text { if } w s_{j}<+w,
\end{array} \quad L^{w} T_{s_{j}}= \begin{cases}L^{w s_{j}}, & \text { if } w<w s_{j}, \\
q L^{w s_{j}}+(q-1) L^{w}, & \text { if } w s_{j}<w,\end{cases} \right. \\
\text { and } \quad X^{w} T_{s_{j}}= \begin{cases}X^{w s_{j}}, \\
q X^{w s_{j}}+(q-1) X^{w}, & \text { if } w s_{j}<0 w .\end{cases}
\end{gathered}
$$

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