A Level 1 Maass Spezialschar for Modular Forms on SO_8.

Finn McGlade

Joint with Jennifer Johnson-Leung, Isabella Negrini, Aaron Pollack, and Manami Roy

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Outline

Contents

Siegel Modular Forms

- The symplectic group

$$\mathrm{Sp}_4 = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_4(\mathbb{R}) \colon AB^t = BA^t,\ CD^t = DC^t, \text{ and } AD^t - BC^t = 1 \right\}$$

acts on the complex three-fold

$$\mathcal{H}_2 = \left\{ \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) \colon \ \mathrm{im}(\tau), \mathrm{im}(\tau'), \mathrm{im}(\tau)\,\mathrm{im}(\tau') - \mathrm{im}(z)^2 > 0 \right\}$$

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Definition (Siegel Modular Forms)

Let $\ell \in \mathbb{Z}_{\geqslant 0}$ and write $M_\ell(\mathrm{Sp}_4(\mathbb{Z}))$ for the space of holomorphic functions

$$F \colon \mathcal{H}_2 \to \mathbb{C}$$

such that $F(\gamma \cdot Z) = \det(CZ + D)^\ell F(Z)\ \forall\, \gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_4(\mathbb{Z})$ and $Z \in \mathcal{H}_2$.

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- (Koecher Principle) $\Longrightarrow F \in M_\ell(\mathrm{Sp}_4(\mathbb{Z}))$ is bounded as $\mathrm{im}(Z) \to \infty$.

The Fourier Expansion of Siegel Modular Forms

- We have the abelian unipotent subgroup

$$N(\mathbb{Z}) = \left\{ \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} : B \in M_2(\mathbb{Z}) \quad \text{and } B^t = B \right\} \subseteq \mathrm{Sp}_4(\mathbb{Z}).$$

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$$F(Z) = \sum_{T \geqslant 0} A[T] \exp(2\pi i \,\mathrm{tr}(TZ))$$

where T runs over $\left\{ \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} : n, r, m \in \mathbb{Z}, n, m, 4nm - r^2 \geq 0 \right\}$.

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Say T is **primitive** if the content $e(T) := \gcd(n, r, m) = 1$.

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- The above generalizes to an arbitrary semi-simple real Lie group G provided the maximal compact $K \leqslant G$ contains a central copy of $\mathbb{C}^1$.

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In 1977 Saito and Kurokawa formulated a conjecture concerning a lift

$$\mathrm{SK}^*\colon M_{\ell-1/2}(\Gamma_0(4)) \to M_\ell(\mathrm{Sp}_4(\mathbb{Z})), \quad h \mapsto \mathrm{SK}^*(h).$$

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Theorem (Maass 1979)

- *Fix h a modular form on $\Gamma_0(4)$ of weight $\ell - 1/2$ with Fourier series*

$$h(z) = \sum_{n=0}^{\infty} c(n) e^{2\pi i n z}, \qquad (z \in \mathbb{C}\colon \mathrm{Im}(z) > 0).$$

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Then the numbers $A[T]$ are the Fourier coefficients of a modular form

$$\mathrm{SK}^*(h) \in M_\ell(\mathrm{Sp}_4(\mathbb{Z})).$$

$\mathrm{SK}^*(h)$ as a theta lift

Given τ in the upper half plane and $Z = \begin{pmatrix} \xi & z \\ z & \xi' \end{pmatrix} \in \mathcal{H}_2$ define

$$\Omega_\ell(Z,\tau) = \sum_{N>0} N^{3/2-\ell} w_{N,\ell}(Z) e^{2\pi i N\tau}.$$

where $w_{N,\ell}(Z) = \sum_{\substack{(a,b,c,d,e)\in\mathbb{Z}^5 \\ 4bd-c^2-4ae=N}} \frac{1}{(a(\xi\xi' - z^2) + b\xi + cz + d\xi' + e)^\ell}$.

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- Using the theory of theta series attached to lattices, Kudla proves

$$\Omega_\ell(\cdot,\tau) \in M_\ell(\mathrm{Sp}_4(\mathbb{Z})) \quad \text{and} \quad \Omega_\ell(Z,\cdot) \in S_{\ell-1/2}(\Gamma_0(4)).$$

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- If $h \in S_{\ell-1/2}(\Gamma_0(4))$ then $\mathrm{SK}^*(h)$ is the Petersson inner product

$$\mathrm{SK}^*(h) : Z \mapsto \int_{\Gamma_0(4)\backslash\mathcal{H}} h(u+iv)\Omega_\ell(Z, u-iv)\, v^{\ell-1/2} \frac{du dv}{v^2}.$$

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- This realizes SK^* in the general theory of theta correspondences. From this perspective, one wants to characterize the image of SK^*.

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Theorem (Maass-Zagier (1980))

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$$A[T] = \sum_{d \in \mathbb{Z}_{\geqslant 1} \,:\, d | \gcd(n,r,m)} d^{\ell-1} A\left[\begin{pmatrix} nm/d^2 & r/(2d) \\ r/(2d) & 1 \end{pmatrix}\right]. \quad \text{(Maass Relation)}$$

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(c) *If $T_1, T_2 \geqslant 0$ are primitive and $\det(T_1) = \det(T_2)$ then $A[T_1] = A[T_2]$.*

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(iv) Since $\begin{pmatrix} nm/d^2 & r/(2d) \\ r/(2d) & 1 \end{pmatrix}$ is also primitive, (ii) and (iii) imply

$$A\left[\begin{pmatrix} nm/d^2 & r/(2d) \\ r/(2d) & 1 \end{pmatrix}\right] = c\left(\frac{-4\det(T)}{d^2}\right). \tag{2}$$

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$$A[T] = \sum_{d \mid e(T)} d^{\ell-1} c\left(\frac{-4\det(T)}{d^2}\right). \quad (1)$$

(ii) So $e(T) = 1$ implies $A[T] = c(-4\det(T))$ depends only on $\det(T)$.

(iii) If $e(T) > 1$ and $d \mid e(T)$ then $\det \begin{pmatrix} nm/d^2 & r/(2d) \\ r/(2d) & 1 \end{pmatrix} = \frac{\det(T)}{d^2}$.

(iv) Since $\begin{pmatrix} nm/d^2 & r/(2d) \\ r/(2d) & 1 \end{pmatrix}$ is also primitive, (ii) and (iii) imply

$$A\left[\begin{pmatrix} nm/d^2 & r/(2d) \\ r/(2d) & 1 \end{pmatrix}\right] = c\left(\frac{-4\det(T)}{d^2}\right). \quad (2)$$

(v) Substituting (2) into (1) gives the Maass relation.

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Let $\mathbb{H} = \mathbb{R}\langle 1, i, j, k\rangle$ and set $V = \mathbb{H} \oplus \mathbb{H}$ equipped with the quadratic form

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Consider the embedding

$$\rho \colon \mathbb{H}^1 \hookrightarrow K^0 \subseteq \mathrm{SO}_8, \qquad (\rho(g) \cdot (q_1, q_2) = (gq_1, q_2)).$$

Then $\mathbb{H}^1 \trianglelefteq K^0$ will substitute for the lack of a central $\mathbb{C}^1 \subseteq K^0$.

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The above generalizes to an arbitrary semi-simple real Lie group G provided the maximal compact $K \leqslant G$ contains a normal copy of $\mathbb{H}^1$.

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$$\Phi_Z(g) = \sum_{\substack{[T_1,T_2] \in \mathrm{M}_2(\mathbb{Z}) \bigoplus \mathrm{M}_2(\mathbb{Z}) \\ \text{such that } Q[T_1,T_2] > 0}} \Lambda[T_1,T_2]\mathcal{W}_{[T_1,T_2]}(g).$$

- $Q(T_1,T_2)$ *is the binary quadratic form*

$$Q[T_1,T_2] = \det(xT_1 - yT_2).$$

- $\mathcal{W}_{[T_1,T_2]}(g)\colon\ \mathrm{SO}_8 \to \mathrm{Sym}^\ell(\mathbb{V})$ *is an explicit special function which depends only on* ℓ, $[T_1,T_2]$, *and the choice of* K.

Contents

From Sp_4 to SO_8.

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Say $[T_1, T_2] \in \mathrm{M}_2(\mathbb{Z}) \oplus \mathrm{M}_2(\mathbb{Z})$ is **primitive** if $\forall\, \gamma \in \mathrm{M}_2(\mathbb{Z})^{\det \neq 0}$,

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Theorem (Johnson-Leung, M, Negrini, Pollack, & Roy)

Suppose $\ell \in \mathbb{Z}_{\geqslant 18}$ *and* $\Phi \in S_\ell(\mathrm{SO}_8(\mathbb{Z}))$ *with Fourier Expansion*

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(a) There exists $F \in S_\ell(\mathrm{Sp}_4(\mathbb{Z}))$ such that $\Phi = \theta(F)^*$.

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(a) There exists $F \in S_\ell(\mathrm{Sp}_4(\mathbb{Z}))$ such that $\Phi = \theta(F)^*$.

(b) If $[T_1,T_2] \in M_2(\mathbb{Z}) \oplus M_2(\mathbb{Z})$ satisfies $Q[T_1,T_2] > 0$ then

$$\Lambda[T_1,T_2] = \sum_{\substack{\gamma \in \mathrm{GL}_2(\mathbb{Z}) \backslash \mathrm{M}_2(\mathbb{Z})^{\det \neq 0} \text{ s.t} \\ [T_1,T_2]\gamma^{-1} \in \mathrm{M}_2(\mathbb{Z}) \oplus \mathrm{M}_2(\mathbb{Z})}} |\det(\gamma)|^{\ell-1} \Lambda \widehat{[T_1,T_2]\gamma^{-1}}$$

Here $\widehat{[T_1,T_2]\gamma^{-1}} \in M_2(\mathbb{Z}) \oplus M_2(\mathbb{Z})$ is a specific primitive element (determined by $[T_1,T_2]\gamma^{-1}$) satisfying $Q\widehat{[T_1,T_2]\gamma^{-1}} = Q[T_1,T_2]\gamma^{-1}$.

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- From this analysis one proves that $\mathrm{FJ}(y)$ naturally determines a scalar valued holomorphic modular form on PGSp_4.

First Fourier Jacobi-Coefficients

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There exists a $F \in S_\ell(\mathrm{Sp}_4(\mathbb{Z}))$ such that if $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} > 0$ then the T-th Fourier coefficient of F satisfies

$$A[T] = \Lambda\left[\begin{pmatrix} n & 0 \\ r & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ m & 0 \end{pmatrix}\right]$$

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- A similar statement holds for general y, however, unlike in the classical case, the level of $\mathrm{FJ}(y)$ may deepens as $\langle y, y\rangle \to \infty$.

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- One could speculate about similar statement for modular forms on $\mathrm{SO}_8(\mathbb{Z})$. Work of Bhargava and Weissman predicts connections between Fourier coefficients of modular forms on $\mathrm{SO}_8(\mathbb{Z})$ and the fine structure of ideal class groups of imaginary quadratic fields.

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Thank you for listening