Cuspidal D_4-Modular Forms are Determined by Their Slice Primitive Fourier Coefficients.

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D_4-Modular Forms

- Let $\ell \in \mathbb{Z}_{\geqslant 1}$.
- Let $G = \mathrm{Spin}_8 \to \mathrm{SO}_8$ be split, simply connected, simple D_4.
- Φ =**weight ℓ[1] cuspidal (quaternionic) modular form on $G(\mathbb{Z})$**.

[1] Φ transforms under $K = \mathrm{SU}(2) \times_{\mu_2} (\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2))$ via $\mathrm{Sym}^{2\ell}\,\mathbb{C}^2 \boxtimes \mathbf{1}_{(\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2))}$

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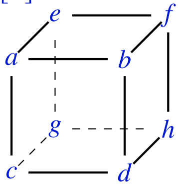
Here B runs over all **Bhargava cubes**



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such that $\mathrm{disc}(B) := (ah + ed - bg - fc)^2 - 4(ad - bc)(eh - fg) < 0$.

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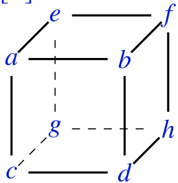

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If F =**cuspidal (holomorphic) modular form on $\mathrm{Sp}_4(\mathbb{Z})$** then

$$F = \sum_{a,b,c \in \mathbb{Z}:\ b^2 - 4ac < 0} A\left[\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\right] q^T$$

(Zagier 80') If $A\left[\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\right] = 0\ \forall a, b, c$ such that $\gcd(a, b, c) = 1$, $F = 0$

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Slice Primitivity

Let (M_i, N_i), $i = 1, 2, 3$, be the pairs of matrices obtained by **slicing** B

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right), \quad \left(\begin{pmatrix} a & e \\ b & f \end{pmatrix}, \begin{pmatrix} c & g \\ d & h \end{pmatrix}\right), \quad \left(\begin{pmatrix} a & c \\ e & g \end{pmatrix}, \begin{pmatrix} b & d \\ f & h \end{pmatrix}\right)$$

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Say B is **i-slice primitive** if $\mathbb{Q}$-span$\{M_i, N_i\} \cap \mathrm{M}_2(\mathbb{Z}) = \mathbb{Z}$-span$\{M_i, N_i\}$.

- If B is i-slice primitive then $\gcd(a, b, c, d, e, f, g, h) = 1$.

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Theorem (M.)

Let $\Phi = \sum \Lambda[B]\mathbf{q}^B$ be a cuspidal modular form on $G(\mathbb{Z})$. If there exists $\mathbf{i} \in \{1, 2, 3\}$ such that $\Lambda[B] = 0$ for all $\mathbf{i}$-slice primitive B then $\Phi = 0$.

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Application (J. Johnson-Leung, I. Negrini, M., A. Pollack, M. Roy)

S_ℓ^* =Mass subspace $\subseteq S_\ell(\mathrm{SO}_8(\mathbb{Z}))$. Let $\pi_i : \mathrm{Spin}_8 \to \mathrm{SO}_8$ be the i-th projection and $Q_i^B = -\det(xM_i - yN_i)$. What is the pullback $\pi_i^* S_\ell^*$?

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$$\pi_i^* S_\ell^* = \left\{ \Phi \in S_\ell(G(\mathbb{Z})) : \begin{array}{c} \text{if } B_1 \text{ and } B_2 \text{ are } \mathbf{i}\text{-slice primitive and} \\ Q_i^{B_1} = Q_i^{B_2} \text{ then } \Lambda[B_1] = \Lambda[B_2] \end{array} \right\}.$$