Fourier Coefficients and Algebraic Cusp Forms on $\mathrm{U}(2,n)$.

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Joint with Anton Hilado and Pan Yan

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Outline

Contents

The Fourier Expansion of Modular Forms on $\mathrm{Sp}_4(\mathbb{Z})$

Suppose $\ell \in \mathbb{Z}_{\geqslant 0}$ is even and define

$$\mathcal{H}_2 = \left\{ \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) : \ \mathrm{im}(\tau), \mathrm{im}(\tau'), \mathrm{im}(\tau)\,\mathrm{im}(\tau') - \mathrm{im}(z)^2 > 0 \right\}.$$

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$$F(Z) = \sum_{T \geqslant 0} A[T] \exp(2\pi i \mathrm{tr}(TZ)).$$

Here T ranges over $\left\{ \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} : n, r, m \in \mathbb{Z}, n, m, 4nm - r^2 \geqslant 0 \right\}.$

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Theorem (Shimura 75')

*The space $M_\ell(\Gamma_2)$ admits a **basis** consisting of algebraic forms.*

Siegel Modular Forms Semi-Classically

- Sp_4 acts **transitively** on $\mathcal{H}_2$ and iI_2 has stabilizer

$$K = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} : AB^t = BA^t, AA^t + BB^t = I_2 \right\}.$$

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- Given an **arbitrary** function $F \colon \mathcal{H}_2 = \mathrm{Sp}_4 / K \to \mathbb{C}$ define

$$\varphi_F \colon \mathrm{Sp}_4 \to \mathbb{C}, \qquad \varphi_F(g) = j(g,i)^{-\ell} F(g \cdot iI_2)$$

where $j(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), i) = \det(Ci + D)$.

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  - (i) If $\gamma \in \Gamma_2$, $g \in \mathrm{Sp}_4$, and $k \in K$ then

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  - (ii) φ_F satisfies a **Cauchy Riemann** type equation $D\varphi_F \equiv 0$.

Contents

Modular Forms on $\mathrm{U}(2,n)$.

Let $n \in \mathbb{Z}_{\geqslant 1}$, $J = \mathrm{diag}(1,-1,\cdots,-1) \in M_n(\mathbb{C})$, and

$$\mathrm{U}(2,n) = \left\{ g \in \mathrm{GL}_{n+2}(\mathbb{C}) : {}^t\overline{g} \begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & J & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix} g = \begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & J & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix} \right\}.$$

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Let $K = \mathrm{U}(2) \times \mathrm{U}(n) \leqslant \mathrm{U}(2,n)$ be a maximal compact subgroup and let

$$\mathbb{V} = (\mathrm{Sym}^2 \mathbb{C}^2 \otimes_{\mathbb{C}} \det{}^{-1}_{\mathrm{U}(2)}) \boxtimes (\mathrm{Trivial})_{\mathrm{U}(n)}.$$

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Definition

Let $\ell \in \mathbb{Z}_{\geqslant 1}$ and fix an arithmetic subgroup $\Gamma \leqslant \mathrm{U}(2,n)$. Write $\mathcal{M}_\ell(\Gamma)$ to denote the space of **smooth**, **moderate** growth functions

$$\Phi \colon \mathrm{U}(2,n) \to \mathrm{Sym}^\ell \mathbb{V} \quad \text{such that}$$

(i) If $\gamma \in \Gamma$, $g \in \mathrm{U}(2,n)$, and $k \in K$ then

$$\Phi(\gamma g) = \Phi(g) \qquad \text{and} \qquad \Phi(gk) = k^{-1}\Phi_F(g).$$

(ii) Φ satisfies a **specific** K-invariant differential equation $D\Phi \equiv 0$.

Some History of Modular Forms on $\mathrm{U}(2,n)$

- **1995: Koseki-Oda** established uniqueness of Whittaker models when $n = 1$ and calculated the local archimedean L-factors for the standard L-functions of elements in $\mathcal{M}_\ell(\Gamma)$.

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- **2024: B. Hu** progress towards **Deligne's Conjecture** for the adjoint L-function of Modular forms on $\mathrm{U}(2,1)$.

The Fourier Expansion of Modular Forms on $\mathrm{U}(2,n)$

Let $N = \left\{ \begin{pmatrix} 1 & * & * \\ \mathbf{0} & I_n & * \\ 0 & \mathbf{0} & 1 \end{pmatrix} \right\}$ and $Z = \left\{ \begin{pmatrix} 1 & \mathbf{0} & * \\ \mathbf{0} & I_n & \mathbf{0} \\ 0 & \mathbf{0} & 1 \end{pmatrix} \right\}$ be the center of N.

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Given $\Phi \in \mathcal{M}_\ell(\Gamma)$, the **constant term** of Φ along $Z = [N, N]$ is

$$\Phi_Z(g) = \int_{(\Gamma \cap Z)\backslash Z} \Phi(zg)\,dg. \qquad (g \in \mathrm{U}(2,n))$$

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Theorem (Koseki, Oda when n=1 (95'), Hilado, M, Yan (24'))

There exists a unique **explicit** family of special functions

$$\{\mathcal{W}_{\mathbf{v}} \colon \mathrm{U}(2,n) \to \mathrm{Sym}^\ell \mathbb{V} \colon \mathbf{v} \in \mathbb{C}^n \text{ such that } {}^t\overline{\mathbf{v}}J\mathbf{v} > 0\}$$

such that if $\Phi \in \mathcal{S}_\ell(\Gamma)$ then Φ_Z **Fourier expands** in characters of N/Z as

$$\Phi_Z(g) = \sum_{\mathbf{v} \in \Lambda \colon {}^t\overline{\mathbf{v}}J\mathbf{v} > 0} A_\Phi[\mathbf{v}]\mathcal{W}_{\mathbf{v}}(g). \qquad (g \in \mathrm{U}(2,n))$$

Here $\Lambda \subseteq \mathbb{C}^n$ is a lattice and $A_\Phi[\mathbf{v}] \in \mathbb{C}$ is a **Fourier Coefficient**.

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Conjecture (after A. Pollack)

The space $\mathcal{S}_\ell^0(\Gamma)^\perp$ admits a basis consisting of **algebraic forms**.

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For modular forms on G_2 of weight $\ell \in 2\mathbb{Z}_{>3}$ this is a result of Pollack (23')

The Saito Kurokawa Lifting

Given $Z = \begin{pmatrix} \xi & z \\ z & \xi' \end{pmatrix} \in \mathcal{H}_2$ and $\tau \in \mathcal{H}_1 \subseteq \mathbb{C}$ in the **upper half plane**, let

$$\Omega_\ell(Z,\tau) = \sum_{N>0} N^{3/2-\ell} w_{N,\ell}(Z) e^{2\pi i N\tau}.$$

where $w_{N,\ell}(Z) = \sum_{\substack{(a,b,c,d,e)\in\mathbb{Z}^5 \\ 4bd-c^2-4ae=N}} \frac{1}{(a(\xi\xi' - z^2) + b\xi + cz + d\xi' + e)^\ell}.$

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- Using the theory of **theta series** attached to lattices, Kudla proves

$$\Omega_\ell(\cdot,\tau) \in M_\ell(\Gamma_2) \quad \text{and} \quad \Omega_\ell(Z,\cdot) \in S_{\ell-1/2}(\Gamma_0(4)).$$

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- Define the **Saito-Kurokawa Lift** $\mathrm{SK}^* : S^+_{\ell-1/2}(\Gamma_0(4)) \to M_\ell(\Gamma_2)$ as

$$h \mapsto \mathrm{SK}^*(h)(Z) := \int_{\Gamma_0(4)\backslash\mathcal{H}_1} h(u+iv)\Omega_\ell(Z, u-iv)\, v^{\ell-1/2} \frac{du\,dv}{v^2}.$$

Rationality of the Saito Kurokawa Lift

Theorem (Andrianov, Maass, Zagier 77'-80')

Fix $h \in S^{+}_{\ell-1/2}(\Gamma_0(4))$ *with* ***Fourier series*** $h(z) = \sum_{n=0}^{\infty} c(n) e^{2\pi i n z}$.

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For $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} > 0$, *let*

$$A[T] = \sum_{d \mid \gcd(n,r,m)} d^{\ell-1} c\left(\frac{4\det(T)}{d^2}\right).$$

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Then $\mathrm{SK}^*(h)$ *is* ***non-zero*** *with Fourier expansion*

$$\mathrm{SK}^*(h)(Z) = \sum_{T>0} A[T] \exp(2\pi i \mathrm{tr}(TZ)).$$

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Fix $h \in S^+_{\ell-1/2}(\Gamma_0(4))$ with ***Fourier series*** $h(z) = \sum_{n=0}^{\infty} c(n)e^{2\pi i n z}$.

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$$A[T] = \sum_{d \mid \gcd(n,r,m)} d^{\ell-1} c\left(\frac{4\det(T)}{d^2}\right).$$

Then $\mathrm{SK}^*(h)$ *is* ***non-zero*** *with Fourier expansion*

$$\mathrm{SK}^*(h)(Z) = \sum_{T>0} A[T]\exp(2\pi i \mathrm{tr}(TZ)).$$

In particular, the mapping $\mathrm{SK}^* : S^+_{\ell-\frac{1}{2}}(\Gamma_0(4)) \hookrightarrow S_\ell(\Gamma_2)$ *is defined over* $\mathbb{Q}$.

Algebraic Theta Lifts on $\mathrm{U}(2,n)$

Let $\mathrm{U}(1,1) = \left\{ \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} h \colon z \in \mathbb{C}^1, g \in \mathbf{SL}_2(\mathbb{R}) \right\}$.

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Theorem (Hilado, M, Yan 2024)

Let $\ell \geqslant n$ and $f\colon \mathrm{U}(1,1) \to \mathbb{C}$ be a weight $2\ell - n + 2$ ***holomorphic*** *modular form of central character $\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \mapsto z^{n+2}$ and* ***level*** *$\Gamma' \leqslant \mathrm{U}(1,1)$.*

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(i) *There exists an arithmetic subgroup $\Gamma \leqslant \mathrm{U}(2,n)$ and an integral kernel*

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such that $\theta^*(f)(g) := \int_{\Gamma' \backslash \mathrm{U}(1,1)} \Omega(g,h)\overline{f(h)}dh$ *lies in* $\mathcal{S}_\ell(\Gamma)$.

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(ii) *If $\Gamma' \backslash \mathcal{H}_1$ is non-compact and the Fourier coefficients of f lie in an **algebraic extension** $E/\mathbb{Q}$ then $\theta^*(f)$ is **algebraic** and*

$$A_{\theta^*(f)}[\mathbf{v}] \in E(\mu_\infty) \quad \forall \mathbf{v} \in \Lambda.$$

Thank you for listening