A Level 1 Maass Spezialschar for Modular Forms on SO_8.

Finn McGlade

Joint with Jennifer Johnson-Leung, Isabella Negrini, Aaron Pollack, and Manami Roy

October, 2023, Omaha

Outline

Contents

The Fourier Expansion of Modular Forms on $\mathrm{Sp}_4(\mathbb{Z})$

Suppose $\ell \in \mathbb{Z}_{\geqslant 0}$ is even and define

$$\mathcal{H} = \left\{ \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) \colon \ \mathrm{im}(\tau), \mathrm{im}(\tau'), \mathrm{im}(\tau)\,\mathrm{im}(\tau') - \mathrm{im}(z)^2 > 0 \right\}.$$

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$$F(Z) = \sum_{T \geqslant 0} A[T] \exp(2\pi i \mathrm{tr}(TZ)).$$

Here T ranges over $\left\{ \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} : n, r, m \in \mathbb{Z}, n, m, 4nm - r^2 \geqslant 0 \right\}$.

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- Alternatively, T runs over positive semi-definite quadratic forms

$$[n, r, m] = nx^2 + rxy + my^2$$

  Write $e(T) = \gcd(n, r, m)$ to denote the *content* T.

The Saito Kurokawa Lift

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Theorem (Maass 1979)

- *Fix h a modular form on $\Gamma_0(4)$ of weight $\ell - 1/2$ with Fourier series*

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Then

$$\mathrm{SK}^*(h)\colon \mathcal{H} \to \mathbb{C}, \qquad Z \mapsto \sum_{T \geqslant 0} A[T] \exp(2\pi i \mathrm{tr}(TZ))$$

is a weight ℓ Siegel modular form on $\mathrm{Sp}_4(\mathbb{Z})$.

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$$A[T] = \sum_{d \in \mathbb{Z}_{\geqslant 1} :\, d | \gcd(n,r,m)} d^{\ell-1} A\left[\begin{pmatrix} nm/d^2 & r/(2d) \\ r/(2d) & 1 \end{pmatrix}\right]. \quad \textit{(Maass Relation)}$$

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- *If* $T_1, T_2 \geqslant 0$ *are primitive and* $\det(T_1) = \det(T_2)$ *then* $A[T_1] = A[T_2]$.

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Definition

A **weight $\ell \in \mathbb{Z}_{\geqslant 0}$ modular form on $\mathrm{SO}_8(\mathbb{Z})$** is a function

$$\Phi\colon\ \mathrm{SO}_8(\mathbb{R}) \to \mathrm{Sym}^{2\ell}(\mathbb{C}^2)$$

such that Φ is smooth, of moderate growth and satisfies

- $\Phi(\gamma g) = \Phi(g)$ for all $\gamma \in \mathrm{SO}_8(\mathbb{Z})$ and $g \in \mathrm{SO}_8(\mathbb{R})$.
- If $k \in K^0$ and $g \in \mathrm{SO}_8(\mathbb{R})$ then $\Phi(gk) = k^{-1}\Phi(g)$.
- The function Φ satisfies a specific differential equation

$$D_\ell \Phi \equiv 0.$$

$S_\ell(\mathrm{SO}_8(\mathbb{Z}))$=space of weight ℓ cuspidal modular forms on $\mathrm{SO}_8(\mathbb{Z})$.

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Theorem (Wallach 2003, Pollack 2020)

Let $\Phi \in \mathcal{S}_\ell(\mathrm{SO}_8(\mathbb{Z}))$, $Z = [N, N]$, *and define* $\Phi_Z(g) := \int_{Z(\mathbb{Z})\backslash Z(\mathbb{R})} \Phi(zg)dz$.

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$$\Phi_Z(g) = \sum_{\substack{[T_1,T_2] \in \mathrm{M}_2(\mathbb{Z}) \oplus \mathrm{M}_2(\mathbb{Z}) \\ \text{such that } Q(T_1,T_2) > 0}} \Lambda[T_1, T_2]\mathcal{W}_{[T_1,T_2]}(g).$$

- *$Q(T_1, T_2)$ is the binary quadratic form*

$$Q(T_1, T_2) = \det(xT_1 - yT_2).$$

- *$\mathcal{W}_{[T_1,T_2]}(g)\colon\ \mathrm{SO}_8(\mathbb{R}) \to \mathrm{Sym}^{2\ell}(\mathbb{C}^2)$ is an explicit special function which depends only on ℓ, $[T_1, T_2]$, and the choice of K.*

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From Sp_4 to SO_8.

Theorem (Pollack 2021)

- *Let* $\ell \in \mathbb{Z}_{\geqslant 18}$ *and suppose* $F \in S_\ell(\mathrm{Sp}_4(\mathbb{Z}))$ *is a cuspidal Siegel modular form on* $\mathrm{Sp}_4(\mathbb{Z})$ *with Fourier expansion*

$$F(Z) = \sum_{T>0} A[T] \exp(2\pi i \,\mathrm{tr}(ZT)).$$

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- *For each $[T_1, T_2] \in \mathrm{M}_2(\mathbb{Z}) \oplus \mathrm{M}_2(\mathbb{Z})$ satisfying $Q(T_1, T_2) > 0$, let*

$$\Lambda[T_1, T_2] = \sum_{\substack{\gamma \in \mathrm{GL}_2(\mathbb{Z}) \backslash \mathrm{M}_2(\mathbb{Z})^{\det \neq 0} \text{ s.t} \\ [T_1, T_2]\gamma^{-1} \in \mathrm{M}_2(\mathbb{Z}) \oplus \mathrm{M}_2(\mathbb{Z})}} |\det(\gamma)|^{\ell-1} A[Q(T_1, T_2)\gamma^{-1}].$$

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Then

$$\theta(F)^*_Z : g \mapsto \sum_{\substack{[T_1, T_2] \in \mathrm{M}_2(\mathbb{Z}) \oplus \mathrm{M}_2(\mathbb{Z}) \\ \text{such that } Q(T_1, T_2) > 0}} \Lambda[T_1, T_2] \mathcal{W}_{T_1, T_2}(g).$$

is the constant term of a unique modular form $\theta(F)^* \in \mathcal{S}_\ell(\mathrm{SO}_8(\mathbb{Z}))$.

Maass Relations on SO_8

Theorem (in progress Johnson-Leung, M, Negrini, Pollack, & Roy)

Suppose $\ell \in \mathbb{Z}_{\geqslant 18}$ *and* $\Phi \in \mathcal{S}_\ell(\mathrm{SO}_8(\mathbb{Z}))$ *with Fourier Expansion*

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Say $[T_1,T_2] \in \mathrm{M}_2(\mathbb{Z}) \oplus \mathrm{M}_2(\mathbb{Z})$ is **primitive** if $\forall\, \gamma \in \mathrm{M}_2(\mathbb{Z})^{\det\neq 0}$,

$$[T_1,T_2]\gamma^{-1} \in \mathrm{M}_2(\mathbb{Z}) \oplus \mathrm{M}_2(\mathbb{Z}) \iff \gamma \in \mathrm{GL}_2(\mathbb{Z}).$$

Some Fourier Jacobi Expansion

Proposition (Johnson-Leung, M, Negrini, Pollack, & Roy)

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- The function $h(\tau) = \phi_m(\tau, 0)$ is a modular form on $\mathrm{SL}_2(\mathbb{Z})$ with Fourier coefficients $c(n) = \sum_{r\in\mathbb{Z}:\ r^2\leqslant 4nm} A\left[\begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}\right]$.

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  where $\mathrm{FJ}(y)(g) = \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \Phi(ug)e^{-2\pi i\langle y,u\rangle}du$.
- $\mathrm{FJ}(y)$ restricts to an automorphic on $\mathrm{Stab}_{\mathrm{SO}_6}(y) \simeq \mathrm{PGSp}_4 \leqslant L$.

Fourier Jacobi Coefficients of Modular Forms on SO_8

- $Q \leqslant \mathrm{SO}_8$ parabolic stabilizing an isotropic line.
- $Q = LU$ with $L \simeq \mathrm{GL}_1 \times \mathrm{SO}_6$ and U a split quadratic space of $\dim$ 6.
- $\Phi \in \mathcal{S}_\ell(\mathrm{SO}_8(\mathbb{Z}))$ Fourier expands along $U(\mathbb{Z})\backslash U(\mathbb{R})$ as

$$\Phi(g) = \sum_{y \in U(\mathbb{Z}):\ (y,y)>0} \mathrm{FJ}(y)(g),$$

  where $\mathrm{FJ}(y)(g) = \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \Phi(ug)e^{-2\pi i\langle y,u\rangle}du$.
- $\mathrm{FJ}(y)$ restricts to an automorphic on $\mathrm{Stab}_{\mathrm{SO}_6}(y) \simeq \mathrm{PGSp}_4 \leqslant L$.
- For appropriate $y_0 \in U(\mathbb{Z})$, one can explicate the Fourier expansion of $\mathrm{FJ}(y_0)|_{\mathrm{PGSp}_4}$ along the Siegel unipotent radical in PGSp_4.

Thank you for listening

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