Positive level, negative level and level zero

A survey of the geometry and representation theory of quantum affine algebras.

F. McGlade A. Ram Y. Yang

- This work is joint with Arun Ram and Yaping Yang.
- The plan is to present a brief survey talk on the geometry and representation theory of quantum affine algebras.

The affine lie algebra $\widehat{\mathfrak{sl}}_2(\mathbb{C})$

- Let $\widehat{\mathfrak{sl}}_2(\mathbb{C})$ be an affine Kac-Moody Lie algebra

$$\widehat{\mathfrak{sl}}_2(\mathbb{C}) := \left(\bigoplus_{k\in\mathbb{Z}} \mathfrak{sl}_2(\mathbb{C})\epsilon^k\right) \oplus \mathbb{C}K \oplus \mathbb{C}d$$

For $x, y \in \mathfrak{sl}_2(\mathbb{C})$ and $k, \ell \in \mathbb{Z}$, we have $[K, x\epsilon^k] = 0$, $[K, d] = 0$, $[d, x\epsilon^k] = kx\epsilon^k$, and

$$[x\epsilon^k, y\epsilon^\ell] = (xy - yx)\epsilon^{k+\ell} + k\delta_{k,-\ell}\langle x, y\rangle K.$$

- The Cartan subalgebra of $\widehat{\mathfrak{sl}}_2(\mathbb{C})$ is

$$\mathfrak{h} = \mathfrak{a} \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

where $\mathfrak{a} \subseteq \mathfrak{sl}_2(\mathbb{C})$ is a fixed Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{C})$.

- Thus $\mathfrak{h}$ has a basis d, h_1, K where h_1 is the Chevalley generator of $\mathfrak{a}$.

The affine Weyl group

Let $\{\delta, \omega_1, \Lambda_0\}$ be the basis of $\mathfrak{h}^*$ dual to the basis $\{d, h_1, K\}$ of $\mathfrak{h}$.

Definition

The *affine Weyl group* W is the subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by the matrices

$$s_0 = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \qquad s_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Equivalently, W is the subgroup generated by

$$t_{k\alpha^\vee} = \begin{pmatrix} 1 & -k & -k^2 \\ 0 & 1 & 2k \\ 0 & 0 & 1 \end{pmatrix}, \qquad s_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad k \in \mathbb{Z}.$$

- $W = \langle s_0, s_1 \rangle$ is the *Coxeter presentation*.
- $W = \langle t_{k\alpha^\vee}, s_1 \mid k \in \mathbb{Z} \rangle$ is the *translation presentation*.

W acting on $\mathfrak{h}_{\mathbb{R}}^*$

$$W = \langle s_0, s_1 \rangle \subseteq \mathrm{GL}(\mathfrak{h}_{\mathbb{R}}^*) \qquad \mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}\text{-span}\{\delta, \omega_1, \Lambda_0\}$$

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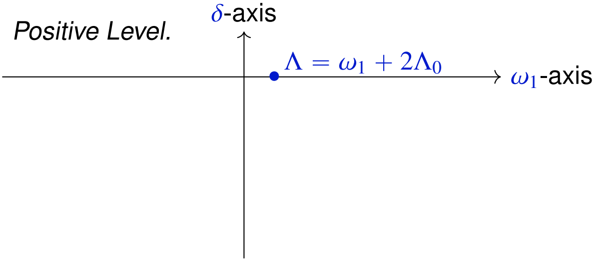

W acting on $\mathfrak{h}^*_{\mathbb{R}}$

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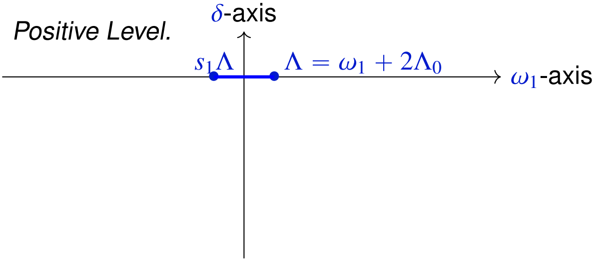

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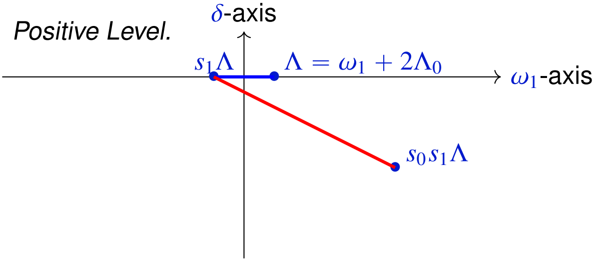

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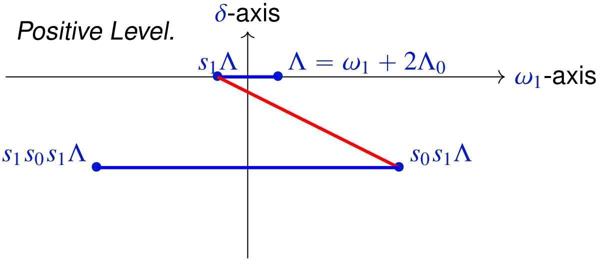

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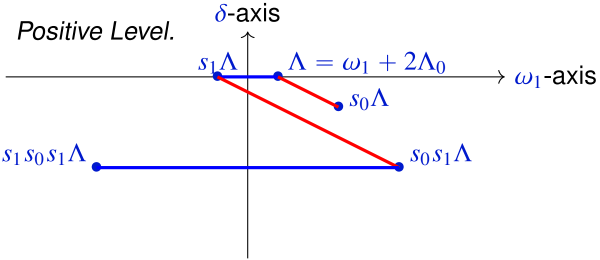

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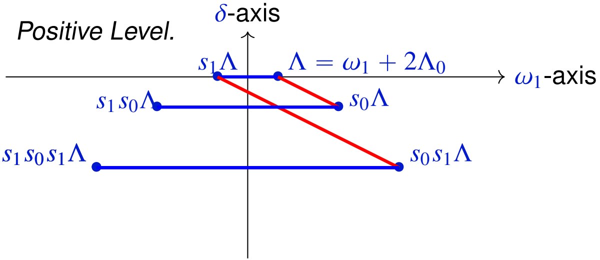

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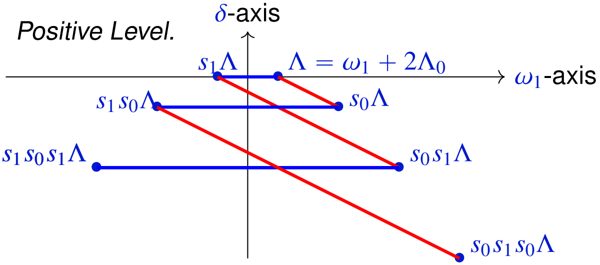

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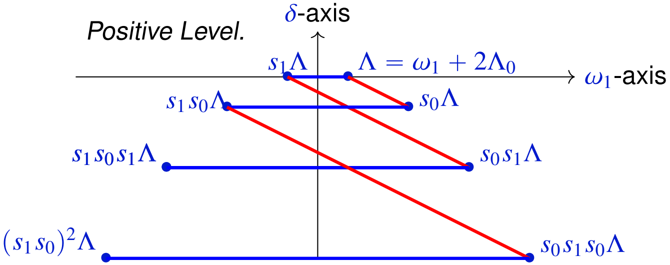

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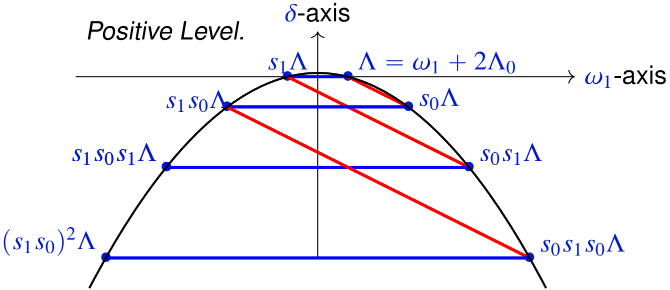
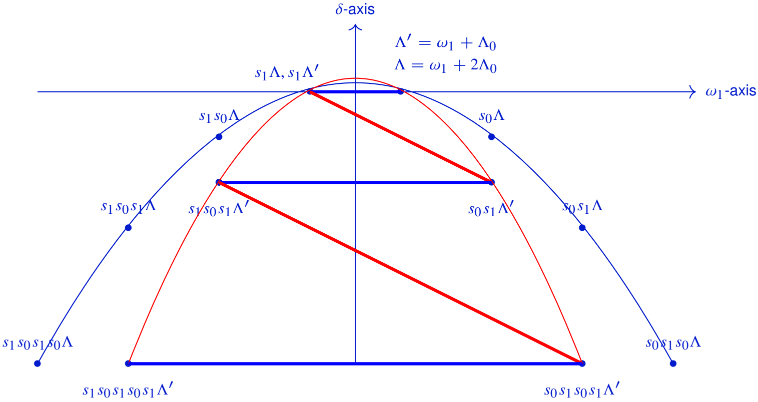


W acting on $\mathfrak{h}^*_{\mathbb{R}}$



The parabolas bounding the orbits $W(\omega_1 + 2\Lambda_0)$ and $W(\omega_1 + \Lambda_0)$ mod Λ_0.

- An *alcove* is a subset $\mathcal{A} \subseteq \mathbb{R}\omega_1 + \Lambda_0$ such that $\mathcal{A} = X \mod \delta$ where X is a fundamental domain for the action of W on $\mathbb{R}\omega_1 + \mathbb{R}\delta + \Lambda_0$

W acting on $\mathfrak{h}_{\mathbb{R}}^*$

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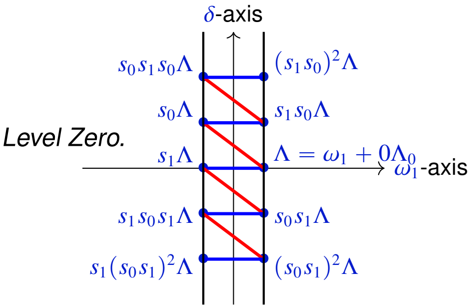

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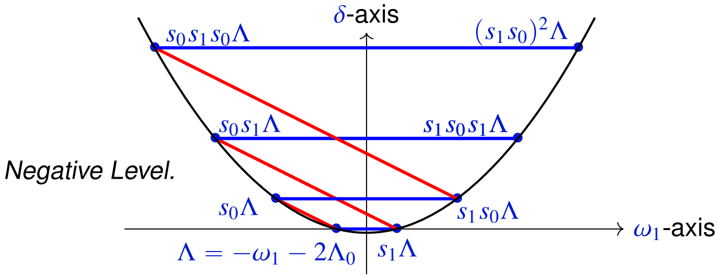
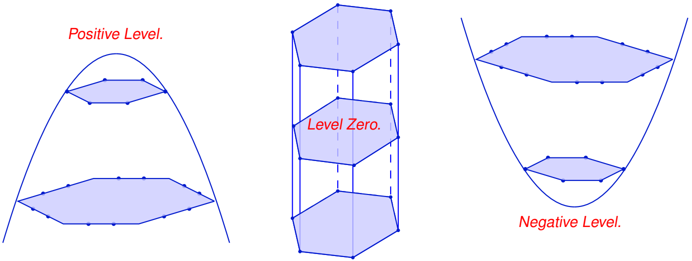
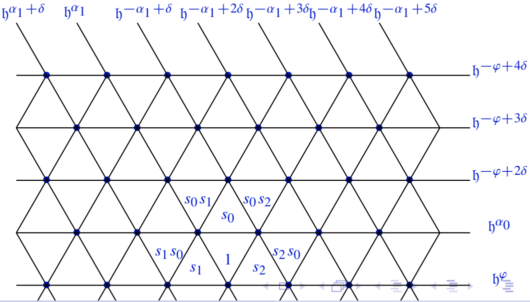


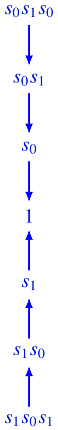
The pictures for $\widehat{\mathfrak{sl}}_3(\mathbb{C})$



By collapsing the δ axis in the positive level picture we obtain a picture of affine Weyl group for $\widehat{\mathfrak{sl}}_3(\mathbb{C})$:

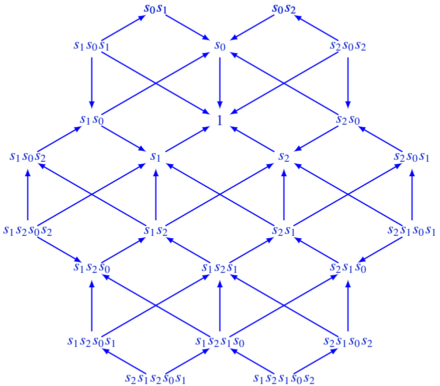


The positive level Bruhat order $\stackrel{+}{\leq}$



The $\stackrel{+}{\leq}$ Bruhat order for $\widehat{\mathfrak{sl}}_2$

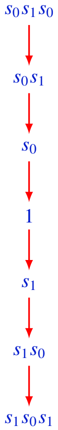
1 is minimal

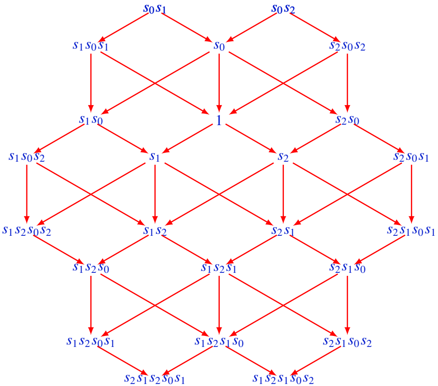


The $\stackrel{+}{\leq}$ Bruhat order for $\widehat{\mathfrak{sl}}_3$

1 is minimal

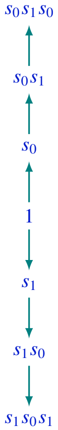
The level zero Bruhat order $\le_0$

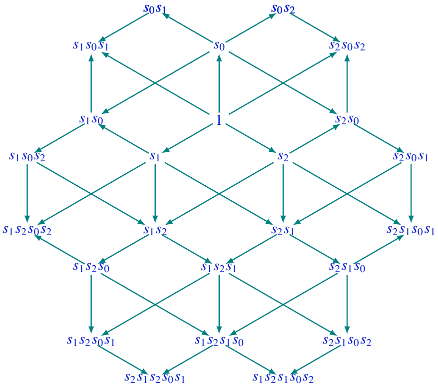




The $\le_0$ Bruhat order for $\widehat{\mathfrak{sl}}_2$ translation invariant

The $\le_0$ Bruhat order for $\widehat{\mathfrak{sl}}_3$ translation invariant





The $\underset{-}{\leq}$ Bruhat order for $\widehat{\mathfrak{sl}}_2$.
1 is maximal

The $\underset{-}{\leq}$ Bruhat order for $\widehat{\mathfrak{sl}}_3$.
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The affine flag varities G/I^+, G/I^0 and G/I^-

- The *positive level (thin) affine flag variety* is $\mathrm{SL}_n(\mathbb{C}((t)))/I^+$ where

$$I^+ = \left\{ \begin{pmatrix} a_1 & & b_{ij} & \\ & a_2 & & \\ c_{ij} & & \ddots & \\ & & & a_n \end{pmatrix} \in \mathrm{SL}_n(\mathbb{C}((t))) \;\middle|\; \begin{matrix} a_i \in \mathbb{C}[\![t]\!]^\times \\ b_{ij} \in \mathbb{C}[\![t]\!] \\ c_{ij} \in t\mathbb{C}[\![t]\!] \end{matrix} \right\}$$

- The *level* 0 *(semi-infinite) affine flag variety* is $\mathrm{SL}_n(\mathbb{C}((t)))/I^0$ where

$$I^0 = \left\{ \begin{pmatrix} a_1 & & b_{ij} & \\ & a_2 & & \\ 0 & & \ddots & \\ & & & a_n \end{pmatrix} \in \mathrm{SL}_n(\mathbb{C}((t))) \;\middle|\; \begin{matrix} a_i \in \mathbb{C}[\![t]\!]^\times \\ b_{ij} \in \mathbb{C}((t)) \end{matrix} \right\}$$

- The *negative level (thick) affine flag variety* is $\mathrm{SL}_n(\mathbb{C}((t)))/I^-$ where

$$I^- = \left\{ \begin{pmatrix} a_1 & & c_{ij} & \\ & a_2 & & \\ b_{ij} & & \ddots & \\ & & & a_n \end{pmatrix} \in \mathrm{SL}_n(\mathbb{C}((t))) \;\middle|\; \begin{matrix} a_i \in \mathbb{C}[t^{-1}]^\times \\ b_{ij} \in \mathbb{C}[t^{-1}] \\ c_{ij} \in t^{-1}\mathbb{C}[t^{-1}] \end{matrix} \right\}$$

Combinatorics in affine flag varieties

- The affine Weyl group W indexes the I^+ orbits in G/I^+, G/I^0 and G/I^-. We obtain decompositions

$$G = \bigsqcup_{x\in W} I^+ x I^+, \qquad G = \bigsqcup_{y\in W} I^0 y I^+, \qquad G = \bigsqcup_{z\in W} I^- z I^+.$$

- Alcove walk combinatorics developed by Parkinson, Ram and Schwer gives a combinatorial description to the *affine Schubert cells*

$$I^+ x I^+/I^+ \subseteq G/I^+, \qquad I^+ y I^0/I^0 \subseteq G/I^0, \qquad I^+ z I^-/I^- \subseteq G/I^-.$$

- The positive level description is

$$\left\{ \begin{array}{l} \text{blue labeled paths of type} \\ w = (i_1, \ldots, i_\ell) \end{array} \right\} \xrightarrow{\sim} (I^+ w I^+)/I^+$$

$$(c_1, \ldots, c_\ell) \longmapsto y_{i_1}(c_1)\ldots y_{i_\ell}(c_\ell) I^+$$

where $y_{i_j}(c)$ is the image of $\begin{pmatrix} c & 1 \\ -1 & 0 \end{pmatrix}$ under the homomorphism $\varphi_j : \mathrm{SL}_2(\mathbb{C}) \to G$, $j \in \{0, \ldots, n\}$.

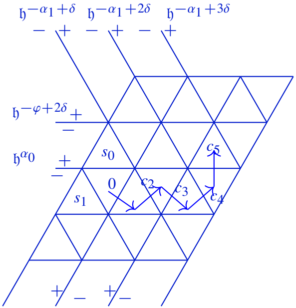
The folding algorithm

To index $I^0 w I^+$ we index each component in the decomposition.

$$I^0 w I^+ = \bigsqcup_{v \in W} (I^+ v I^+ \cap I^0 w I^+)$$

The folding algorithm takes a point in $I^+ v I^+$ and re-expresses it as a point in $I^0 w I^+$ for some w. The implementation of the algorithm relies on the level zero Bruhat order and the hyperplane labels.

$$(I^+ s_1 s_2 s_1 s_0 s_2 I^+ \cap I^0 s_1 s_0 s_2 s_0 I^+)/I^+ \simeq \mathbb{C}^\times \times \mathbb{C}^3.$$



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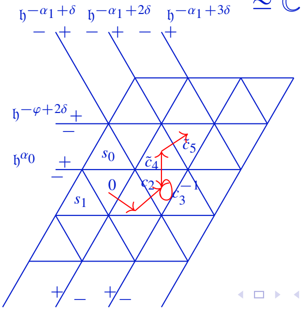


The quantum affine algebra $\mathbf{U}$

- The *quantum affine algebra* $\mathbf{U}$ is the $\mathbb{C}(q)$-algebra generated by
$$E_0, E_1, F_0, F_1, K_0^{\pm}, K_1^{\pm}, C^{\pm\frac{1}{2}}, D^{\pm 1},$$
with Chevalley-Serre type relations corresponding to the Dynkin diagram of $\hat{\mathfrak{sl}}_2(\mathbb{C})$.
- For $i \in \{0, 1\}$ let $\mathbf{U}_{(i)}$ be the subalgebra of $\mathbf{U}$ generated by $\{E_i, F_i, K_i^{\pm 1}\}$.

Definition

An $\mathbf{U}$-module M is *integrable* if $\mathrm{Res}^{\mathbf{U}}_{\mathbf{U}_{(i)}}(M)$ is a direct sum of finite dimensional $\mathbf{U}_{(i)}$-modules for $i \in \{0, 1\}$.

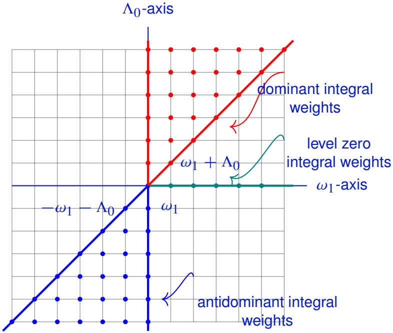
- Integrable modules play a central role in the classification of finite dimesional simple $\mathbf{U}$-modules.
- To carry out this classification introduce a generalization of highest weight modules and consider extremal weight modules.

Extremal weight modules

The (extremal) weights of integrable **U**-modules always lie in the set

$$\mathfrak{h}_{\mathbb{Z}}^* = \mathbb{C}\delta + \mathbb{Z}\omega_1 + \mathbb{Z}\Lambda_0.$$

For $\Lambda \in \mathfrak{h}_{\mathbb{Z}}^*$, write $L(\Lambda)$ for the (universal) integrable *extremal weight module* of extremal weight Λ.



$\left(\mathfrak{h}_{\mathbb{Z}}^*/W\right)$ mod δ

If $\Lambda \in \mathfrak{h}_{\mathbb{Z}}^* \backslash \mathbb{C}\delta$ then

- $L(\Lambda)$ is a simple highest weight module $\Leftrightarrow$ Λ is dominant.
- $L(\Lambda)$ is neither a highest weight nor a lowest weight module $\Leftrightarrow$ Λ is level zero.
- $L(\Lambda)$ is a simple lowest weight module $\Leftrightarrow$ Λ is antidominant.

If $w \in W$ then $L(\Lambda) \cong L(w\Lambda)$.

Thank You!