MATH 20D Spring 2023 Lecture 23.

Impulse Response Functions and Systems of Differential Equations.

Announcements

- Homework 7 is due tomorrow at 10pm.

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- Homework 8 is available and due next Saturday at 10pm. No late submissions will be accepted.

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- Midterm 2 grades are available, regrade request window closing Friday June 2nd at 11:59pm.

Outline

Contents

Last Time

- The **Dirac delta "function"** $\delta(t) = \lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(t)$ where

$$\mathcal{F}_\varepsilon(t) = \begin{cases} 1/\varepsilon, & 0 < t < \varepsilon, \\ 0, & t > \varepsilon. \end{cases}$$

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Key Property

If $a \geqslant 0$ is constant and $f\colon [0,\infty) \to \mathbb{R}$ is continuous at $x = a$ then

$$\int_0^\infty f(t)\delta(t-a)dt = f(a).$$

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In particular $\mathscr{L}\{\delta(t-a)\}(s) = e^{-as}$.

- Under $\mathscr{L}$ the initial value problem

$$y'(t) = \delta(t-a), \qquad y(0) = 0$$

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- Hence $y(t) = \mathscr{L}^{-1}\{e^{-as}/s\}(t) = u(t-a)$ solves the IVP above.

Definition

Given a differential equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t)$$

the ***impulse response function*** *is the solution to the initial value problem*

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Example

Determine the impulse response function associated to the differential equation

$$y'' - 6y' + 13y = g(t)$$

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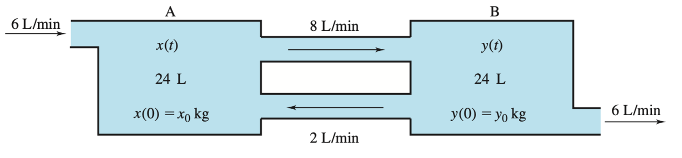
Example

Two tanks, each holding 24 liters of a brine solutions, are interconnected by pipes as shown below. Fresh water solution water is pumped into the tanks at a rate of 6 L/min, and fluid is drained out of the system at the same rate



If $x(t)$ denotes the amount of salt in tank A at time t and $y(t)$ denotes the amount of salt in tank B at time t, then

$$x'(t) = -x(t)/3 + y(t)/12$$
$$y'(t) = x(t)/3 - y(t)/3.$$

- As we will see, it is convenient to recast the system of differential equation

$$x'(t) = -x(t)/3 + y(t)/12$$
$$y'(t) = x(t)/3 - y(t)/3.$$

as a single **first order matrix differential equation**

$$\mathbf{x}'(t) = \begin{pmatrix} -1/3 & 1/12 \\ 1/3 & -1/3 \end{pmatrix} \mathbf{x}(t)$$

where the unknown $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is a **vector valued function**.

- This trick can also be used to transform higher order equations into first order systems of equations with vector valued unknowns.

Example

Express the higher order differential below as an equation of the form $\mathbf{x}'(t) = A\mathbf{x}(t)$ where $\mathbf{x}(t)$ is a vector valued function and A is matrix.

(a) $y'' + 4y = 0$ **(b)** $y''' + 2y'' + y' = 0$

- Our usual way of writing a general second order linear ODE is as

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t). \tag{1}$$

- If $\mathbf{x}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$ then $\mathbf{x}'(t) = \begin{pmatrix} y'(t) \\ y''(t) \end{pmatrix}$.

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- If $\mathbf{x}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$ then $\mathbf{x}'(t) = \begin{pmatrix} y'(t) \\ y''(t) \end{pmatrix}$.Since

$$y''(t) = -q(t)y'(t) - p(t)y(t) + g(t)$$

equation (1) can be re-expressed as

$$\mathbf{x}'(t) = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 0 \\ g(t) \end{pmatrix}.$$

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Definition

We say that a matrix differential equation is in ***normal form*** *if is expressed as*

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)$$

where $A(t)$ is a matrix valued function and $\mathbf{f}(t)$ is a vector valued function.

Example

Rewrite the system of differential equations below as a first order matrix differential equation in normal form

$$x'(t) = \cos(t)x(t) - \sin(t)y(t) + e^t$$
$$y'(t) = \sin(t)x(t) + \cos(t)y(t) - te^{-2t}.$$