MATH 202A
APPLIED ALGEBRA I
FALL 2019

Homework week 9

Due by the beginning of class on Monday 2nd December (hand in via Gradescope).

1. Let V, W be finite-dimensional vector spaces. Prove that the map

$$\mathcal{L}(V,W) \to \mathcal{L}(W,V)^*$$
$$\phi \mapsto \big(\psi \mapsto \operatorname{tr}(\phi \circ \psi) = \operatorname{tr}(\psi \circ \phi)\big)$$

is an isomorphism.

[So, a matrix norm on $\mathcal{L}(V,W)$ has a corresponding dual norm on $\mathcal{L}(W,V)$, etc..]

2. Consider $V = \mathbb{C}^n$, $W = \mathbb{C}^m$. For a linear map $\phi\colon V \to W$ represented my an $m \times n$ matrix A, prove that:—

(a) $\|\phi\|_{\ell^1 \to \ell^1} = \max_{j=1}^n \sum_{i=1}^m |A_{ij}|$;

(b) $\|\phi\|_{\ell^\infty \to \ell^\infty} = \max_{i=1}^m \sum_{j=1}^n |A_{ij}|$;

(c) $\|\phi\|_{\ell^1 \to \ell^\infty} = \|A\|_{v,\infty}$;

(d) $\|\phi\|_{\ell^\infty \to \ell^1} = \|A\|_{v,1}$.

3. Let U, V, W be inner product spaces, and let $\phi \in \mathcal{L}(U,V)$. Define

$$\Phi\colon \mathcal{L}(V,W) \to \mathcal{L}(U,W)$$
$$\psi \mapsto \psi \circ \phi.$$

Prove that the operator norm $\|\Phi\|_{\mathrm{Frob}(\mathcal{L}(V,W)) \to \mathrm{Frob}(\mathcal{L}(U,W))}$ with respect to the Frobenius norms on $\mathcal{L}(V,W)$ and $\mathcal{L}(U,W)$, is exactly the operator norm $\|\phi\|_{U \to V}$.

4. Let V be a finite-dimensional inner product space.

(a) Prove that if $\phi, \psi \in \mathcal{L}(V,V)$ are both self-adjoint then $\langle \phi, \psi \rangle_{\mathrm{Frob}}$ is real.

Given $v \in V$, let $\theta_v \in \mathcal{L}(V,V)$ denote the linear map $u \mapsto \langle u, v \rangle\, v$. Consider the subset $\mathcal{C} \subseteq \mathcal{L}(V,V)$ given by

$$\mathcal{C} = \left\{ \sum_{i=1}^m a_i \theta_{v_i} : m \geq 0,\ v_1, \ldots, v_m \in V,\ a_1, \ldots, a_m \geq 0 \right\};$$

i.e., the set of non-negative sums of elements θ_v.

(b) Let $\phi \in \mathcal{L}(V,V)$ be self-adjoint. Prove that the following are equivalent:—

(i) ϕ is non-negative definite;

(ii) $\phi \in \mathcal{C}$;

(iii) for all $\psi \in \mathcal{C}$ we have $\langle \phi, \psi \rangle_{\mathrm{Frob}} \geq 0$.

[If you wish, you can say that "the non-negative definite cone $\mathcal{C}$ is self-dual".]